Applications of Regularly Varying Functions in Study of Cosmological Parameters

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Abstract. Most of the cosmological parameters, such as the scale factor $a(t)$, the energy density $\rho(t)$ and the pressure of the material in the universe $p(t)$ under usual circumstances satisfy asymptotically the power law. On the other hand the quantities that satisfy the power law are best modeled by regularly varying functions. The aim of this paper is to apply the theory of regularly varying functions to study Friedmann equations and their solutions which are in fact mentioned cosmological parameters. In particular we shall consider possible formulas for cosmological parameters of the dual universe.

Keywords: regular variation, cosmological parameter, Friedmann equations, dual universe.

1 Introduction

Theory of regularly varying functions was started by J. Karamata in [13] and sometimes it is also called Karamata theory of regular variation. Many other mathematicians further developed this theory, see Bingham et al. [2] and Seneta [22]. At the present time this theory is used in many areas, including asymptotic analysis of functions, Tauberian theory, probability, differential equations and analytic number theory. There were several attempts to use this theory in cosmology, particularly in the study of asymptotic behavior of cosmological parameters, eg Mijajlovic et al., [18], [19], but also by Molchanov [20] and Stern [23]. Barrow in [3] and Barrow and Show in [4] used a theory of Hardy and Fowler which preceded the theory of regular variation in studies of asymptotic behavior of solutions to the Einstein equations describing expanding universes.

In [18] we introduced a new constant Γ related to Friedmann equations. Determining the values of Γ one can obtain the asymptotical behavior of the solutions, i.e. of the expansion scale factor $a(t)$ and terms $\rho(t)$ and $p(t)$. It appears that the instance $\Gamma < 1/4$ is appropriate for both cases, a spatially flat and an open universe, and gives a sufficient and necessary condition for the solutions to be regularly varying. In describing cosmological parameters we used the theory of regularly varying solutions of linear second order differential equations, see

Marić [16], which gives necessary and sufficient conditions for the existence of such solutions. From the theory of regular variation it follows that the solutions under usual assumptions include a multiplicative term which is a slowly varying function. We also present a set of formulas that can be assigned to cosmological parameters of the dual universe. These formulas correspond to the second fundamental solution of the acceleration equation.

We shall shortly review definitions and properties of regularly varying functions. In particular we shall use some theorems on regularly varying solutions of the second order differential equation

$$\ddot{y} + f(t)y = 0, \quad f(t) \text{ is continuous on } [\alpha, \infty]. \tag{1}$$

The notion of regular variation is related to the power law distribution represented by the following relationship between some quantities F and t:

$$F(t) = t^r(\alpha + o(1)), \quad \alpha, r \in \mathbb{R}. \tag{2}$$

This definition of power law is in a close relation to the notion of a slowly varying function. A real positive continuous function[1] $L(t)$ defined for $x > x_0$ which satisfies

$$\frac{L(\lambda t)}{L(t)} \to 1 \quad \text{as} \quad t \to \infty, \quad \text{for each real } \lambda > 0. \tag{3}$$

is called a slowly varying (SV) function.

Definition 1. *A function F(t) is said to satisfy a generalized power law if*

$$F(t) = t^r L(t) \tag{4}$$

where $L(t)$ is a slowly varying function and r is a real constant.

Logarithmic function $\ln(x)$ and iterated logarithmic functions $\ln(\ldots\ln(x)\ldots)$ are examples of slowly varying functions. More complicated examples are provided in [2], [22] and [16].

A positive continuous function F defined for $t > t_0$, is a regularly varying (RV) function of an index r, if and only if it satisfies

$$\frac{F(\lambda t)}{F(t)} \to \lambda^r \quad \text{as} \quad t \to \infty, \quad \text{for each } \lambda > 0. \tag{5}$$

It immediately follows that a regularly varying function $F(t)$ has the form (4). Therefore $F(t)$ is regularly varying if and only if it satisfies the generalized power law. By $\mathcal{R}_\alpha$ we denote the class of regularly varying functions of an index α. Hence $\mathcal{R}_0$ is the class of all slowly varying functions. By $\mathcal{Z}_0$ we shall denote the class of zero functions at ∞, i.e. $\varepsilon \in \mathcal{Z}_0$ if and only if $\lim_{t\to+\infty} \varepsilon(t) = 0$. The following theorem [13] describes the fundamental property of these functions.

[1] Continuing the works of G.H. Hardy, J.L. Littlewood and E. Landau, Karamata [13] originally defined and studied this notion for continuous functions. Later this theory was extended to measurable functions. Due to physical constraints, we are dealing here only with continuous functions.

Theorem 1. (*Representation theorem*) $L \in \mathcal{R}_0$ *if and only if there are measurable functions* $h(x)$, $\varepsilon \in \mathcal{Z}_0$ *and* $b \in \mathbb{R}$ *so that*

$$L(x) = h(x)e^{\int_b^x \frac{\varepsilon(t)}{t}dt}, \quad x \geq b, \tag{6}$$

and $h(x) \to h_0$ *as* $x \to \infty$, h_0 *is a positive constant.*

If $h(x)$ is a constant function, then $L(x)$ is called normalized. Let $\mathcal{N}$ denote the class of normalized slowly varying functions. The next fact on $\mathcal{N}$-functions will be useful for our later discussion. If $L \in \mathcal{N}$ and there is $\ddot{L}$, then ε in (6) has the first order derivative $\dot{\varepsilon}$. This follows from the identity $\varepsilon(t) = t\dot{L}(t)/L(t)$.

For our study of Friedmann equations we need the next result [9], [16] on solutions of equation (1). This theorem gives necessary and sufficient conditions for equation $\ddot{y} + f(t)y = 0$ to have regularly varying solutions.

Theorem 2. *(Howard-Marić) Let* $-\infty < \Gamma < 1/4$, *and let* $\alpha_1 < \alpha_2$ *be two roots of the equation*

$$x^2 - x + \Gamma = 0. \tag{7}$$

Further let L_i, *i=1,2 denote two normalized slowly varying functions. Then there are two linearly independent regularly varying solutions of* $\ddot{y} + f(t)y = 0$ *of the form*

$$y_i(t) = t^{\alpha_i}L_i(t), \quad i = 1, 2, \tag{8}$$

if and only if $\lim_{x\to\infty} x\int_x^\infty f(t)dt = \Gamma$. *Moreover,* $L_2(t) \sim \dfrac{1}{(1-2\alpha_1)L_1(t)}$. $\square$

The limit integral in the theorem is not easy to compute. As $\lim_{t\to\infty} t^2 f(t) = \Gamma$ implies $\lim_{x\to\infty} x\int_x^\infty f(t)dt = \Gamma$, we see that

$$\lim_{t\to\infty} t^2 f(t) = \Gamma \tag{9}$$

gives a useful sufficient condition for the existence of regular solutions of the equation $\ddot{y} + f(t)y = 0$ as described in the previous theorem.

2 Cosmological parameters

Cosmological parameters are usually defined as some general physical quantities related to the Universe. Such approach for Lambda cold dark matter model of Universe (ΛCDM model) is presented in the standard literature, for example in [12], [10] and [21]. Here our approach is somewhat formalistic. For cosmological parameters we take primarily solutions of Fiedmann equations [7]:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \text{Friedmann equation,}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \qquad \text{Acceleration equation,}$$
$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \qquad \text{Fluid equation.}$$

and any functions derived from these solutions. Therefore, the scale factor $a(t)$, the energy density $\rho(t)$ and the pressure of the material in the universe $p(t)$ are basic cosmological parameters. We remind that Friedmann equations are derived from the Einstein field equations. These three equations are not independent. For example, the fluid equation can be inferred from the other two equations. Therefore, for solving of these system which consists essentially of two equations and three unknowns some additional condition is needed. Usually equation of state $p = w\rho c^2$ is assumed.

Suppose $\bar{a}(t)$, $\bar{\rho}(t)$ and $\bar{p}(t)$ are some definite solutions of Friedmann equations. Taking

$$\mu(t) = \frac{4\pi G}{3} t^2 \left(\bar{\rho}(t) + \frac{3\bar{p}(t)}{c^2}\right), \tag{10}$$

we see that then $\bar{a}(t)$ is a solution of the second order linear differential equation:

$$\ddot{a} + \frac{\mu(t)}{t^2} a = 0. \tag{11}$$

It is easy to check that in fact *any* solution $b(t)$ of (11) jointly with $\bar{\rho}(t)$ and $\bar{p}(t)$ is a solution of all three Friedmann equations. Therefore, in search for RV solutions of the acceleration equation and so of the Friedmann equations, we can use the Howard-Marić theorem 2. We just did this in our previous work [18]. We review some results from there we need in our further discussion.

First observe that the integral limit in the Howard-Marić theorem for the equation (11) is given by:

$$\mathbf{M}(\mu) = \lim_{x\to\infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt. \tag{12}$$

The functions for which this integral limit converges define so called Marić class of functions $\mathcal{M}$. Then $\mathbf{M}$ is a real functional defined on $\mathcal{M}$. Also, in view of (9) we have

$$\text{If} \quad \lim_{t\to\infty} \mu(t) = \Gamma \quad \text{then} \quad \mathbf{M}(\mu) = \Gamma. \tag{13}$$

We note that the opposite of (13) does not hold, see [18], [19]. There RV solutions of Friedmann equations are found (theorems 3.2 and 3.3) and appropriate cosmological parameters for non-oscillatory universe are determined. Assuming that the integral limit $\mathbf{M}(\mu)$ is convergent, say $\mathbf{M}(\mu) = \Gamma$, there is proved:

- If $\Gamma < 1/4$ then the universe is non-oscillatory.
- The converse is almost true, namely, if the universe is non-oscillatory then $\Gamma \leq 1/4$.
- If $\Gamma < 1/4$ and in some special cases for $\Gamma = 1/4$, the scale factor $a(t)$, a solution of Friedmann equations, is an RV function.

In view of these properties it is justified to call the constant Γ a threshold constant. Assume that α is a root of the polynomial $x^2 - x + \Gamma$. Then

$$\Gamma = \alpha(1 - \alpha) \tag{14}$$

In this case cosmological parameters are represented as follows:

Scale factor $a(t)$: $a(t) = t^\alpha L(t)$, $\alpha \neq 0$ and L is an RV function. In other words, $a(t)$ is a regularly varying function of an index α.

Hubble parameter $H(t) = \dot{a}(t)/a(t)$:

$$H(t) = \frac{\alpha}{t} + \frac{\varepsilon}{t}, \quad \varepsilon \in \mathcal{Z}_0. \tag{15}$$

Deceleration parameter $q(t)$:

$$q(t) = \frac{\mu(t)}{\alpha^2}(1+\eta) = \frac{1-\alpha}{\alpha} - \frac{t\dot{\varepsilon}}{\alpha^2}(1+\eta) + \tau, \quad \varepsilon, \eta, \tau \in \mathcal{Z}_0. \tag{16}$$

Assuming that the scale factor $a(t)$ satisfies the generalized power law one can introduce a new constant w. It will appear that w is in fact the equation of state parameter. Assuming $a(t) = t^\alpha L(t)$, $L \in \mathcal{N}$ and $\alpha \neq 0$, we define w by

$$w \equiv w_\alpha = \frac{2}{3\alpha} - 1. \tag{17}$$

Then the cosmological parameters can be put in the following form:

$$\begin{aligned} \alpha &= \frac{2}{3(1+w)}, \qquad a(t) = a_0 t^{\frac{2}{3(1+w)}} L(t) \\ H(t) &\sim \frac{2}{3(1+w)t}, \quad \mathbf{M}(q) = \frac{1+3w}{2} \end{aligned} \tag{18}$$

Formulas for the exponent α and the Hubble parameter $H(t)$ are widely found in the literature. Formulas for $a(t)$ and $q(t)$ are also reduced to the standard form if $L(t)$ and $q(t)$ are constant at infinity, or if the equation of state $p = w\rho c^2$ is assumed, or $\lim_{t\to\infty} t\dot{\varepsilon}(t) = 0$. We did not assumed in derivation of (18) any of these assumptions. In fact, we found asymptotics for solutions of Friedmann equations only assuming $\mathbf{M}(\mu) = \mathit{\Gamma} < 1/4$, and in certain cases for $\mathit{\Gamma} = 1/4$. As far as we know, it is implicitly widely assumed that the limit $\lim_{t\to\infty} \mu(t)$ exists and is finite, what is much stronger assumption than that the integral limit $\mathbf{M}(\mu)$ is convergent.

We note, if basic cosmological parameters satisfy power law under definition (17), then for the universe with the flat curvature the following weak form of the equation of state holds:

There are functions $\xi, \zeta \in \mathcal{Z}_0$ *such that* $p = \hat{w}\rho c^2$, *where* $\hat{w}(t) = w - t\dot{\xi} + \zeta$.

Therefore, if $t\dot{\xi} \to 0$ as $t \to \infty$, then $\hat{w}(t) \approx w$, what leads to $p = w\rho c^2$, the standard equation of state and classical asymptotics for cosmological parameters. In [18] is also found

$$\mathbf{M}(\mu) = \mathit{\Gamma} = \frac{2}{9} \cdot \frac{1+3w}{(1+w)^2}. \tag{19}$$

As the Friedmann equations are invariant under translation transformation, the above formulas also hold for the expanding universe with the cosmological constant $\mathit{\Lambda}$.

3 Cosmological parameters for dual universe

In the previous section we have seen that $q(t)$ and $p(t)$ may vary, depending on the limit of the hidden parameter $t\dot{\varepsilon}(t)$ as $t \to \infty$. As indicated in [19] one can speculate that this variation is an effect of the existence of the dual universe. We remind that one of the concepts of string theory and hence M-theory is that the big bang was a collision between two membranes. The outcome was the creation of two universes, one in the surface of each membrane. Using the Large Hadron Collider (LHC) located in CERN, some data are collected that might lead to the conclusion that the parallel universe exist. Specifically, if the LHC detects the presence of miniature black holes at certain energy levels, then it is believed [6] that these would be the fingerprints of multiple universes. Collected data are still analyzed.

We will not enter here into a full discussion on the existence of the multiverse. But if the existence the parallel universe is assumed, we can explicitly find a set of formulas that might represent cosmological parameters of the dual universe. We obtain them using the second fundamental solution $L_2(t)$ in Howard - Marić theorem applied to the acceleration equation. To find the second fundamental solution and therefore the dual set of these formulas we take the second root $\beta = 1 - \alpha$ of the quadratic equation $x^2 - x + \Gamma = 0$ appearing in this theorem. To avoid singularities, we assume $\alpha, \beta \neq 0$. Now we use β instead of α for the index of RV solution $a(t)$ - scale factor and for determination of other constants and cosmological parameters. As in (17) we introduce $w_\beta = \frac{2}{3\beta} - 1$. Then we have the following symmetric identity for the equation of state parameters:

$$w_\alpha + w_\beta + 3w_\alpha w_\beta = 1 \tag{20}$$

For our universe we have $w = w_\alpha$, while for the dual universe the corresponding equation of state parameter is w_β. Then the dual formulas are obtained by replacing α with β and w_α with w_β in (15), (16) and (18). If one wants to give any physical meaning to the so obtained dual set of functions, it is rather natural to interpret them as the cosmological parameters of the dual universe.

As we shall see these two universes are isomorphic in the sense that there is an isomorphism which maps cosmological parameters into their dual forms. In this derivation we shall use some elements of the Galois theory. For the basics of this theory the reader may consult for example [11].

Our assumption that $\Gamma < \frac{1}{4}$ and that the solutions α and β of the equation (7) differ, say $\alpha < \beta$, introduces the following kind of symmetry. Let $F = \mathbf{R}(t, \Gamma)$ be the extension algebraic field where $\mathbf{R}$ is the field of real numbers and t and Γ are letters (variables). It is easy to see that for such Γ the polynomial $x^2 - x + \Gamma$ is irreducible over the field F. Hence, the Galois group $\mathbf{G}$ of the equation (7) is of the order 2 and has a nontrivial automorphism σ. Let α and β be the roots of the polynomial $x^2 - x + \Gamma$. Then $\sigma(\alpha) = \beta$ and $\sigma(\beta) = \alpha$. Further, let $\Gamma = \frac{2}{9} \cdot \frac{1+3w}{(1+w)^2}$ where w is a parameter. Then we can take $\alpha = \frac{2}{3(1+w)}$ and $\beta = \frac{1+3w}{3(1+w)}$. Let $w_\alpha \equiv w$ and $w_\beta \equiv \frac{1-w}{1+3w}$. Then $\sigma(w_\alpha) = w_\beta$ since w_α and w_β are rational expressions respectively in α and β. Further, the time t and the constant Γ are invariant under σ i.e. $\sigma(t) = t$ and $\sigma(\Gamma) = \Gamma$ since t and Γ are the elements of the ground field F. The cosmological parameters (15), (16) and (18) are rational expressions of w so if P_α is the corresponding parameter to the solution α, then $\sigma(P_\alpha) = P_\beta$. For example, for the Hubble parameters we have $\sigma(H_\alpha) = H_\beta$. Hence, not only solutions (isomorphic via σ) come into the pairs but the sets of all cosmological parameters come as well. At this point one may speculate about two dual universes having the same time t and the constant Γ and the conjugated parameters w_α and w_β connected by the relation (20).

Of course, there is a question what are the values of the constants appearing in cosmological parameters, for example of $w = w_\alpha$. Most results in the literature see e.g. [25], are consistent with the w = -1 cosmological constant case. Results from experimental cosmology, such as the Baryon Oscillation Spectroscopic Survey (BOSS) of Luminous Red Galaxies (LRGs) in the Sloan Digital Sky Survey (SDSS) are consistent with w = -1, the dark energy equation of state, [1]. However, the value $w = -1$ yields singularity in (18). For such w there is no corresponding α neither Γ. Equation of state is $p = -\rho c^2$ and then by fluid equation we have $\dot{\rho} = 0$, i.e ρ is a constant. This case corresponds to the cosmological constant, so $\rho = \rho_\Lambda = \frac{\Lambda}{8\pi G}$. In the absences of α and β for dual w_β of $w = w_\alpha$ we may take (20) for defining relation . Putting $w_\alpha = -1$ in this identity we obtain $w_\beta = -1$. Hence, dual universe is also equipped with a cosmological constant and its expansion is governed with the dark energy.

The other values of w are also considered. For example if $w = 1/3$ then $\alpha = \beta = 1/2$, $\Gamma = 1/4$ and in this case Howard-Marić theorem cannot be applied since functions $L_1(t)$ and $L_2(t)$ from this theorem are not fundamental solutions. But there is a variant of this theorem appropriate for this case [16], and applying it one can show that $a(t)$ is regularly varying of index $\frac{1}{2}$ if and only if $w \sim \frac{1}{3}$ as $t \to \infty$, i.e. $p \sim \frac{1}{3}c^2\rho$ holds asymptotically. This is the second classic cosmological solution. For more details one can consult [18].

4 Conclusion

A detailed analysis of Friedmann equations and cosmological parameters from the point of view of regular variation is presented. The central role in this analysis has the acceleration equation since it can be considered as a linear second order differential equation and that the theory of regularly varying solutions of such equations is well developed [16]. We introduced in a formal way certain constants such as the threshold constant Γ and the equation of state parameter w. Both constants have the fundamental role in describing asymptotics of cosmological parameters and evolution of the Universe. We also inferred formulas that might represent the cosmological parameters of the dual universe.

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