Some Exact Solutions of a Heat Wave Type of a Nonlinear Heat Equation

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Abstract. The exact solutions of the nonlinear heat (porous medium) equation are constructed. We obtain a new class of the heat wave type solutions the construction of which is reduced to the Cauchy problems for nonlinear second order differential equations with a singularity. For these problems we prove a new existence and uniqueness theorem in the class of analytic functions. A special case of the heat wave front is considered in details. The results of numerical experiments are presented and discussed.

Keywords: nonlinear heat equation, porous medium equation, exact solutions, heat waves, Cauchy problem, existence and uniqueness theorem.

1 Introduction

The heat equation [1,2] is one of the well-known objects of classical mathematical physics. If a thermal conductivity does not depend on temperature, we have the linear equation. This case is well studied and we do not consider it. In this paper we deal with the nonlinear heat equation when the coefficient of thermal conductivity has a power-law dependence on the temperature. Besides heat conduction this equation also describes the ideal polytropic gas filtration in a porous medium. Therefore, in the literature it is also called “the porous medium equation” [2, 3].

Solutions of a heat wave type are an important and interesting class of nonlinear heat equation solutions. Description of the process of the heat wave spread across the cold background at a finite speed, and the first examples of heat wave type solution were given by Ya.B. Zel’dovich in [4]. In the class of analytical functions the boundary-value problem with degeneration (Sakharov’s problem of the initiation of the heat wave) was first considered by A.F. Sidorov in [5]. The inverse problem, where for a given edge of the heat wave solution is recovered, including the boundary regime, was studied by S.P. Bautin in [6]. There are certain papers of the scientific Sidorov's school members, which are devoted to this problem [7–9]. The numerical methods for the construction of a heat wave are proposed in [10, 11].

In this paper we construct exact solutions of the heat wave type for the nonlinear one-dimensional heat equation. The construction reduces to the Cauchy problem for nonlinear ordinary differential equations of second order with a singularity at the highest derivative. In the literature such solutions of nonlinear partial differential equations are called "the exact solutions" [12, 13]. The obtained exact solutions allow us to find some of global properties of heat waves.

2 Problem Statement

We consider the nonlinear parabolic equation

$$T_t = \operatorname{div}(k\nabla T) ,$$

in the case of $k = T^\sigma$, $\sigma \in \mathbb{R}^{>0}$ (the porous medium equation) [2, 3], i. e.

$$T_t = \operatorname{div}(T^\sigma \nabla T) . \tag{1}$$

Here T is a function (temperature), depending on the time $t \geqslant 0$ and $\mathbf{x} \stackrel{\text{def}}{=} (x_1, x_2, x_3) \in \mathbb{R}^3$ be a vector of spatial variables. Operators div and ∇ act on $\mathbf{x}$.

If there are the symmetries, Eq. (1) can be converted to the form of one-dimensional heat equation

$$u_\tau = uu_{\rho\rho} + \frac{1}{\sigma}u_\rho^2 + \frac{\nu}{\rho}uu_\rho , \quad \nu \in \{0, 1, 2\} , \tag{2}$$

where $u : D \to \mathbb{R}$ is a unknown function, defined on a set $D \subset \mathbb{R}^2$. It depends on the time variable $\tau \geqslant 0$ and the space variable

$$\rho \stackrel{\text{def}}{=} ||\mathbf{x}|| = \left(\sum_{k=1}^{\nu+1} x_k^2 \right)^{\frac{1}{2}} .$$

If $\nu \neq 0$, it should be noted that $\rho \neq 0$.

The values of the parameter ν correspond to the heat propagation on the line, on the plane and in the space of symmetrically with regard to the origin.

In this paper we construct and study the exact heat wave-type solutions of Eq. (2), which satisfy the condition

$$u|_{\rho=f(\tau)} = 0 , \tag{3}$$

where $\rho = f(\tau)$ is a front of the heat wave, defined in the plane of the variables (τ, ρ). We have found that the boundary problem (2), (3), besides the trivial solution $u(\tau, \rho) = 0$, which is obvious, has some nontrivial classes of exact solutions.

Similar one-dimensional nonlinear heat conduction problems with the heat flux at the origin specified in the form of an exponential time dependence are considered in paper [14]. We construct exact (automodel) and approximate solutions of this problem.

3 Construction of Exact Solutions

This section is dedicated to finding non-trivial heat wave-type solutions of Eq. (2), the construction of which is associated with the solution of ordinary differential equations.

We assume that

$$u(\tau,\rho)=\psi(\tau,\rho)w(\xi)\ ,\quad \xi\stackrel{\text{def}}{=}\xi(\tau,\rho)\ , \tag{4}$$

where ψ, ξ and w are twice continuously differentiable functions of their variables, such that $\psi_\tau\xi_\tau\xi_\rho\not\equiv 0$. Now we substitute (4) in (2) and find the acceptable expression for $\psi(\tau,\rho)$ and $\xi(\tau,\rho)$. After dividing the resulting equation by $\psi^2\xi_\rho^2\not\equiv 0$, we have

$$ww''+\frac{1}{\sigma}(w')^2+\left[2\left(\frac{1}{\sigma}+1\right)\frac{\psi_\rho}{\psi\xi_\rho}+\frac{\xi_{\rho\rho}}{\xi_\rho^2}+\nu\frac{1}{\rho\xi_\rho}\right]ww'+$$
$$+\left(\frac{1}{\sigma}\frac{\psi_\rho^2}{\psi^2\xi_\rho^2}+\frac{\psi_{\rho\rho}}{\psi\xi_\rho^2}+\nu\frac{\psi_\rho}{\rho\psi\xi_\rho^2}\right)w^2-\frac{\xi_\tau}{\psi\xi_\rho^2}w'-\frac{\psi_\tau}{\psi^2\xi_\rho^2}w=0\ .$$

In order that the obtained expression becomes an ODE we should solve an overdetermined system of partial differential equations

$$\frac{\psi_\rho}{\psi\xi_\rho}=a_1\ ,\quad \frac{\xi_{\rho\rho}}{\xi_\rho^2}=a_2\ ,\quad \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2}=a_3\ ,\quad \frac{\xi_\tau}{\psi\xi_\rho^2}=a_4\ ,\quad \frac{\psi_\tau}{\psi^2\xi_\rho^2}=a_5\ ,$$
$$\frac{\psi_\rho}{\rho\psi\xi_\rho^2}=a_6\ ,\quad \frac{1}{\rho\xi_\rho}=a_7\ , \tag{5}$$

where $a_l\in\mathbb{R}$, $l=\overline{1,7}$.

Proposition 1. *Let $\nu\neq 0$, then the system* (5) *is solvable if*

$$2a_2=-a_1=-2a_7\ ,\quad 2a_2^2=a_3=a_6\ ,\quad a_2\neq 0\ .$$

Proof. **1°.** Let $a_4\neq 0$. We have $\psi(\tau,\rho)=\xi_\tau/(a_4\xi_\rho^2)$ from the fourth equation of system (5). We can find $\xi(\tau,\rho)$ from the second and seventh equations. These two equations are solvable, only if $a_2=-a_7\neq 0$. In this case $\xi(\tau,\rho)=\ln[f(\tau)\rho]^{-1/a_2}$. Substituting $\psi(\tau,\rho)$ and $\xi(\tau,\rho)$ in (5), we obtain the relations $2a_2=-a_1=-2a_7$, $2a_2^2=a_3=a_6$ and solvable by quadratures ODE

$$ff''+\left(\frac{a_5}{a_2a_4}-1\right)(f')^2=0\ ,$$

which determines

$$f(\tau) = \begin{cases} C_2 e^{C_1 \tau} , & \text{if } a_5 = 0 , \\ (C_1\tau + C_2)^{\frac{a_2 a_4}{a_5}} , & \text{if } a_5 \neq 0 . \end{cases}$$

Consequently, the system (5) is solvable.

2°. Let $a_4 = 0$. From the fourth equation of system (5) we have $\xi(\tau,\rho) = \xi(\rho)$. Then second and seventh equations provide that $a_2 = -a_7 \neq 0$ and we have $\xi(\rho) = \ln[c\rho]^{-1/a_2}$. Substituting $\xi(\rho)$ in (5), we obtain the system of equations

$$\frac{\rho\psi_\rho}{\psi} = -\frac{a_1}{a_2} , \quad \frac{\rho^2\psi_{\rho\rho}}{\psi} = \frac{a_3}{a_2^2} , \quad \frac{\rho^2\psi_\tau}{\psi^2} = \frac{a_5}{a_2^2} , \quad \frac{\rho\psi_\rho}{\psi} = \frac{a_6}{a_2^2} . \tag{6}$$

Equations (6) have solutions

$$\psi(\tau,\rho) = f_1(\tau)\rho^{-\frac{a_1}{a_2}} , \quad \psi(\tau,\rho) = f_2(\tau)\rho^{\frac{a_6}{a_2^2}} ,$$

$$\psi(\tau,\rho) = f_3(\tau)\rho^{\frac{a_2+\sqrt{a_2^2+4a_3}}{2a_2}} + f_4(\tau)\rho^{\frac{a_2-\sqrt{a_2^2+4a_3}}{2a_2}} , \quad \psi(\tau,\rho) = \frac{a_2^2\rho^2}{a_2^2 f_5(\rho)\rho^2 - a_5\tau} .$$

Thus, it is obvious that for the compatibility of (6) and, as a consequence, the system (5) as well, it is required that

$$-\frac{a_1}{a_2} = \frac{a_6}{a_2^2} = \frac{a_2 \pm \sqrt{a_2^2+4a_3}}{2a_2} = 2 \iff 2a_2 = -a_1 , \ 2a_2^2 = a_6 = a_3 .$$

The proposition is proved. □

The case $\nu = 0$ deserves a special attention. Here the system for $\xi(\tau,\rho)$ and $\psi(\tau,\rho)$ consists of five equations:

$$\frac{\psi_\rho}{\psi\xi_\rho} = a_1 , \quad \frac{\xi_{\rho\rho}}{\xi_\rho^2} = a_2 , \quad \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2} = a_3 , \quad \frac{\xi_\tau}{\psi\xi_\rho^2} = a_4 , \quad \frac{\psi_\tau}{\psi^2\xi_\rho^2} = a_5 , \tag{7}$$

where $a_l \in \mathbb{R}$, $l = \overline{1,5}$.

Proposition 2. *Let $\nu = 0$, then (7) is solvable if*

$$2a_2 = -a_1 , \quad 2a_2^2 = a_3 .$$

Proof. **1°. a)** Let $a_2, a_4 \neq 0$. We have $\psi(\tau,\rho) = \xi_\tau/(a_4\xi_\rho^2)$ from the fourth equation of system (7) and $\xi(\tau,\rho) = \ln[f(\tau)\rho + g(\tau)]^{-1/a_2}$ from the second one. Thus, substituting the expression for $\psi(\tau,\rho)$ and $\xi(\tau,\rho)$ in (7), we get the system of ODE's for $f(\tau)$ and $g(\tau)$:

$$\frac{(2\rho f + g)f' + fg'}{f(\rho f' + g')} = -\frac{a_1}{a_2} , \quad \frac{(\rho f + g)f'}{f(\rho f' + g')} = \frac{a_3}{2a_2^2} ,$$

$$\frac{\rho f(\rho f + g)f'' + f(\rho f + g)g'' - \rho(\rho f + 2g)(f')^2 + f(g')^2 - 2gf'g'}{f(\rho f' + g')^2} = \tag{8}$$

$$= -\frac{a_5}{a_2 a_4} .$$

In order to get rid of the variable ρ in (8) we demand that $g(\tau) \equiv 0$. Then from the first and second equations we have $a_1 = -2a_2$ and $a_3 = 2a_2^2$, respectively, and the third one is converted to the exactly solvable ODE

$$ff'' + \left(\frac{a_5}{a_2 a_4} - 1\right)(f')^2 = 0 \ ,$$

which determines

$$f(\tau) = \begin{cases} C_2 e^{C_1 \tau} \ , & \text{if } a_5 = 0 \ , \\ (C_1 \tau + C_2)^{\frac{a_2 a_4}{a_5}} \ , & \text{if } a_5 \neq 0 \ . \end{cases}$$

Consequently, the system (7) is solvable.

b) Let $a_2 \neq 0, \ a_4 = 0$. We have $\xi(\tau, \rho) = \xi(\rho)$ from the fourth equation of system (7). With this in mind we obtain $\xi(\rho) = \ln[c_1\rho + c_2]^{-1/a_2}$ from the second equation of system (7). Substituting $\xi(\rho)$ in (7), we obtain the system of equations

$$\frac{(\rho + c)\psi_\rho}{\psi} = -\frac{a_1}{a_2} \ , \quad \frac{(\rho + c)^2 \psi_{\rho\rho}}{\psi} = \frac{a_3}{a_2^2} \ , \quad \frac{(\rho + c)^2 \psi_\tau}{\psi^2} = \frac{a_5}{a_2^2} \ , \tag{9}$$

where $c = c_2/c_1$. Equations (9) have solutions

$$\psi(\tau, \rho) = f_1(\tau)(\rho + c)^{-\frac{a_1}{a_2}} \ ,$$

$$\psi(\tau, \rho) = f_2(\tau)(\rho + c)^{\frac{a_2 + \sqrt{a_2^2 + 4a_3}}{2a_2}} + f_3(\tau)(\rho + c)^{\frac{a_2 - \sqrt{a_2^2 + 4a_3}}{2a_2}} \ ,$$

$$\psi(\tau, \rho) = \frac{a_2^2 (\rho + c)^2}{a_2^2 f_4(\rho)(\rho + c)^2 - a_5 \tau} \ .$$

Thus, it is obvious that for the compatibility of (9) and, as a consequence, of the system (7) it is required that

$$-\frac{a_1}{a_2} = \frac{a_2 \pm \sqrt{a_2^2 + 4a_3}}{2a_2} = 2 \iff 2a_2 = -a_1, \ 2a_2^2 = a_3 \ .$$

2°. a) Let $a_2 = 0, \ a_4 \neq 0$. We have $\psi(\tau, \rho) = \xi_\tau/(a_4 \xi_\rho^2)$ from the fourth equation of system (7) and $\xi(\tau, \rho) = f(\tau)\rho + g(\tau)$ from the second one. Thus, substituting the expression for $\psi(\tau, \rho)$ and $\xi(\tau, \rho)$ in (7), we get $a_3 = 0$ and the system of ODE's for $f(\tau)$ and $g(\tau)$:

$$\frac{f'}{f(\rho f' + g')} = a_1 \ , \quad \frac{\rho f f'' + f g'' - 2\rho (f')^2 - 2f'g'}{f(\rho f' + g')^2} = \frac{a_5}{a_4} \ . \tag{10}$$

To eliminate the variable ρ in (10) we have to demand that $f(\tau) \equiv \text{const}$. Then from the first equation we have $a_1 = 0$, and the second one is converted to the exactly solvable ODE

$$g'' - \frac{a_5}{a_4}(g')^2 = 0 \ ,$$

which determines

$$g(\tau) = \begin{cases} C_1\tau + C_2 \ , & \text{if } a_5 = 0 \ , \\ \ln(C_1\tau + C_2)^{-\frac{a_4}{a_5}} \ , & \text{if } a_5 \neq 0 \ . \end{cases}$$

Consequently, the system (7) is solvable.

b) Let $a_2, a_4 = 0$. We have $\xi(\tau,\rho) = \xi(\rho)$ from the fourth equation of system (7). Given this, we obtain $\xi(\rho) = c_1\rho + c_2$ from the second equation of system (7). Substituting $\xi(\rho)$ in (7), we obtain the system of equations

$$\frac{\psi_\rho}{\psi} = c_1 a_1 \ , \quad \frac{\psi_{\rho\rho}}{\psi} = c_1^2 a_3 \ , \quad \frac{\psi_\tau}{\psi^2} = c_1^2 a_5 \ . \tag{11}$$

Equations (11) have solutions

$$\psi(\tau,\rho) = f_1(\tau)e^{c_1 a_1 \rho} \ , \quad \psi(\tau,\rho) = f_2(\tau)e^{c_1\sqrt{a_3}\rho} + f_3(\tau)e^{-c_1\sqrt{a_3}\rho} \ ,$$
$$\psi(\tau,\rho) = \frac{1}{f_4(\rho) - c_1^2 a_5 \tau} \ .$$

Thus, it is obvious that for the compatibility of the system of equations (11) and, as a consequence, of the system (7) it is required to $a_1 = a_3 = 0$.

The proposition is proved. □

Using the obtained results we can present the following non-trivial exact solution of the equation (2):

$$u(\tau,\rho) = f'(\tau)w(\xi) \ , \qquad \xi = \rho - f(\tau) \ , \qquad f(\tau) = \begin{cases} C_1\tau + C_2 \ , \\ \ln(C_1\tau + C_2)^\alpha \ ; \end{cases} \tag{12}$$

$$u(\tau,\rho) = \frac{f'(\tau)}{f(\tau)}\rho^2 w(\xi) \ , \quad \xi = \ln[\rho/f(\tau)] \ , \quad f(\tau) = \begin{cases} C_2 e^{C_1\tau} \ , \\ (C_1\tau + C_2)^\alpha \ , \end{cases} \tag{13}$$

where $\alpha \neq 0$, $|C_1| + |C_2| > 0$, and (12) takes place only when $\nu = 0$. Note that $w(\xi)$ in (12) satisfies the ODE

$$ww'' + \frac{1}{\sigma}(w')^2 + w' + K(\alpha)w = 0 \ , \tag{14}$$

where $K(\alpha)$ is equal to zero or α^{-1} if f is a linear or logarithmic function, respectively. (ξ) in (13) satisfies the ODE

$$ww'' + \frac{1}{\sigma}(w')^2 + \left(\nu + 3 + \frac{4}{\sigma}\right)ww' + w' + \left(2\nu + 2 + \frac{4}{\sigma}\right)w^2 + K(\sigma)w = 0 \ , \tag{15}$$

where $K(\alpha)$ is equal to zero or α^{-1} if f is exponential or power-law function, respectively.

It is obvious that the solutions of (12) and (13) are of heat wave type solutions and satisfy the boundary condition (3) if and only if the solutions $w(\xi)$ of (14) and (15) satisfy the initial conditions

$$w|_{\xi=0} = 0 \, , \quad w'|_{\xi=0} = -\sigma \, , \tag{16}$$

Thus, in this section we obtain exact solutions of the heat wave type (12) and (13), the procedure of construction is reduced to the solution of the Cauchy problem (14), (16) and (15), (16) respectively. Next the important question concerning the solvability of these problems will be investigated.

Remark 1. If $f(\tau) = C_1\tau + C_2$ we have the known linear heat wave type solution

$$u(\tau, \rho) = \sigma C_1(C_1\tau - \rho + C_2) \, .$$

Indeed, in this case $K(\sigma) = 0$ and the Cauchy problem (14), (16) have a unique solution $w(\xi) = -\sigma\xi$. Then from (12) we obtain a linear function.

Remark 2. If $f(\tau) = (C_1\tau + C_2)^\alpha, \ \alpha = 1, \ \nu = 0$ we have a linear heat wave as well.

4 The Existence and Uniqueness of Solutions

The Cauchy problem for ordinary differential equations, which in the previous section was reduced to the construction of exact solutions of the equation (2), have a singularity, since $\xi = 0$ degenerates the order of the equations. Therefore, the existence of their solutions requires additional study, which will be done in to this section. Consider the general form of the problem

$$\begin{gathered} ww'' + \frac{1}{\sigma}(w')^2 + w' + K_1 ww' + K_2 w^2 + K_3 w = 0 \, , \\ w|_{\xi=0} = 0 \, , \quad w'|_{\xi=0} = -\sigma \, , \end{gathered} \tag{17}$$

where $K_i \in \mathbb{R}$, $i = \overline{1,3}$. We have the following theorem.

Theorem 1. *The Cauchy problem* (17) *has a unique nontrivial analytic solution in a neighborhood of* $\xi = 0$.

Proof. The proof is presented briefly because it is carried out by standard procedure of the majorants method.

The solution of the Cauchy problem (17) is constructed in the form of a power series

$$w(\xi) = \sum_{n=0}^{+\infty} a_n \xi^n \, , \quad a_n \stackrel{\text{def}}{=} \left. \frac{w^{(n)}(\xi)}{n!} \right|_{\xi=0} \, . \tag{18}$$

In this case, $a_0 \equiv 0, \ a_1 \equiv -\sigma$, and the remaining coefficients of the series (18) are uniquely determined according to the recurrence formula

$$a_{n+1} = \frac{1}{\sigma(\sigma n+1)(n+1)} \left[\sigma \sum_{k=0}^{n-2} (k+1)(k+2) a_{k+2} a_{n-k} + \right.$$
$$+ \sum_{k=1}^{n-1} (k+1)(n-k+1) a_{k+1} a_{n-k+1} +$$
$$\left. + K_1 \sum_{k=0}^{n-1} (k+1) a_{k+1} a_{n-k} + K_2 \sum_{k=1}^{n-1} a_k a_{n-k} + K_3 a_n \right] , \; n \in \mathbb{N} \, .$$

Next, we move to a new function $v \stackrel{\text{def}}{=} v(\xi)$ by the formula

$$w(\xi) = -\sigma\xi + \xi^2 v(\xi) \, .$$

Thus, we have the Cauchy problem

$$\begin{gathered} Av + B\xi v' + C\xi^2 v'' = D + \xi g_1(\xi, v) + \xi^2 g_2(\xi, v, v') + \xi^3 g_3(\xi, v, v', v''), \\ v|_{\xi=0} = v_0 \, , \quad v'|_{\xi=0} = v_1 \, , \end{gathered} \tag{19}$$

where $A, B, C \in \mathbb{R}_+$, $D \in \mathbb{R}$, and $g_{1,2,3}$ are analytic functions of their arguments (a specific type of these constants and functions is irrelevant for the proof).

Majorant Cauchy problem for (19) has the form

$$\begin{gathered} V'' = E[(G_1)_\xi + (G_1)_V V' + G_2 + \xi G_3] \, , \\ V|_{\xi=0} = V_0 \, , \; V'|_{\xi=0} = V_1 \, , \end{gathered} \tag{20}$$

where

$$E = \max_{n \in \mathbb{Z}_{\geqslant 0}} \left[\frac{(n-1)n+1}{A + nB + (n-1)nC} \right] ,$$
$$G_1 \stackrel{\text{def}}{=} G_1(\xi, V) \, , \; G_2 \stackrel{\text{def}}{=} G_2(\xi, V, V') \, , \; G_3 \stackrel{\text{def}}{=} G_3(\xi, V, V', V'') \, ,$$
$$V_0 > v_0 \, , \; V_1 > v_1 \, , \; G_i > g_i \, , \; i = \overline{1,3} \, .$$

It is easy to show that the Cauchy problem (20) in a neighborhood of $\xi = 0$ has a unique analytic solution majorizing zero. Consequently, the functions v and w are also analytical.

The theorem is proved. □

Therefore, the local solvability of the Cauchy problem (17) in the class of analytic functions is proved.

5 Particular Case

5.1 Evaluation of the Interval of Existence of a Solution

Theorem 1 provides local solvability of the Cauchy problem in the class of analytic functions. However, it does not allow to evaluate the interval of convergence of the series. In this section, this complex and substantive problem is investigated for a particular case. Consider the Cauchy problem

$$\begin{gathered} ww'' + \frac{1}{\sigma}(w')^2 + w' + \frac{1}{\alpha}w = 0 \ , \\ w|_{\xi=0} = 0 \ , \quad w'|_{\xi=0} = -\sigma \ . \end{gathered} \tag{21}$$

Construction of the solution of (2) is reduced to the problem (21), when $\nu = 0$ and the heat front has the form $f(\tau) = \ln(C_1\tau + C_2)^\alpha$.

We construct the solution of (21) in the form of a power series

$$w(\xi) = \sum_{n=0}^{+\infty} a_n \xi^n \ , \quad a_n \stackrel{\text{def}}{=} \frac{w^{(n)}(\xi)}{n!}\bigg|_{\xi=0} \ . \tag{22}$$

In this case $a_0 \equiv 0,\ a_1 \equiv -\sigma$. The remaining coefficients of the series (22) are determined from the recurrence formula

$$a_{n+1} = \frac{1}{(\sigma n + 1)(n+1)} \left[\sum_{k=0}^{n-2} \left(k + 1 + \frac{n-k}{\sigma} \right) (k+2) a_{k+2} a_{n-k} + \right. \\ \left. + \frac{a_n}{\alpha} \right] \ , \ n \in \mathbb{N} \ . \tag{23}$$

Proposition 3. *Power series* (22) *is convergent if* $|\xi| \leqslant |\alpha|\sigma,\ \sigma \geqslant 1$.

The proof is cumbersome and is not given here. However, the idea of the proof is simple and consists in the construction of the estimates for (23) with two well-known inequalities

$$\sum_{k=1}^{n-3} \frac{1}{k+1} \leqslant \ln\left(\frac{2n-1}{5}\right) \ , \quad \sum_{k=1}^{n-3} \frac{1}{(k+1)^2} < \frac{\pi^2}{6} - \frac{5}{4} \ .$$

Proposition 3 allows us to specify the area of existence of analytical solutions of the Cauchy problem (21). This result means that the analytical solution exists and is unique in the segment $\xi \in [-|\alpha|\sigma,\ |\alpha|\sigma]$. Let us see what conclusions this fact leads to for the original problem.

Let us recall that in the present case

$$u(\tau, \rho) = \frac{\alpha C_1}{C_1\tau + C_2} w(\xi) \ , \quad \xi = \rho - \ln(C_1\tau + C_2)^\alpha \ .$$

We assume that the heat wave starts from the origin. For this purpose the heat wave front $f(\tau) = \ln(C_1\tau + C_2)^\alpha$ must satisfy the condition $f|_{\tau=0} = 0$. Thus, $C_2 = 1$, therefore, $f(\tau) = \ln(C_1\tau + 1)^\alpha$. Since $\tau \geqslant 0$, then $f(\tau)$ is analytical for $0 \leqslant \tau \leqslant 1/C_1,\ C_1 > 0$.

It should be noted that depending on the sign of the parameter α the heat wave may have two directions of motion. Let $\alpha > 0$, then $u(\tau, \rho) \geqslant 0$ if and only if $\xi \leqslant 0$. Since we are interested in the analytical solution, we assume $-\alpha\sigma \leqslant \xi \leqslant 0$. In this case, the heat wave moves to the right, and the area of the existence of an analytic solution is $0 \leqslant \tau \leqslant 1/C_1$, $0 \leqslant \rho \leqslant \alpha \ln 2$. In the case of $\alpha < 0$ heat wave moves to the left.

Remark 3. The constraint $\sigma \geqslant 1$, violates the generality, however, is it physicaly motivated, because in filtration problems σ is a measure of the gas adiabatic, which is known [15] to be greater than one.

Remark 4. For the coefficients a_{n+1} of (22) we have

$$a_{n+1} = (-1)^n \frac{\sigma b_{n+1}}{(n+1)!\alpha^n(\sigma+1)^n \prod\limits_{m=2}^{n} (m\sigma+1)^{\lfloor \frac{n}{m} \rfloor}}, \; n \in \mathbb{N}, \tag{24}$$

where $\lfloor x \rfloor \stackrel{\text{def}}{=} \max\{\, n \in \mathbb{Z} \mid n \leqslant x \,\}$, $b_1 \equiv -1$, and the remaining coefficients b_{n+1} are determined from the recurrence formula

$$b_{n+1} = \frac{1}{\sigma} \sum_{k=0}^{n-2} \binom{n}{k} \left(\sigma + \frac{n-k}{k+1}\right) \prod_{m=2}^{n-1} (m\sigma+1)^{\lfloor \frac{n}{m} \rfloor - \lfloor \frac{k+1}{m} \rfloor - \lfloor \frac{n-k-1}{m} \rfloor} b_{k+2} b_{n-k} -$$
$$- \prod_{m=1}^{n-1} (m\sigma+1)^{\lfloor \frac{n}{m} \rfloor - \lfloor \frac{n-1}{m} \rfloor} b_n \, , \; n \in \mathbb{N} \, .$$

Note that $b_{n+1} \in \mathbb{Z}[\sigma]$, and

$$\deg(b_{n+1}) = \sum_{k=2}^{n-1} \left\lfloor \frac{n}{k} \right\rfloor .$$

For example,

$$b_2 = 1 \, , \quad b_3 = 1 \, , \quad b_4 = 3\sigma + 5 \, , \quad b_5 = 36\sigma^3 + 132\sigma^2 + 143\sigma + 41 \, ,$$
$$b_6 = 360\sigma^4 + 1824\sigma^3 + 3203\sigma^2 + 2232\sigma + 469 \, , \quad \ldots$$

The leading coefficients of the polynomials b_{n+1} are calculated according to the formula

$$\frac{(n-1)!}{2^{n-1}} \prod_{m=2}^{n} m^{\lfloor \frac{n}{m} \rfloor} .$$

Using the representation (24) it can be assumed that the interval of convergence of the series (22) is $|\xi| < 2|\alpha| N(\sigma)$, where $N(\sigma) \underset{\sigma \to +\infty}{\sim} \sigma$.

5.2 Numerical Research

Finally, we present the results of numerical research of the problem (21). Using the fourth order Runge-Kutta method in increments of $h = 10^{-4}$ the numerical solution of problem (21) is constructed. Calculations show that the solution $w(\xi)$ has a singular point: in the case of $\alpha > 0$ it can't be extended to the left of some point $-\xi_0$ (fig. 1 (a)), and in the case $\alpha < 0$ it can't be extended to the right of the point ξ_0 (fig. 1 (b)).

Fig. 1. The behavior of the numerical solution of the Cauchy problem (21): (a) if $\alpha > 0$, (b) if $\alpha < 0$.

In table 1 we present calculations, illustrating the behavior of the studied numerical solutions of the Cauchy problem near the point ξ_0 for some values of the parameters α and σ. Here ξ_* is some point close to ξ_0.

From the results of numerical calculations which are presented in table 1 it can be assumed that the position of the singular point ξ_0 on the ξ–axis is defined as $\xi_0 = 2\alpha N(\sigma)$, where $N(\sigma) \underset{\sigma\to+\infty}{\sim} \sigma$.

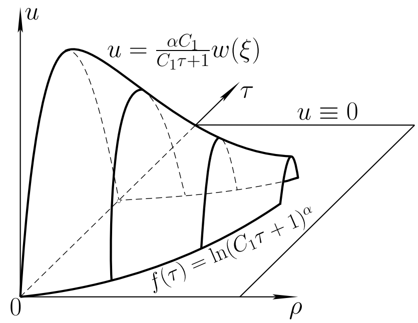

Fig. 2. Heat wave with the front $f(\tau) = \ln(C_1\tau + 1)^\alpha$.

The presented in this section results are easy to interpret in terms of the original problem (2), (3). In this case, we have a heat wave (in assumption that its movement starts from the origin and $\alpha > 0$) with the front of $f(\tau) = \ln(C_1\tau+1)^\alpha$. The behavior of this wave is shown schematically in figure 2. It should be noted here that in this case we observe an effect of the heat wave separation.

6 Conclusion

The authors obtained exact heat wave type solutions of the nonlinear heat equation (2), satisfying the boundary condition (3). The procedure for constructing

Table 1. Numerical calculations

$\alpha = 0.5$				
σ	0.5	1	2	3
$\lvert\xi_*\rvert$	0.68609	1.16480	2.13212	3.11027
$\lvert w(\xi_*)\rvert$	$9.53136 \cdot 10^{-5}$	$9.43393 \cdot 10^{-6}$	$5.69455 \cdot 10^{-7}$	$1.70947 \cdot 10^{-7}$
$\lvert w'(\xi_*)\rvert$	$1.88372 \cdot 10^{6}$	$7.43198 \cdot 10^{4}$	$5.34697 \cdot 10^{3}$	$1.35322 \cdot 10^{3}$
$\alpha = 1$				
σ	0.5	1	2	3
$\lvert\xi_*\rvert$	1.37218	2.32960	4.26425	6.22054
$\lvert w(\xi_*)\rvert$	$2.53957 \cdot 10^{-4}$	$1.52447 \cdot 10^{-5}$	$5.79874 \cdot 10^{-7}$	$4.86545 \cdot 10^{-8}$
$\lvert w'(\xi_*)\rvert$	$1.06136 \cdot 10^{6}$	$9.19834 \cdot 10^{4}$	$7.49431 \cdot 10^{3}$	$2.59468 \cdot 10^{3}$
$\alpha = 2$				
σ	0.5	1	2	3
$\lvert\xi_*\rvert$	2.74436	4.65920	8.52851	12.44108
$\lvert w(\xi_*)\rvert$	$1.88666 \cdot 10^{-4}$	$1.06928 \cdot 10^{-5}$	$9.91851 \cdot 10^{-7}$	$1.37783 \cdot 10^{-7}$
$\lvert w'(\xi_*)\rvert$	$7.69236 \cdot 10^{6}$	$2.62281 \cdot 10^{5}$	$8.10400 \cdot 10^{3}$	$2.31033 \cdot 10^{3}$
$\alpha = 3$				
σ	0.5	1	2	3
$\lvert\xi_*\rvert$	4.11654	6.98881	12.79277	18.66183
$\lvert w(\xi_*)\rvert$	$4.06943 \cdot 10^{-4}$	$1.87411 \cdot 10^{-5}$	$1.10758 \cdot 10^{-6}$	$2.39348 \cdot 10^{-7}$
$\lvert w'(\xi_*)\rvert$	$3.72015 \cdot 10^{6}$	$2.24469 \cdot 10^{5}$	$9.39279 \cdot 10^{3}$	$2.19988 \cdot 10^{3}$

these solutions is reduced to the Cauchy problem for nonlinear ordinary differential equations of second order with a singularity. We establish the solvability of the obtained problems in the class of analytic functions (theorem 1).

Unlike solutions in form of power series [8–11], obtained exact solutions have several advantages. For example, it is possible to get comprehensive information on the properties of the heat waves. In this paper we have obtained an estimates for the area of analyticity (proposition 3) of heat wave type solution with front $f(\tau) = \ln(C_1\tau + 1)^\alpha$. It's behavior have been studied by numerical methods.

Note that the heat wave type solutions of the nonlinear heat equation are important both from a theoretical point of view and in connection with applications. For instance, heat waves propagating with a finite rate, can be used to describe high-temperature processes in plasma [4].

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