On the Inverse Problem of a Creeping Motion in Thin Layers

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Abstract. The new partially invariant solution of two-dimential motions of heated viscous liquid equations is considered. For factor-system arised the initial boundary value problem is formulated. This problem is inverse one and describing of common motion of two immiscible liquids in a plane channel under the action of thermocapillary forces. As Marangoni number is small (so-called creeping flow) the problem becomes the linear one. Some a priori estimates are obtained and input data conditions when solution tends to stationary one are found. In Laplace transforms the exact solution is obtained as quadratures and some numerical results of velocities behavior in layers are presented.

Keywords: Thermocapillarity, a priori estimates, conjugate initial-boundary value problem, asymptotic behaviour, numerical simulation

1 Introduction

It is well known that in a non-uniformly heated liquid a motion can arise. In some applications of liquid flows, a joint motion of two or more fluids with surfaces takes place. If the liquids are not soluble in each other, they form a more or less visual interfaces. The petroleum-water system is a typical example of this situation. At the present time modelling of multiphase flows taking into account different physical and chemical factors is needed for designing of cooling systems and power plants, in biomedicine, for studying the growth of crystals and films, in aerospace industry [1-4].

Nowadays, there are exact solutions of the Marangoni convection [5-7]. One of the first solutions was obtained in [8]. This is the Poiseuille stationary flow of two immiscible liquids in an inclined channel. As a rule, all such flows were considered steady and unidirectional. The stability of such flows was investigated in [9, 10]. As for non-stationary thermocapillary flows, studying of them began recently [11, 12].

Thermocapillary convection problem for two incompressible liquids separated by a closed interface in a container was investigated in [13]. Local (in time) unique solvability of the problem was obtained in Holder classes of functions. The problem of thermalcapillary 3D motion of a drop was studied in [14]. Moreover, its unique solvability in Holder spaces with a power-like weight at infinity was established. Velocity vector field decreases at infinity in the same way as the initial data and mass forces, the temperature diverges to the constant which is the limit of the initial temperature at infinity. The present work is devoted to studying of solutions of a conjugate boundary value problem arising as a result of linearization of the Navier-Stokes system supplemented with temperature equation. The description of the 2D creeping joint motion of two viscous heat conducting fluids in flat layers is also provided here. The motion arises due to thermocapillary forces imposed along two interfaces, after which the unsteady Marangoni convection begins. Such kind of convection can dominate in flows under microgravity conditions or in motions of thin liquid films.

2 Statement of the Problem

The 2D motion of a viscid incompressible heat-conducting liquid in the absence of mass forces is described by the system of equations

$$u_{1t} + u_1 u_{1x} + u_2 u_{1y} + \frac{1}{\rho} p_x = \nu(u_{1xx} + u_{1yy}), \tag{2.1}$$

$$u_{2t} + u_1 u_{2x} + u_2 u_{2y} + \frac{1}{\rho} p_y = \nu(u_{2xx} + u_{2yy}), \tag{2.2}$$

$$u_{1x} + u_{2y} = 0, \tag{2.3}$$

$$\theta_t + u_1\theta_x + u_2\theta_y = \chi(\theta_{xx} + \theta_{yy}). \tag{2.4}$$

Here $u_1(x, y, t)$ and $u_2(x, y, t)$ are the components of the velocity vector, $p(x, y, t)$ is the pressure, $\theta(x, y, t)$ is the temperature, $\rho > 0$ is the density, $\nu > 0$ is the kinematic viscosity and $\chi > 0$ is the thermal conductivity of the liquid. The quantities $\rho > 0$, $\nu > 0$ and $\chi > 0$ are constant.

The system of equation (2.1)–(2.4) admits a four-dimential Lie subalgebra $G_4 = \langle \partial_x, \partial_{u_1} + t\partial_x, \partial_p, \partial_\theta \rangle$. Its invariants are t, y, u_2 and a partially invariant solution of rank 2 and defect 3 should be sought for in the form

$$u_1 = u_1(x, y, t), \quad u_2 = v(y, t), \quad p = p(x, y, t), \quad \theta = \theta(x, y, t).$$

Inserting the exact form of the solution into the equations (2.1)–(2.3) yields

$$u_1 = w(y, t)x + g(y, t), \quad w + v_y = 0,$$

$$w_t + v w_y + w^2 = f(t) + \nu w_{yy}, \quad \frac{1}{\rho} p = d(y, t) - \frac{f(t)x^2}{2}, \tag{2.5}$$

$$d_y = \nu v_{yy} - v_t - v v_y, \quad g_t + v g_y + w g = 0$$

with some function $f(t)$ that is arbitrary so far.

Regarding the temperature field, we assume that equation (2.4) has the solution of the form

$$\theta = a(y, t)x^2 + m(y, t)x + b(y, t). \tag{2.6}$$

As we see below, (2.6) is in good accord with conditions on the interface.

The stationary solution of the Navier-Stokes equations in the form (2.5) for $g = 0$ for pure viscous fluid was found for the first time by [15]. It describes the liquid impingement from infinity on the plane $y = 0$ under the no slip condition on it. In the paper [16], this solution for the flow between two plates or for the flow in a cylindrical tube (axisymmetric analogue of solution (2.5)) was applied.

It is known that the temperature dependence of the surface tension coefficient is the one of the most important factors leading to the dynamic variety of the interfacial surface. In the papers [17, 18] the stationary solutions in form (2.5), (2.6) was found at $a(y,t) \equiv 0$, $b = \text{const}$ for a flat layer with a free boundary $y = l = \text{const}$ and a solid wall $y = 0$. The non-uniqueness of solution depending on the physical parameters of the problem was revealed. A similar problem in the case of half space was investigated in [19].

We assume for simplicity that $g(y,t) \equiv 0$, $m(y,t) \equiv 0$. The latter condition means that the temperature field has an extremum at $x = 0$, more exactly, a maximum for $a(y,t) < 0$ and a minimum for $a(y,t) > 0$.

Let us apply the solution of the form (2.5), (2.6) to described joint motion of two immiscible liquids in the flat layer $0 < y < h$ considering that the wall $y = 0$ and $y = h$ are solid and the line Γ: $y = l(x,t)$ is their common interface, see Fig. 1.

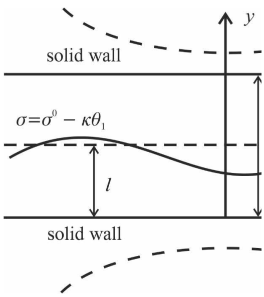

Fig. 1. Geometry of the Marangoni convection problem

Introduction the index $j = 1, 2$ for the liquids and using (2.5) and (2.6), we come to the conclusion that the unknowns satisfy the equations

$$w_{jt} + v_j w_{jy} + w_j^2 = \nu_j w_{jyy} + f_j(t), \quad w_j + v_{jy} = 0, \tag{2.7}$$

$$\frac{1}{\rho_j} p_j = d_j(y,t) - \frac{f_j(t)x^2}{2}, \quad d_{jy} = \nu_j v_{jyy} - v_{jt} - v_j v_{jy}, \tag{2.8}$$

$$a_{jt} + 2w_j a_j + v_j a_{jy} = \chi_j a_{jyy}, \quad b_{jt} + v_j b_{jy} = \chi_j b_{jyy} + 2\chi_j a_j \tag{2.9}$$

in domain $0 < y < l(x,t)$ for $j = 1$ and in domain $l(x,t) < y < h$ for $j = 2$.

At the interface $y = l(x,t)$ the conditions hold [1]

$$w_1(l(x,t),t) = w_2(l(x,t),t), \quad v_1(l(x,t),t) = v_2(l(x,t),t), \tag{2.10}$$

$$l_t + x w_1(l(x,t),t) l_x = v_1(l(x,t),t), \tag{2.11}$$

$$a_1(l(x,t),t) = a_2(l(x,t),t), \quad k_1 \frac{\partial a_1}{\partial n} = k_2 \frac{\partial a_2}{\partial n}\,,$$

$$b_1(l(x,t),t) = b_2(l(x,t),t), \quad k_1 \frac{\partial b_1}{\partial n} = k_2 \frac{\partial b_2}{\partial n}\,,$$

$k_1 > 0$, $k_2 > 0$ are the heat conductivity coefficients and $\mathbf{n} = (1+l_x^2)^{-1/2}(-l_x, 1)$ is the normal to the line $y = l(x,t)$.

The dynamic condition for Γ has a vector form [1]

$$(p_1 - p_2)\mathbf{n} + 2[\mu_2 D(\mathbf{u}_2) - \mu_1 D(\mathbf{u}_1)]\mathbf{n} = 2\sigma K \mathbf{n} + \nabla_\Gamma \sigma, \quad \mu_j = \rho_j \nu_j. \tag{2.12}$$

In (2.12) $D(\mathbf{u})$ is the strain-rate tensor, $\sigma(\theta_1)$ is the surface tension coefficient, K is the mean curvature of the interface, whereas $\nabla_\Gamma = \nabla - \mathbf{n}(\mathbf{n}\cdot\nabla)$ on the right-hand side designates the surface gradient. For most of real liquid media the dependence $\sigma(\theta_1)$ is approximated well by the linear function

$$\sigma(\theta_1) = \sigma^0 - \kappa\theta_1, \tag{2.13}$$

where $\sigma^0 > 0$ and $\kappa > 0$. They are assumed constant and determent by experimental methods. Projecting condition (2.12) to the tangent direction $\boldsymbol{\tau} = (1 + l_x^2)^{-1/2}(1, l_x)$, and using (2.13), (2.6) we obtain

$$l_x[\mu_2(v_{2y} - w_2) - \mu_1(v_{1y} - w_1)] + \frac{x}{2}\,(1 - l_x^2)(\mu_2 w_{2y} - \mu_1 w_{1y})$$
$$= -\kappa(\theta_{1x} + l_x\theta_{1y}) = -\kappa[2a_1 x + l_x(a_{1y}x^2 + b_{1y})]. \tag{2.14}$$

The projection (2.12) to the normal $\mathbf{n}$ yields

$$\rho_1 d_1 - \rho_2 d_2 + \frac{[\rho_2 f_2(t) - \rho_1 f_1(t)]x^2}{2} + 2[\mu_2 D(\mathbf{u}_2) - \mu_1 D(\mathbf{u}_1)]\mathbf{n}\cdot\mathbf{n}$$
$$= [\sigma^0 - \kappa(a_1 x^2 + b_1)]\frac{l_{xx}}{(1 + l_x^2)^{3/2}}\,. \tag{2.15}$$

The boundary conditions on the solid walls have the form

$$w_1(0,t) = 0, \quad v_1(0,t) = 0, \quad w_2(h,t) = 0, \quad v_2(h,t) = 0, \tag{2.16}$$

$$a_1(0,t) = a_{10}(t), \quad a_2(h,t) = a_{20}(t), \tag{2.17}$$

$$b_1(0,t) = b_{10}(t), \quad b_2(h,t) = b_{20}(t), \tag{2.18}$$

with some given functions $a_{j0}(t)$ and $b_{j0}(t)$.

The initial conditions for the velocities are zero because of we study the properties of the solution of the problem simulating the motion only under the action of thermocapillary forces $w_j(y,0) = 0$, $v_j(y,0) = 0$. Besides, $l(x,0) = l_0(x)$, $a_j(y,0) = a_j^0(y)$, $b_j(y,0) = b_j^0(y)$.

Note several specific features of the formulated problem. This is a nonlinear and inverse one since the functions $f_j(t)$ are unknowns also. It is easy to understand if we exclude the functions $v_j(y,t)$ from the equations of mass conservation. Then the problem reduces to the conjugate problem for the functions $w_j(y,t)$, $a_j(y,t)$ and $l(x,t)$. The problem for $b_j(y,t)$ given $v_j(y,t)$ and $a_j(y,t)$ can be separated. The functions $d_j(y,t)$ can be recovered by quadrature from the second equation (2.8) up to a function of time. The last condition in (2.10) and the fourth from (2.16) are the additional conditions on $f_j(t)$, $j = 1,2$.

Let us introduce the characteristic scales of length and time as well as functions w_j, v_j, a_j, d_j and f_j, namely, the quantities l^0, l^{02}/ν_1, $\kappa a^0 l^0/\mu_1$, $\kappa a^0 l^{02}/\mu_1$, a^0, $\kappa a^0 l^0/\rho_1$, $\kappa a^0/(\rho_1 l^0)$, where $l^0 = \mathrm{const} > 0$ is the average value of thickness of the first layer of the liquid at $t = 0$, $a^0 = \max\limits_{t \geqslant 0} |a_{20}(t) - a_{10}(t)| > 0$, or $a^0 = \max\limits_{j} \max\limits_{y} |a_{j0}(y)| > 0$, if $a_{20}(t) = a_{10}(t)$. In the dimensionless variables, some factor appears at the nonlinear terms in (2.7), the Marangoni number

$$\mathrm{M} = \kappa a^0 l^{03}/(\mu_1 \nu_1). \tag{2.19}$$

The same applies to the kinematic condition (2.11)

$$\bar{l}_{\bar{t}} + \bar{x}\mathrm{M}\bar{w}(\bar{l}(\bar{x},\bar{t}),\bar{t})\bar{l}_{\bar{x}} = \mathrm{M}\bar{v}_1(\bar{l}(\bar{x},\bar{t}),\bar{t}). \tag{2.20}$$

Assume that the $\mathrm{M} \ll 1$. The latter holds either in the thin layers or large viscosities. Then the nonlinear terms in the equations can be neglected and the latter become linear. In particular, the kinematic condition (2.20) has the form $\bar{l}_{\bar{t}} = 0$, i.e. $\bar{l} = \bar{l}(x)$. Let us turn to (2.15). After transition to the dimensionless variables on the right-hand side the Weber number $\mathrm{We} = \sigma^0/(\kappa a^0 l^{02})$ appears instead of σ^0. In the real conditions $\mathrm{We} \gg 1$ for the most of liquid media; for example, for the water–air system $\mathrm{We} \sim 10^6$.

Therefore, for these Weber numbers, (2.14) assume the form $\bar{l}_{\bar{x}\bar{x}} = 0$, i.e. $\bar{l} = \alpha x + l^0$. We assume later that $\alpha = 0$ and the interface is the plane $y = l^0 < h$ parallel to the solid walls $y = 0$ and $y = h$; in what follows, the index 0 for l^0 will be omitted.

3 A priori Estimates

Let us present the so-obtained linear problem in its entirely in dimensional form

$$w_{jt} = \nu_j w_{jyy} + f_j(t), \tag{3.1}$$

$$w_j(y,0) = 0, \tag{3.2}$$

$$w_1(0,t) = 0, \quad w_2(h,t) = 0, \tag{3.3}$$

$$w_1(l,t) = w_2(l,t), \tag{3.4}$$

$$\mu_2 w_{2y}(l,t) - \mu_1 w_{1y}(l,t) = -2\kappa a_1(l,t), \tag{3.5}$$

$$\int\limits_0^l w_1(z,t)\,dz = 0, \quad \int\limits_l^h w_2(z,t)\,dz = 0, \tag{3.6}$$

where $0 < y < l$ for $j = 1$ and $l < y < h$ for $j = 2$. The first equality in (3.6) follows from (2.10) whereas the last in the no-slip condition $v_2(h,t) = 0$.

Let us write the problem for the functions $a_j(y,t)$

$$a_{jt} = \chi_j a_{jyy}, \tag{3.7}$$

$$a_j(y,0) = a_j^0(y), \tag{3.8}$$

$$a_1(0,t) = a_{10}(t), \quad a_2(h,t) = a_{20}(t), \tag{3.9}$$

$$a_1(l,t) = a_2(l,t), \quad k_1 a_{1y}(l,t) = k_2 a_{2y}(l,t). \tag{3.10}$$

In order to obtain a priori estimates for $w_j(y,t)$, $f_j(t)$ of the solution of (3.1)–(3.5), it is necessary firstly to infer the estimates for the solutions of initial-boundary value problem (3.7)–(3.10). We perform the change of variables

$$\begin{aligned} a_1(y,t) &= \bar a_1(y,t) + \frac{a_{10}(t)(y-l)^2}{l^2}, \quad 0 \leqslant y \leqslant l^0 \equiv l, \\ a_2(y,t) &= \bar a_2(y,t) + \frac{a_{20}(t)(y-l)^2}{(h-l)^2}, \quad l \leqslant y \leqslant h. \end{aligned} \tag{3.11}$$

The functions $\bar a_j(y,t)$ in their domains satisfy the equations

$$\bar a_{1t} = \chi_1 \bar a_{1yy} + \frac{2\chi_1 a_{10}(t)}{l^2} - \frac{a_{10}'(t)(y-l)^2}{l^2} \equiv \chi_1 \bar a_{1yy} + g_1(y,t), \tag{3.12}$$

$$\bar a_{2t} = \chi_2 \bar a_{2yy} + \frac{2\chi_2 a_{20}(t)}{(h-l)^2} - \frac{a_{20}'(t)(y-l)^2}{(h-l)^2} \equiv \chi_2 \bar a_{2yy} + g_2(y,t), \tag{3.13}$$

where the prime denotes differentiation with respect to time. Boundary conditions (3.9) for $\bar a_1$ and $\bar a_2$ become homogeneous, whereas (3.10) preserve it form. Initial conditions (3.8) for $\bar a_1$ and $\bar a_2$ change

$$\begin{aligned} \bar a_1(y,0) &= a_1^0(y) - \frac{a_{10}(0)(y-l)^2}{l^2} \equiv \bar a_1^0(y), \\ \bar a_2(y,0) &= a_2^0(y) - \frac{a_{20}(0)(y-l)^2}{l^2} \equiv \bar a_2^0(y). \end{aligned} \tag{3.14}$$

Let us multiply (3.1), (3.2) by $\rho_1 c_1 \bar a_1$ and $\rho_2 c_2 \bar a_2$ c_1, c_2 and integrate over the segments $[0,l]$, $[l,h]$ taking into account (3.8) and (3.9). Then add up the result. We infer that

$$\frac{dA(t)}{dt} + k_1 \int\limits_0^l \bar a_{1y}^2\,dy + k_2 \int\limits_l^h \bar a_{2y}^2\,dy = \rho_1 c_1 \int\limits_0^l g_1 \bar a_1\,dy + \rho_2 c_2 \int\limits_l^h g_2 \bar a_2\,dy, \tag{3.15}$$

$$A(t) = \frac{\rho_1 c_1}{2} \int\limits_0^l \bar{a}_1^2 \, dy + \frac{\rho_2 c_2}{2} \int\limits_l^h \bar{a}_2^2 \, dy, \tag{3.16}$$

where c_j are the coefficients of the specific heat capacity. Along with (3.15) there is another identity

$$\rho_1 c_1 \int\limits_0^l \bar{a}_{1t}^2 \, dy + \rho_2 c_2 \int\limits_l^h \bar{a}_{2t}^2 \, dy + \frac{1}{2} \frac{\partial}{\partial t} \left[k_1 \int\limits_0^l \bar{a}_{1y}^2 \, dy + k_2 \int\limits_l^h \bar{a}_{2y}^2 \, dy \right]$$

$$= \rho_1 c_1 \int\limits_0^l g_1 \bar{a}_{1t} \, dy + \rho_2 c_2 \int\limits_l^h g_2 \bar{a}_{2t} \, dy. \tag{3.17}$$

From (3.15) and (3.17) we obtain the inform estimates in y

$$|a_j(y,t)| \leqslant \left(\frac{8\chi_j}{k_j^2} \, F(t) A(t) \right)^{1/4} + |a_{j0}(t)|, \tag{3.18}$$

where

$$F(t) = k_1 \int\limits_0^l \bar{a}_{10}^2(y) \, dy + k_2 \int\limits_l^h \bar{a}_{20}^2(y) \, dy + \frac{2k_1}{\chi_1} \left[\frac{4\chi_1}{l^3} \int\limits_0^t a_{10}^2(\tau) \, d\tau + \frac{l}{5} \int\limits_0^t (a'_{10}(\tau))^2 \, d\tau \right]$$

$$+ \frac{2k_2}{\chi_2} \left[\frac{4\chi_2}{(h-l)^3} \int\limits_0^t a_{20}^2(\tau) \, d\tau + \frac{h-l}{5} \int\limits_0^t (a'_{20}(\tau))^2 \, d\tau \right] \equiv F(t), \tag{3.19}$$

$$A(t) \leqslant e^{-2\delta t} \left[\sqrt{A(0)} + \sqrt{\frac{k_1}{\chi_1}} \left(\frac{2\chi_1}{\sqrt{l^3}} \int\limits_0^t e^{\delta\tau} |a_{10}(\tau)| \, d\tau + \sqrt{\frac{l}{5}} \int\limits_0^t e^{\delta\tau} |a'_{10}(\tau)| \, d\tau \right) \right.$$

$$\left. + \sqrt{\frac{k_2}{\chi_2}} \left(\frac{2\chi_2}{\sqrt{(h-l)^3}} \int\limits_0^t e^{\delta\tau} |a_{20}(\tau)| \, d\tau + \sqrt{\frac{h-l}{5}} \int\limits_0^t e^{\delta\tau} |a'_{20}(\tau)| \, d\tau \right) \right]^2. \tag{3.20}$$

As to functions $w_j(y,t)$, $f_j(t)$ the following estimates hold

$$|w_1(y,t)| \leqslant 2 \left[\frac{E(t)}{\nu_1} \left(F(t) + \frac{4\kappa^2 l a_1^2(l,t)}{5\mu_1} \right) \right]^{1/4}, \tag{3.21}$$

$$|w_2(y,t)| \leqslant \left(\frac{8}{\nu_2} \, E(t) F_2(t) \right)^{1/4}, \tag{3.22}$$

$$|f_1(t)| \leqslant 2 \left[\frac{E_1(t)}{\nu_1} \left(F_3(t) + \frac{4\kappa^2 l (a'_1(l,t))^2}{5\mu_1} \right) \right]^{1/4} + \frac{12\nu_1}{l^2} \left(\frac{8E(t)}{\nu_1} \, F_2(t) \right)^{1/4},$$

$$|f_2(t)| \leqslant \left(\frac{8E_1(t)}{\nu_2}\,F_3(t)\right)^{1/4} + \frac{12\nu_2}{(h-l)^2}\left(\frac{8E(t)}{\nu_2}\,F_2(t)\right)^{1/4}, \tag{3.23}$$

where

$$E(t) \leqslant e^{-4\delta_1 t}\int\limits_0^t H(\tau)e^{4\delta_1\tau}\,d\tau, \tag{3.24}$$

$$H(t) = \frac{2\kappa}{\varepsilon}\left[\left(\frac{8\chi_1}{k_1^2}\,F(t)A(t)\right)^{1/2} + a_{10}^2(t)\right]. \tag{3.25}$$

The functions $E_1(t)$, $F_1(t)$, $F_2(t)$ and $F_3(t)$ have the same structures as $E(t)$, $F(t)$.

4 Stationary Flow

The problem (3.1)–(3.10) has the stationary solution $w_j^s(y)$, $a_j^s(y)$, f_j^s

$$w_1^s(y) = \frac{\kappa(1-\gamma)Ah(3y^2/h^2 - 2\gamma y/h)}{2\gamma\mu_2[\gamma+\mu(1-\gamma)]},$$

$$w_2^s(y) = \frac{\kappa\gamma Ah(3y^2/h^2 - 2(2+\gamma)y/h + 1 + 2\gamma)}{2(1-\gamma)\mu_2[\gamma+\mu(1-\gamma)]},$$

$$a_1^s = \frac{(a_{20}^s - a_{10}^s)}{[\gamma + k(1-\gamma)]}\,\frac{y}{h} + a_{10}, \tag{4.1}$$

$$a_2^s = \frac{1}{\gamma + k(1-\gamma)}\,\left[k(a_{20}^s - a_{10}^s)\frac{y}{h} + ka_{10}^s + \gamma(1-k)a_{20}^s\right],$$

$$f_1^s = -\frac{3\kappa\nu(1-\gamma)A}{\gamma h\rho_2[\gamma+\mu(1-\gamma)]}, \quad f_2^s = -\frac{3\kappa\gamma A}{(1-\gamma)h\rho_2[\gamma+\mu(1-\gamma)]},$$

$a_1^s(0) = a_{10}^s$, $a_2^s(h) = a_{20}^s$, $k = k_1/k_2$, $\nu = \nu_1/\nu_2$, $\gamma = l/h < 1$, $\mu = \mu_1/\mu_2$,

$$A = \frac{(a_{20}^s - a_{10}^s)\gamma}{\gamma + k(1-\gamma)}; \tag{4.2}$$

$$v_1^s(y) = -\frac{\kappa(1-\gamma)Ah}{2\gamma\mu_2[\gamma+\mu(1-\gamma)]}\,\left(\frac{y^3}{h^3} - \frac{\gamma y^2}{h^2}\right),$$

$$v_2^s(y) = -\frac{\kappa\gamma Ah^2}{2(1-\gamma)\mu_2[\gamma+\mu(1-\gamma)]}\left[\left(\frac{y^3}{h^3} - \gamma^3\right)\right. \tag{4.3}$$
$$\left. -(2+\gamma)\left(\frac{y^2}{h^2} - \gamma^2\right) + (1+2\gamma)\left(\frac{y}{h} - \gamma\right)\right].$$

Introducing the differences

$$N_j(y,t) = a_j^s(y) - a_j(y,t), \quad M_j(y,t) = w_j^s(y) - w_j(y,t) \tag{4.4}$$

and carrying out the calculations analogous to those in Section 2, we can prove that the solution of the nonstationary problem reaches the steady regime $w_j^s(y)$, $a_j^s(y)$ and f_j^s under the conditions of convergence of the integrals

$$\int\limits_0^\infty e^{\delta\tau}|a_{j0}^s - a_{j0}(\tau)|\,d\tau, \quad \int\limits_0^\infty e^{\delta\tau}|a'_{j0}(\tau)|\,d\tau, \quad \int\limits_0^\infty e^{\delta\tau}|a''_{j0}(\tau)|\,d\tau. \tag{4.5}$$

More exactly, $\|w_j(y,t) - w_j^s(y)\| \leqslant d_j e^{-\delta_1 t}$, $\|a_j(y,t) - a_j^s(y)\| \leqslant l_j e^{-\delta_2 t}$, $\|f_j(t) - f_j^s\| \leqslant N e^{-\delta_3 t}$ with the positive constant d_j, l_j, N, δ_1, δ_2, δ_3 depending on physical parameters of liquid and layers thicknesses.

5 Nonstationary Motion and Numerical Results

To describe the nonstationary motion of two viscous thermally conducting liquids the Laplace transform will be applied to problem (3.1)–(3.10). As a result we come to boundary value problem for images $\hat{a}_j(y,p)$ of functions $a_j(y,t)$

$$\hat{a}_{yy} - \frac{p\hat{a}}{\chi_j} = -\frac{a_j^0(y)}{\chi_j}\,, \tag{5.1}$$

$$\hat{a}_1(0,p) = \hat{a}_{10}(p), \quad \hat{a}_2(h,p) = \hat{a}_{20}(p), \tag{5.2}$$

$$\hat{a}_1(l,p) = \hat{a}_2(l,p), \quad k_1\hat{a}_{1y}(l,p) = k_2\hat{a}_{2y}(l,p), \tag{5.3}$$

and images $\hat{w}_j(y,p)$ and $\hat{f}_j(p)$ of functions $w_j(y,t)$, $f_j(t)$

$$\hat{w}_{jyy} - \frac{p}{\nu_j}\,\hat{w}_j = -\frac{\hat{f}_j(p)}{\nu_j}\,, \tag{5.4}$$

$$\hat{w}_1(0,p) = 0, \quad \hat{w}_2(h,t) = 0, \tag{5.5}$$

$$\hat{w}_1(l,p) = \hat{w}_2(l,p), \tag{5.6}$$

$$\mu_2\hat{w}_{2y}(l,p) - \mu_1\hat{w}_{1y}(l,p) = -2\kappa\hat{a}_1(l,p), \tag{5.7}$$

$$\int\limits_0^l \hat{w}_1(y,p)\,dy = 0, \quad \int\limits_l^h \hat{w}_2(y,p)\,dy = 0. \tag{5.8}$$

In condition (5.2) and equation (5.4) $\hat{a}_{j0}(p)$, $\hat{f}_j(p)$ are images of functions $a_{j0}(t)$, $f_j(t)$ respectively. The solutions of problem (5.1)–(5.8) can be written as

$$\hat{a}_j(y,p) = C_j^1\,\mathrm{sh}\sqrt{\frac{p}{\chi_j}}\,y + C_j^2\,\mathrm{ch}\sqrt{\frac{p}{\chi_j}}\,y - \frac{1}{\sqrt{p\chi_j}}\int\limits_{y_j}^{y} a_j^0(z)\,\mathrm{sh}\sqrt{\frac{p}{\chi_j}}\,(y-z)\,dz, \tag{5.9}$$

$$\hat{w}_j(y,p) = -2\kappa\hat{a}_1(l,p)\left[D_j^1\,\mathrm{sh}\sqrt{\frac{p}{\nu_j}}\,y + D_j^2\,\mathrm{ch}\sqrt{\frac{p}{\nu_j}}\,y + \frac{L_j(p)}{p}\right], \tag{5.10}$$

where $\hat{f}_j(p) = -2\kappa \hat{a}_1(l,p)L_j(p)$.

The values C_j^1, C_j^2, D_j^1, D_j^2 and $\hat{f}_j(p)$ determined from the boundary conditions (5.2), (5.3), (5.5)–(5.8). Due to the cumbersome the type of these values is not given here.

Let us assume that $\lim\limits_{t\to\infty} a_{j0}(t) = a_{j0}^s$, $j = 1,2$, using the formulas (5.9), (5.10) and presenting for the values C_j^1, C_j^2, D_j^1, D_j^2 and $\hat{f}_j(p)$ we can prove the limit equalities

$$\lim_{t\to\infty} a_j(y,t) = a_j^s(y), \quad \lim_{t\to\infty} w_j(y,t) = w_j^s(y),$$

$$\lim_{t\to\infty} f_j(t) = f_j^s,$$

where $a_j^s(y)$, $w_j^s(y)$, f_j^s are determined by formulas (4.1), (4.2).

Let us apply the numerical method of inversion of Laplace transformation to obtained formulas (5.9), (5.10). The graphs only for the velocities are given because the have a real physical meanings. All numerical calculations were made for the system of liquid silicon–water. Thickness of the layers is the same and equal to 1 mm. The corresponding values of the defining parameters are given in Table 1.

Table 1. Physical properties of liquids

Item	liquid silicon	water
ρ, kg/m^3	956	998
$\nu \times 10^{-6}$, m^2/s	10.2	1.004
k, kg $\cdot$ m/s$^3 \cdot$ K	0.133	0.597
$\chi \times 10^{-6}$, m^2/s	0.0675	0.143
æ $\times 10^{-5}$, kg/s$^2 \cdot$ K	6.4	15.14

Figure 2–5 show the profiles of the dimensionless functions $\bar{w}_j(\xi,\tau) = w_j(y,t)\mu_2/(\kappa A)$ ($\xi = y/l$, $\tau = \nu_1 t/l^2$ are the dimensional variables) and transverse velocity $\bar{v}_j(\xi,\tau) = v_j(y,t)\mu_2/(\kappa A h)$ with $a_{20}(t) = 0$. In particular, the functions $\bar{w}_j$ are negative, so reverse flows arise here. Figure 2, 3 show the results of calculations when $a_{10}(\tau) = \sin\tau$, $a_{20}(\tau) = 0$. That is the limit of $a_{10}(\tau)$ at $\tau \to \infty$ does not exist and the velocity field does not converge to a stationary one.

Figure 4, 5 show an evolution of the convergence of functions $\bar{w}_j$ and transverse velocities $\bar{v}_j$ to stationary regime for the case $a_{10}(\tau) = 1 + e^{-\tau}\cos(10\tau)$, $a_{20}(\tau) = 0$. These results are good agreement with the a priori estimates were obtained in Section 4.

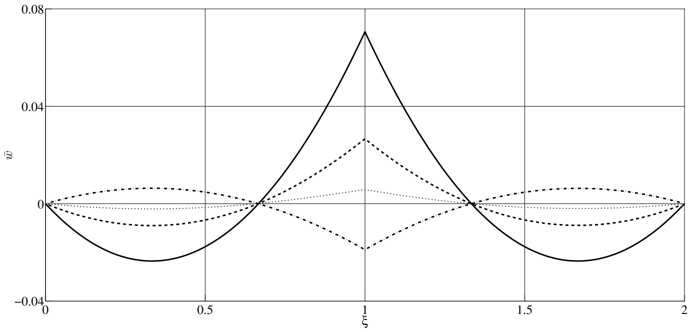

Fig. 2. Evolution of functions $\bar{w}_j$ for $a_{10}(\tau) = \sin\tau$. Total line is the stationary profiles, $-- - \tau = 4$, $- \cdot - - \tau = 5$, $\cdots - \tau = 7$

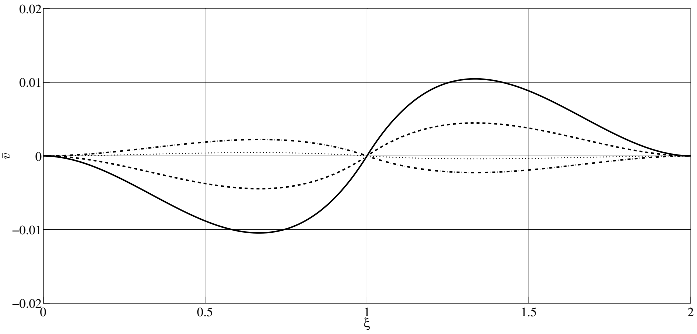

Fig. 3. Evolution of functions $\bar{v}_j$ for $a_{10}(\tau) = \sin\tau$. Total line is the stationary profiles, $-- - \tau = 3$, $- \cdot - - \tau = 6$, $\cdots - \tau = 8$

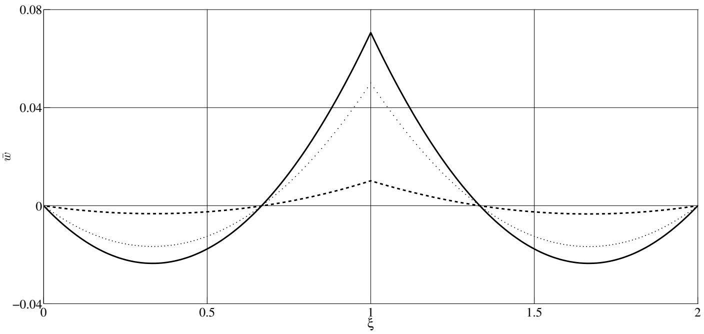

Fig. 4. Evolution of functions $\bar{w}_j$ for $a_{10}(\tau) = 1 + e^{-\tau}\cos(10\tau)$. Total line is the stationary profiles, $-- - \tau = 1$, $\cdots - \tau = 4$

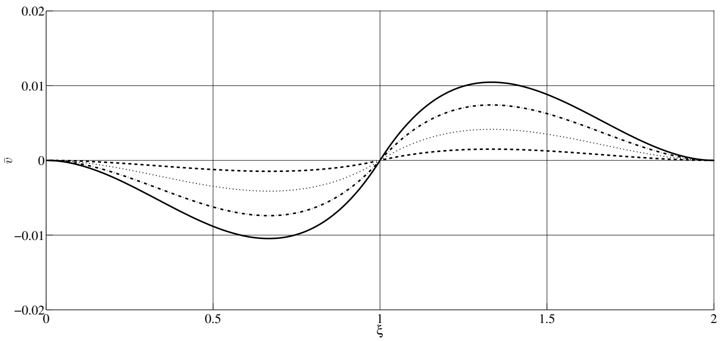

Fig. 5. Evolution of functions $\bar{v}_j$ for $a_{10}(\tau) = 1 + e^{-\tau}\cos(10\tau)$. Total line is the stationary profiles, $---\ \tau = 1$, $-\cdot--\ \tau = 4$, $\cdots - \tau = 2$

6 Conclusion

The two-dimensional horizontal layer is a matter of great importance in connection with the theory of convective stability applications in the design of cooling systems, in studying the growth of crystals and films, or in the aerospace industry. We have presented a theoretical and numerical study of a creeping flow of two immiscible viscous heat conducting liquids in thin layers. The flow arises due to heat exchange with the localized parabolic heating of the borders and through the thermocapillary forces on the interface. The following results are obtained: (1) the exact solution describing the stationary thermocapillary convective flow is found; (2) a priori estimates of the initial boundary value problem are established and sufficient conditions on input data when solution tends to stationary one are obtained; (3) the solution of the non-stationary problem in the form of final analytical formulas in the Laplace representation is found and some numerical results of velocities behaviour in layers are presented.

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