Minimal Cut and Minimal Path Vectors in Reliability Analysis of Binary- and Multi-State Systems

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Abstract. Minimal Cut Vectors (MCVs) and Minimal Path Vectors (MPVs) are one of the principal tools of reliability engineering. MCVs represent situations in which repair/improvement of any system component results in functioning/ improvement of the system. MPVs coincide with circumstances under which failure/degradation of any system component causes system failure/degradation. MCVs and MPVs allow us to compute a specific measure known as Fussell-Vesely's Importance (FVI), which is used to evaluate importance of system components for system operation. The FVI has originally been developed for analysis of binary-state systems. In this paper, we propose several generalizations of this measure for multi-state systems. Furthermore, we summarize results from several papers focusing on identification of the MCVs and MPVs in multi-state systems and combine them with the proposed measures to develop a complex procedure for importance analysis of multi-state systems. The tool used for identification of the MCVs and MPVs is logical differential calculus.

Keywords: Multi-State System, Structure Function, Minimal Cut Vector, Minimal Path Vector, Logical Differential Calculus

Key Terms. Reliability, Model, Approach, Methodology, Scientific Field

1 Introduction

One of the basic tasks of reliability analysis is investigation of importance of individual system components for system activity. Generally, two approaches can be used for solving this task. The first one is based on identification of critical state vectors that describe circumstances under which a change of a component activity results in a change of system performance. Another possibility is to detect minimal scenarios that ensure that the system can accomplish its mission or minimal scenarios whose occurrence causes that the system cannot satisfy the requested objectives. These scenarios are known as Minimal Path Sets (MPSs) and Minimal Cut Sets (MCSs) respectively [1], [2], [3], [4].

MCSs and MPSs or their equivalents known as Minimal Cut Vectors (MCVs) and Minimal Path Vectors (MPVs), respectively, have been introduced in reliability analysis of Binary-State Systems (BSSs). In [5], the terminology of MCSs and MPSs has been generalized for Multi-State Systems (MSSs). One of the principal tasks in the analysis based on MCSs or MPSs is their identification. In case of BSSs, a lot of algorithms have been developed for this task, e.g. [6], [7], [8]. Using MCSs and MPSs, several approaches have been proposed to perform the quantitative analysis of BSSs. These approaches allow estimating system availability or computing importance of individual system components [9], [10], [11], [12]. In case of MSSs, MCVs and MPVs are more frequently used than MCSs and MPSs respectively. They have primarily been used in the qualitative and quantitative analysis of network structures, e.g. distribution networks and, therefore, most of the algorithms developed for their identification in MSSs, e.g. [13], [14], [15], have been based on the assumption that the structure of a MSS can be expressed in the form of a network. However, not every MSS can be expressed as a graph structure. Because of that, more general algorithm has been proposed in [16], [17]. This algorithm permits detection of MCVs or MPVs in a MSS of any type. This allows us to apply the concept of MCVs and MPVs not only in the analysis of network systems but also in the investigation of other types of MSSs. This idea has been considered in [18] where one of the typical measures investigating importance of system components in BSSs known as Fussell-Vesely's Importance (FVI) [9], [10] has been generalized for investigation of importance of individual states of components in a MSS. The generalization has been done based on the concept of MCVs and using logical differential calculus.

In this paper, we summarize results presented in [16], [17], [18] and propose an approach for investigation of MSSs using MCVs and MPVs. We define some more general types of FVI that allow us to investigate importance of the entire component (not only of a specific component state). The achieved results are illustrated based on the analysis of the service system considered in [18], but this approach could also be applied in reliability analysis of information systems, especially in the analysis of distributed systems, such as distributed temporal database systems studied in [19].

2 Reliability Analysis

The principal step in investigation of system reliability is creation of its model. As a rule, two types of mathematical models are used in reliability analysis: BSSs and MSSs. A BSS allows defining only two states in system/components performance – functioning (presented as number 1) and failure (represented by number 0). These models are useful in the investigation of consequences of system failure, but they do not allow us to study processes that gradually results in system failure. For this purpose, MSSs are more suitable because they permit defining more than two states to describe system/components performance – from perfectly functioning to complete failure. The dependency of system state on states of its components is defined by a map known as structure function. For MSSs, this function has the next form [2]:

$$\phi(\boldsymbol{x}): \{0,1,\ldots, m_1 -1\}\times\{0,1,\ldots, m_2 -1\}\times\ldots\times\{0,1,\ldots, m_n -1\}\rightarrow\{0,1,\ldots, m -1\}\ , \qquad (1)$$

where n denotes number of system components, m agrees with the number of system states (state 0 means that the system is completely failed, while state m -1 agrees with perfect functioning), m_i denotes number of states of component i (state 0 corresponds to complete failure and state m_i -1 to perfect functioning), for $i = 1,2,\ldots, n$, x_i is a variable defining state of component i, and $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ is a vector defining states of the system components (state vector). Specially, if $m_1 = m_2 =\ldots= m$, then the system is identified as homogeneous [2], [3]. Special type of homogeneous systems is a BSS for which $m_1 = m_2 =\ldots= m = 2$ [1].

Based on the properties of the structure function, two basic classes of systems can be recognized – coherent and noncoherent. A system is coherent if its structure function is non-decreasing in all its variables, i.e. there exist no circumstances under which a degradation of a system component can result in system improvement. If this assumption is not satisfied, then the system is noncoherent. In what follows, only coherent systems will be considered.

The structure function describes layout of the system components and, therefore, it allows us to investigate topological properties of the system. However, this function carries no information about state probabilities of individual system components and, therefore, only its knowledge is not sufficient for the analysis of system reliability. So, the state probabilities of the system components represent other information that has to be included in the model [2], [3]:

$$p_{i,s} = \Pr\{x_i = s\},\ s = 0,1,\ldots, m_i -1\ . \tag{2}$$

For BSSs, $p_{i,0}$ is denoted as q_i, and it is known as component unavailability because it coincides with time during which the component is unavailable (failed). Similarly, $p_{i,1}$ is denoted only as p_i, and it is known as component availability because it agrees with proportion of time during which the component is available (working).

If a system model is known, the analysis can be performed. Some of the basic characteristics evaluating reliability of a system under consideration are system states probabilities, system availability and unavailability. For BSSs, system availability agrees with the probability that the system is in state 1, while the unavailability corresponds to the probability that it is in state 0. This is not true for MSSs where system availability (unavailability) is defined as the probability that the system can (cannot) satisfy a specific requirement. If we know a minimal system state, e.g. state j, that allows satisfying the requirement, then system availability (unavailability) is defined as the probability that the system is at least in (below) state j [2]:

$$A^{\geq j} = \Pr\{\phi(\boldsymbol{x}) \geq j\},\ U^{\geq j} = \Pr\{\phi(\boldsymbol{x}) < j\},\ \text{ for } j \in \{1,2,\ldots,m-1\}\ . \tag{3}$$

2.1 Minimal Path Sets and Minimal Path Vectors

MPSs have been introduced in reliability analysis of BSSs. They represent minimal sets of components whose simultaneous functioning ensures system functioning [1], [12]. In terms of state vectors they can be expressed as so-called MPVs that agree with situations in which a failure of any working component causes system failure. More formally, a state vector $\boldsymbol{x}$ is a MPV if $\phi(\boldsymbol{x}) = 1$ and $\phi(\boldsymbol{y}) = 0$ for any $\boldsymbol{y} < \boldsymbol{x}$. Please note that relation $\boldsymbol{y} < \boldsymbol{x}$ between state vectors $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2,\ldots, y_n)$

means that $y_i \le x_i$ for all $i \in \{1,2,\ldots, n\}$ and there exists at least one i such that $y_i < x_i$. Specially, if only the first part of the definition is satisfied for a state vector $\boldsymbol{x}$, i.e. $\phi(\boldsymbol{x}) = 1$, then the state vector is known as a path vector.

The concept of MPSs and MPVs has been generalized for homogeneous MSSs in [5]. Using that paper, we can generalize this concept on nonhomogeneous systems in the following way.

Let us consider a nonhomogeneous MSS of n components and denote $M = \max_{i \in \{1,2,\ldots,n\}} \{m_i\}$. Next, let us denote a set of all system components that are in state s as N_s for $s = 0,1,\ldots, M$ -1 (please note that if no system component is in state s, then the set N_s is empty). Using this, we can define a partition $\eta = (\mathrm{N}_0, \mathrm{N}_1,\ldots, \mathrm{N}_M)$ of n system components into M sets such that $i \in \mathrm{N}_s \Rightarrow s < m_i$ (this implies that component i can occur in set N_s if and only if its maximal possible state is not less than s). Clearly, there exist $\prod_{i=1}^{n} m_i$ different partitions, and every partition corresponds to a state vector of the system structure function. Moreover, there is one-to-one correspondence between state vectors of the structure function and partitions of the system. This correspondence is based on the rule that the i-th system component is present in set N_s of partition η if and only if a state vector corresponding to the partition η has the form of $(s_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, s, x_{i+1},\ldots, x_n)$. The state vector that agrees with partition η will be denoted as $\boldsymbol{x}(\eta)$. Using these conventions, a MPS for level j of system availability is defined as a partition η for which $\phi(\boldsymbol{x}(\eta)) \ge j$ and $\phi(\boldsymbol{y}) < j$ for any $\boldsymbol{y} < \boldsymbol{x}(\eta)$. Please note that this definition implies that MPSs can be recognized with respect to m -1 different states of the system, i.e. for $j = 1,2,\ldots, m$ -1.

The previous definition of a MPS for level j of system availability uses vectors corresponding to MPSs. As in the case of BSSs, these state vectors are known as MPVs [2], [5]. More formally, a state vector $\boldsymbol{x}$ is a MPV for level j of system availability if $\phi(\boldsymbol{x}) \ge j$ and $\phi(\boldsymbol{y}) < j$ for any $\boldsymbol{y} < \boldsymbol{x}$. Specially, if only the first part of the definition is satisfied by a vector $\boldsymbol{x}$, then it is known as a path vector for level j of system availability. Clearly, MPVs for system availability level j correspond to situations in which a degradation of any system component that can degrade, i.e. that is not completely failed, results in decrease in system state below value j. Please note that the definitions of MPSs and MPVs are very similar but the latter is not based on term “partition” and, therefore, it is probably clearer.

2.2 Minimal Cut Sets and Minimal Cut Vectors

Another concept that is closely related to MPSs and MPVs is a concept of MCSs and MCVs. These terms have also been introduced firstly in the analysis of BSSs. For a BSS, a MCS represents a minimal set of components whose simultaneous failure results in system failure. A state vector that corresponds to a MCS is known as a MCV. Unlike a MCS, a MCV describes a situation in which a repair of any failed component causes system functioning. Mathematically, a state vector $\boldsymbol{x}$ is a MCV if $\phi(\boldsymbol{x}) = 0$ and $\phi(\boldsymbol{y}) = 1$ for any $\boldsymbol{y} > \boldsymbol{x}$. Please note that relation $\boldsymbol{y} > \boldsymbol{x}$ between two state vectors $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2,\ldots, y_n)$ has a similar meaning as in the case of MPVs, i.e. it means that $y_i \ge x_i$ for all $i \in \{1,2,\ldots, n\}$ and there exists at least one i such that $y_i > x_i$. Specially, if only the first part of the definition is satisfied for a state vector $\boldsymbol{x}$, i.e. $\phi(\boldsymbol{x}) = 0$, then the state vector can be recognized as a cut vector.

In case of MSSs, we can generalize definition of a MCS introduced in [5] for nonhomogeneous systems in the similar way as in the case of MPSs. However, this has no practical sense for the purpose of this paper and, therefore, we only introduce definition of a MCVs presented in [2]. So, a state vector $\boldsymbol{x}$ is a MCV for level j of system availability if $\phi(\boldsymbol{x}) < j$ and $\phi(\boldsymbol{y}) \geq j$ for any $\boldsymbol{y} > \boldsymbol{x}$. Furthermore, every state vector $\boldsymbol{x}$ satisfying $\phi(\boldsymbol{x}) < j$ is known as a cut vector for level j of system availability. This definition implies that a MCV for level j of system availability represents a situation in which an improvement of any system component that can be improved, i.e. that is not perfectly working, causes that the system reaches at least state j.

3 Reliability Analysis based on Minimal Path and Cut Vectors

MPVs (MCVs) are very useful in both – qualitative and quantitative analysis. In the qualitative analysis of BSSs they describe circumstances under which a failure of any working (repair of any failed) component results in system failure (functioning). For MSSs, they correspond to situations in which a minor degradation (improvement) of any functioning (non-perfectly working) component causes that the system will not be able (will be able) to accomplish its mission. In the quantitative analysis, they can be used to estimate some global characteristics, e.g. system availability, system state probabilities [1], [2], [3], or to quantify importance of individual system components (or their states in case of MSSs) [9], [10], [11], [12], [18].

Firstly, let us focus on the global reliability characteristics. For MSSs, system availability and unavailability can be calculated based on MPVs and MCVs in the following way [20]:

$$\begin{aligned} A^{\geq j} &= \Pr\left\{\bigcup_{l=1}^{N_p^{\geq j}} \left\{\boldsymbol{x} \geq \mathrm{MPV}_l^{\geq j}\right\}\right\}, \\ U^{\geq j} &= \Pr\left\{\bigcup_{l=1}^{N_c^{\geq j}} \left\{\boldsymbol{x} \leq \mathrm{MCV}_l^{\geq j}\right\}\right\}, \end{aligned} \tag{4}$$

where $N_p^{\geq j}$ ($N_c^{\geq j}$) agrees with the number of MPVs (MCVs) for system state j, $\mathrm{MPV}_l^{\geq j}$ $\left(\mathrm{MCV}_l^{\geq j}\right)$ denotes the l-th MPV (MCV) for level j of system availability, and event $\left\{\boldsymbol{x} \geq \mathrm{MPV}_l^{\geq j}\right\}$ $\left(\left\{\boldsymbol{x} \leq \mathrm{MCV}_l^{\geq j}\right\}\right)$ means that an arbitrary state vector $\boldsymbol{x}$ is greater (less) than or equal to the l-th MPV (MCV) for level j of system availability. Please note that relation "≥" ("≤") between two state vectors has similar meaning as relation ">" ("<") used in definitions of MPVs and MCVs, and the only difference is that it admits equality of state vectors.

Using system availabilities or unavailabilities, the system state probability can be computed as follows:

$$\Pr\{\phi(\boldsymbol{x}) = j\} = \begin{cases} 1 - A^{\geq 1} & = U^{\geq 1} & \text{if } j = 0, \\ A^{\geq j} - A^{\geq j+1} & = U^{\geq j+1} - U^{\geq j} & \text{if } j \in \{1, 2, \dots, m-2\}, \\ A^{\geq m-1} & = 1 - U^{\geq m-1} & \text{if } j = m-1. \end{cases} \quad (5)$$

Another possibility how MPVs and MCVs can be used is investigation of influence of individual components (or their states in case of MSSs) on system activity. For this task, a lot of measures have been proposed. One of them is FVI.

The FVI has been introduced in reliability analysis of BSSs as the probability that a failure of a given component contributes to system unavailability [9], [10], [12]. This agrees with the probability that at least one MCS containing the considered component is failed given that the system is failed. (Please note a MCS is failed if all components forming the MCS are failed.) Another option is to define the FVI in such a way that it allows us to identify contribution of functioning of component i to system availability. This idea has been presented in [11] where this measure has been defined as the probability that at least one MPS containing component i is working given that the system is functioning. (A MPS is working if all components forming it are functioning.) In [20], these measures have been defined using MCVs and MPVs instead of MCSs or MPSs respectively.

Definitions in [20] can be generalized for MSSs in several ways, which allow us to define several types of FVI measures. Firstly, let us focus on FVI measures based on MCVs. In this case, we can define $\mathrm{FVI}_{i,s\downarrow}^{c,\geq j}$ for state s of component i with respect to level j of system availability as the probability that a degradation of a given component state contributes to system unavailability $U^{\geq j}$ [18]:

$$\mathrm{FVI}_{i,s\downarrow}^{c,\geq j} = \frac{\Pr\{\exists \mathrm{MCV}^{\geq j}((s-1)_i) \in \mathrm{MCVs}^{\geq j}; \boldsymbol{x} \leq \mathrm{MCV}^{\geq j}((s-1)_i)\}}{U^{\geq j}}, \quad (6)$$

where $\mathrm{MCVs}^{\geq j}$ is a set of all MCVs for level j of system availability, $\mathrm{MCV}^{\geq j}((s-1)_i)$ denotes a MCV for level j of system availability in which $x_i = s-1$, and event $\{\exists \mathrm{MCV}^{\geq j}((s-1)_i) \in \mathrm{MCVs}^{\geq j}; \boldsymbol{x} \leq \mathrm{MCV}^{\geq j}((s-1)_i)\}$ means that there is at least one MCV with $x_i = s-1$ that is greater than or equal to an arbitrary state vector $\boldsymbol{x}$. The previously introduced meaning of this definition results from the fact that if state vector $((s-1)_i, \boldsymbol{x}) = (x_1, x_2, \dots, x_{i-1}, s-1, x_{i+1}, \dots, x_n)$ is a MCV for level j of system availability, then it follows that degradation of component i from state s to lower one contributes to level j of system unavailability.

Formula (6) admits that component i can degrade more than one state. However, some approaches used in importance analysis of MSSs are based on the assumption that a component can degrade only one state. For this purpose, we can modify formula (6) in the following way:

$$\mathrm{FVI}_{i,s}^{c,\geq j} = \Pr\{\exists \mathrm{MCV}^{\geq j}((s-1)_i) \in \mathrm{MCVs}^{\geq j}; ((s-1)_i, \boldsymbol{x}) \leq \mathrm{MCV}^{\geq j}((s-1)_i)\} \frac{p_{i,s-1}}{U^{\geq j}}, \quad (7)$$

where event $\{\exists \mathrm{MCV}^{\geq j}((s-1)_i) \in \mathrm{MCVs}^{\geq j}; ((s-1)_i, \boldsymbol{x}) \leq \mathrm{MCV}^{\geq j}((s-1)_i)\}$ means that there exists at least one MCV with $x_i = s-1$ that is greater than or equal to an arbitrary state vector $((s-1)_i, \boldsymbol{x})$.

The previous formula focuses on contribution of a specific component state to system unavailability. If we sum $\mathrm{FVI}_{i,s}^{c,\ge j}$ measures through all possible values of *s*, i.e. for $s = 1,2,\ldots, m_i$ -1, then we can quantify the total contribution of component *i* to system unavailability $U^{\ge j}$:

$$\mathrm{FVI}_{i}^{c,\ge j} = \sum_{s=1}^{m_i-1} \mathrm{FVI}_{i,s}^{c,\ge j} . \tag{8}$$

In the similar way, we can generalize FVI based on MPVs introduced in [20] for MSSs. In this case, we obtain the next formulae:

$$\mathrm{FVI}_{i,s}^{p,\ge j} = \Pr\left\{\exists \mathrm{MPV}^{\ge j}\left((s+1)_i\right) \in \mathrm{MPVs}^{\ge j}; ((s+1)_i, \boldsymbol{x}) \ge \mathrm{MPV}^{\ge j}\left((s+1)_i\right)\right\} \frac{p_{i,s+1}}{A^{\ge j}} , \tag{9}$$

$$\mathrm{FVI}_{i}^{p,\ge j} = \sum_{s=0}^{m_i-2} \mathrm{FVI}_{i,s}^{p,\ge j} . \tag{10}$$

Formula (9) quantifies contribution of a minor improvement of state *s* of component *i* to system availability $A^{\ge j}$, while $\mathrm{FVI}_{i,s}^{p,\ge j}$ measure can be used to estimate the total contribution of component *i* to system availability $A^{\ge j}$.

3.1 Direct Partial Logic Derivatives

Logical differential calculus is a special tool developed for analysis of dynamic properties of logic functions. The central term of this tool is a logic derivative. Several types of logic derivatives exist, and one of them is Direct Partial Logic Derivative (DPLD) [21]. This derivative has originally been defined for MVL functions. Since the structure function of a homogeneous MSS can be viewed as a MVL function [21], DPLD can also be applied in reliability analysis of such systems. Furthermore, it has been shown in [22] that this derivative can also be used in the analysis of nonhomogeneous systems. For this purpose, definition of a DPLD of function $\phi(\boldsymbol{x})$ with respect to variable x_i has been generalized in the following way:

$$\frac{\partial\phi(j \to h)}{\partial x_i(s \to r)} = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) = j \text{ and } \phi(r_i, \boldsymbol{x}) = h \\ 0, & \text{otherwise} \end{cases},$$
$$\text{for } s,r \in \{0,1,\ldots,m_i\text{-}1\}, s \ne r, \quad j,h \in \{0,1,\ldots,m\text{-}1\}, j \ne h, \tag{11}$$

where $\phi(a_i, \boldsymbol{x}) = \phi(x_1, x_2,\ldots, x_{i-1}, a, x_{i+1},\ldots, x_n)$ for $a \in \{s, r\}$. This implies that the DPLD can be used to find circumstances under which a degradation/improvement of the *i*-th system component results in degradation/improvement of the whole system. Since only coherent systems are considered in this paper, only DPLDs in which $j > h$ and $s > r$ or in which $j < h$ and $s < r$ can be nonzero. The former identify state vectors $(\cdot_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, x_{i+1},\ldots, x_n)$ at which degradation of component *i* from state *s* to *r* results in degradation of system from state *j* to *h*, while the nonzero values of the latter agree with state vectors $(\cdot_i, \boldsymbol{x})$ at which improvement of system state from value *j* to *h* results from improvement of state *s* of component *i* to value *r*.

3.2 Integrated Direct Partial Logic Derivatives

DPLDs are useful to identify situations in which a specific change of a state of a given component results in the specified change of system state. However, a problem is that a lot of DPLDs can be defined with respect to one component, i.e. if the component has m_i and the system has m different states, then $m_i(m_i -1)m(m -1)$ DPLDs can be defined with respect to this component. Assuming that the system is coherent and using the fact that $\partial\phi(j \to h)/\partial x_i(s \to r) = \partial\phi(h \to j)/\partial x_i(r \to s)$ [21], only $m_i(m_i -1)m(m -1)/4$ DPLDs are needed to be computed. This can be still quite a lot. Another fact is that these DPLDs have a lot of zero values and, therefore, they usually carry little information. Furthermore, if we consider derivative $\partial\phi(j \to h)/\partial x_i(s \to r)$ that is nonzero for a state vector $(s_i, \boldsymbol{x})$, then all other DPLDs $\partial\phi(j' \to h')/\partial x_i(s \to r)$, where $j' \neq j$ or $h' \neq h$, have to take value 0 for the considered state vector. Due to these facts, new types of logic derivatives have been introduced in [16], [22]. These derivatives were named as Integrated Direct Partial Logic Derivatives (IDPLDs), because they combine several DPLDs together. Three types of IDPLDs can be defined. For the purpose of this paper, the most important ones are IDPLDs of type III that can be used to find state vectors at which degradation (improvement) of a given component state causes degradation (improvement) of a given level of system availability. In notation of system degradation, this derivative is defined as follows:

$$\frac{\partial\phi(h_{\geq j} \to h_{<j})}{\partial x_i(s \to r)} = \bigcup_{h_u=j}^{m-1}\bigcup_{h_d=0}^{j-1}\frac{\partial\phi(h_u \to h_d)}{\partial x_i(s \to r)} = \begin{cases} 1 & \text{if } \phi(s_i, \boldsymbol{x}) \geq j \text{ and } \phi(r_i, \boldsymbol{x}) < j \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } s, r \in \{0,1,\ldots,m_i\text{-}1\}, s \neq r,\ j \in \{1,2,\ldots,m\text{-}1\}, \tag{12}$$

where notation $h_{\geq j}$ ($h_{<j}$) means that all system states that are greater than or equal to (less than) j are taken into account. If we want to focus on system improvement, then this IDPLD has the following form:

$$\frac{\partial\phi(h_{<j} \to h_{\geq j})}{\partial x_i(s \to r)} = \bigcup_{h_d=0}^{j-1}\bigcup_{h_u=j}^{m-1}\frac{\partial\phi(h_d \to h_u)}{\partial x_i(s \to r)} = \begin{cases} 1 & \text{if } \phi(s_i, \boldsymbol{x}) < j \text{ and } \phi(r_i, \boldsymbol{x}) \geq j \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } s, r \in \{0,1,\ldots,m_i\text{-}1\}, s \neq r,\ j \in \{1,2,\ldots,m\text{-}1\}. \tag{13}$$

It can be shown simply that the following relation holds between the IDPLDs introduced above:

$$\frac{\partial\phi(h_{\geq j} \to h_{<j})}{\partial x_i(s \to r)} = \frac{\partial\phi(h_{<j} \to h_{\geq j})}{\partial x_i(r \to s)}, \tag{14}$$

but the principal difference between these two integrated derivatives is that IDPLD $\partial\phi(h_{\geq j} \to h_{<j})/\partial x_i(s \to r)$ can be computed only at state vectors of the form of $(s_i, \boldsymbol{x})$, while IDPLD $\partial\phi(h_{<j} \to h_{\geq j})/\partial x_i(r \to s)$ can be calculated only at state vectors of the form of $(r_i, \boldsymbol{x})$ (Fig. 1).

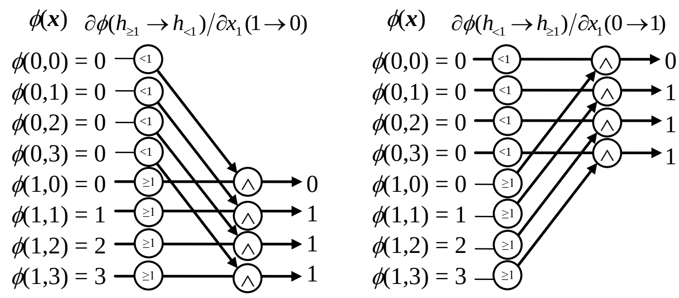

Fig. 1. Example of existence and computation of IDPLDs of type III for structure function $\phi(x_1, x_2) = x_1 x_2$ for $x_1 \in \{0,1\}$ and $x_2 \in \{0,1,2,3\}$.

3.3 Calculation of Minimal Cut and Path Vectors using Integrated Direct Partial Logic Derivatives

According to previous text, a MCV for level j of system availability agrees with a state vector at which the structure function takes a value less than j and at which a minor improvement (improvement by one state) of any non-perfectly working component causes that the system achieves at least state j. Identification of state vectors at which a minor improvement of component i results in such change of system state can be done using IDPLDs $\partial\phi(h_{<j} \to h_{\geq j})/\partial x_i(s \to s+1)$ for $s = 0,1,\ldots, m_i$ -2. If we compute these IDPLDs for all system components and combine them together using some special conjunction, then we can identify all MCVs for level j of system availability. This idea has been studied in [16], [17], where the following algorithm for computation of MCVs has been formulated:

1. Repeat the next two steps for all system components:
   1.1. Compute expanded IDPLDs $\partial_e\phi(h_{<j} \to h_{\geq j})/\partial_e x_i(s \to s+1)$ for $s = 0,1,\ldots, m_i$ -2.
   1.2. Calculate Π-conjunction $\prod_{s=0}^{m_i-2} \partial_e\phi(h_{<j} \to h_{\geq j})/\partial_e x_i(s \to s+1)$ of the expanded IDPLDs computed in the previous step.
2. Calculate Π-conjunction of the Π-conjunctions computed in step 1 and identify state vectors for which it takes value 1. These state vectors agree with the MCVs for level j of system availability.

This algorithm does not use directly IDPLDs of type III but rather their expanded versions that are defined also for state vectors at which IDPLDs cannot be computed. In these situations, the expanded IDPLD takes value "*":

$$\frac{\partial_e\phi(h_{<j} \to h_{\geq j})}{\partial_e x_i(s \to r)} = \begin{cases} 1 & \text{if } x_i = s \text{ and } \phi(s_i, \boldsymbol{x}) < j \text{ and } \phi(r_i, \boldsymbol{x}) \geq j \\ 0 & \text{if } x_i = s \text{ and } \left(\phi(s_i, \boldsymbol{x}) \geq j \text{ or } \phi(r_i, \boldsymbol{x}) < j\right), \\ * & \text{if } x_i \neq s \end{cases}$$
$$\text{for } s, r \in \{0,1,\ldots,m_i\text{-}1\}, s \neq r,\ \ j \in \{1,2,\ldots,m\text{-}1\}. \tag{15}$$

The Π-conjunction used in the algorithm is computed for two expanded IDPLDs based on the rules defined in Table 1 [16], [17].

Table 1. Π-conjunction of two expanded IDPLDs of type III.

Π-conjunction		$\frac{\partial_e \phi(h_{<j} \to h_{\geq j})}{\partial_e x_{i_2}(s_2 \to r_2)}$		
		*	0	1
$\frac{\partial_e \phi(h_{<j} \to h_{\geq j})}{\partial_e x_{i_1}(s_1 \to r_1)}$	*	*	0	1
	0	0	0	0
	1	1	0	1

The similar algorithm can also be used to find all MPVs for level j of system availability [17]. The only difference is that expanded IDPLDs of the form of $\partial_e \phi(h_{\geq j} \to h_{<j}) / \partial_e x_i(s \to s-1)$, for $s = 1,2,\ldots, m_i$ -1, have to be used. These expanded derivatives are defined similarly as (15).

After obtaining the MCVs or MPVs based on logical differential calculus, the quantitative analysis of the investigated system can be performed. This is done simply by computing FVI measures introduced in the previous part.

4 Hand Calculation Example

Previous parts of this paper presented a complex approach for investigation of MSSs. This approach includes 1) computation of expanded IDPLDs, 2) identification of MCVs or MPVs based on the derivatives, 3) evaluation of importance of states of the system components (formulae (7) and (9)) or the total importance of the system components (formulae (8) and (10)) for level j of system availability. Now, we will illustrate some of these steps.

Let us consider the service system used in [18]. It is composed of 3 components: service point 1 (component 1), service point 2 (component 2), and infrastructure (component 3). The system has 4 states with the following meaning: 0 – nonfunctioning (no customer is satisfied by the service points), 1 – partially functioning (some customers are satisfied), 2 – partially nonfunctioning (some customers are not satisfied), 3 – perfectly functioning (all customers are satisfied). The service points can be only functioning (state 1) or failed (state 0). The infrastructure is modeled as a 4-state component where value 0 means that the infrastructure is poor while value 3 agrees with its perfect quality. The system structure function and the state probabilities of the components are defined in Table 2.

If we want to investigate importance of the system components using the proposed method, firstly the MCVs and MPVs of the system have to be computed. This can be done using the algorithm described in the previous section. According to this algorithm, firstly, expanded versions of IDPLDs of type III have to be computed. In the second step, their Π-conjunction has to be calculated. Let us assume that we want to find all MCVs and MPVs for level 1 of system availability. For this purpose, expanded IDPLDs $\partial_e \phi(h_{<1} \to h_{\geq 1}) / \partial_e x_i(s \to s+1)$ and $\partial_e \phi(h_{\geq 1} \to h_{<1}) / \partial_e x_i(s \to s-1)$

have to be calculated for all system components, i.e. for $i = 1,2,3$, and for all components states for which they can be computed. In the next step, Π-conjunctions including all relevant component states have to be calculated for every system component. Please note components 1 and 2 have only 2 possible states, what implies that the Π-conjunctions are equal to expanded derivatives $\partial_e\phi(h_{<1} \to h_{\geq 1})/\partial_e x_1(0 \to 1)$ and $\partial_e\phi(h_{<1} \to h_{\geq 1})/\partial_e x_2(0 \to 1)$ respectively in the case of MCVs identification and derivatives $\partial_e\phi(h_{\geq 1} \to h_{<1})/\partial_e x_1(1 \to 0)$ and $\partial_e\phi(h_{\geq 1} \to h_{<1})/\partial_e x_2(1 \to 0)$ respectively in the case of MPVs identification. Therefore, there is no need to compute Π-conjunctions for these two components. However, the 3-rd component has 4 states and, therefore, the Π-conjunction has to be calculated. So, if we are interested in MCVs, then expression $\prod_{s=0}^{2}\partial_e\phi(h_{<1} \to h_{\geq 1})/\partial_e x_3(s \to s+1)$ has to be formed from the expanded IDPLDs, while conjunction $\prod_{s=1}^{3}\partial_e\phi(h_{\geq 1} \to h_{<1})/\partial_e x_3(s \to s-1)$ has to be found to identify the MPVs. Finally, using Π-conjunction we can combine the expanded IDPLDs computed with respect to components 1 and 2 with the Π-conjunction calculated for component 3 and identify all MCVs (MPVs) for level 1 of system availability. For MCVs, these calculations are shown in Table 3. According to this table, the system has two MCVs for the considered level of system availability, i.e. (0,0,3) and (1,1,0). Similarly, we can compute that the MPVs are (0,1,1) and (1,0,1). The similar procedure can be used to identify the MCVs and MPVs for levels 2 and 3 of system availability. The final results are in Table 4.

Table 2. Structure function of the service system and state probabilities of its components.

Components states		x_3 ($p_{3,s}$)			
x_1 ($p_{1,s}$)	x_2 ($p_{2,s}$)	0 (0.20)	1 (0.60)	2 (0.10)	3 (0.10)
0 (0.30)	0 (0.20)	0	0	0	0
0 (0.30)	1 (0.80)	0	1	1	2
1 (0.70)	0 (0.20)	0	1	1	2
1 (0.70)	1 (0.80)	0	2	3	3

MCVs and MPVs identified in the previous step can be used to investigate importance of individual states of the system components for a given level of system availability. This can be done using FVI measures (7) and (9). For illustration, let us compute $\mathrm{FVI}_{3,1}^{c,\geq 1}$, which quantifies contribution of minor degradation of state 1 of component 3 to system unavailability $U^{\geq 1}$. Using MCVs, this measure can be calculated in the following way:

$$\mathrm{FVI}_{3,1}^{c,\geq 1} = \Pr\{(x_1, x_2, 0) \leq (1,1,0)\}\frac{p_{3,0}}{U^{\geq 1}}, \tag{16}$$

where unavailability $U^{\geq 1}$ can be computed using (4) as follows:

$$U^{\geq 1} = \Pr\{\boldsymbol{x} \leq (0,0,3) \vee \boldsymbol{x} \leq (1,1,0)\}$$
$$= \Pr\{\boldsymbol{x} \leq (0,0,3)\} + \Pr\{\boldsymbol{x} \leq (1,1,0)\} - \Pr\{\boldsymbol{x} \leq (0,0,0)\}. \quad (17)$$

Using the numbers presented in Table 2, we obtain that $\mathrm{FVI}_{3,1}^{c,\geq 1} = 0.8065$. In the similar way, we can compute all other measures of the form of $\mathrm{FVI}_{3,s}^{c,\geq j}$ (Table 5). According to the data presented in Table 5, we can state that a degradation of state 1 of component 3 contributes mainly to unavailabilities $U^{\geq 1}$ and $U^{\geq 2}$, a degradation of state 2 to unavailabilities $U^{\geq 2}$ and $U^{\geq 3}$, and a degradation of state 3 contributes only to unavailability $U^{\geq 3}$.

Table 3. Computation of the minimal cut vectors of the service system using expanded integrated direct partial logic derivatives.

x_1	x_2	x_3	$\phi(\boldsymbol{x})$	$\frac{\partial_e \phi(h_{<1} \to h_{\geq 1})}{\partial_e x_1(0 \to 1)}$	$\frac{\partial_e \phi(h_{<1} \to h_{\geq 1})}{\partial_e x_2(0 \to 1)}$	$\prod_{s=0}^{2} \frac{\partial_e \phi(h_{<1} \to h_{\geq 1})}{\partial_e x_3(s \to s+1)}$	**Π-conjunction**
0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	0
0	0	2	0	1	1	0	0
0	**0**	**3**	0	1	1	*	**1**
0	1	0	0	0	*	1	0
0	1	1	1	0	*	0	0
0	1	2	1	0	*	0	0
0	1	3	2	0	*	*	0
1	0	0	0	*	0	1	0
1	0	1	1	*	0	0	0
1	0	2	1	*	0	0	0
1	0	3	2	*	0	*	0
1	**1**	**0**	0	*	*	1	**1**
1	1	1	2	*	*	0	0
1	1	2	3	*	*	0	0
1	1	3	3	*	*	*	*

Table 4. Minimal cut vectors and minimal path vectors for individual levels of availability of the service system.

System availability level	**Minimal cut vectors for given level of system availability**	**Minimal path vectors for given level of system availability**
1	(0,0,3), (1,1,0)	(0,1,1), (1,0,1)
2	(0,0,3), (0,1,2), (1,0,2), (1,1,0)	(0,1,3), (1,0,3), (1,1,1)
3	(0,1,3), (1,0,3), (1,1,1)	(1,1,2)

In the next step, we can sum all measures $\mathrm{FVI}_{3,s}^{c,\geq j}$ for given level j of system availability. This allows us to evaluate the total contribution of component 3 to system unavailability $U^{\geq j}$. Results of these calculations are presented in the right column of Table 5, and they imply that component 3 has the greatest influence on level 1 of system availability and the lowest on level 2. The similar investigations could be done for the remaining two components or using MPVs.

Table 5. Fussell-Vesely's importance measures for component 3 based on minimal cut vectors.

Component 3		Component state			Sum
		1	2	3	
System availability level	1	0.8065	0	0	0.8065
	2	0.3891	0.3750	0.0856	0.4747
	3	0	0.6757	0	0.6757

5 Conclusion

In this paper, a complex approach for investigation of MSSs based on MCVs, MPVs, and logical differential calculus was presented. The approach allows identifying all MCVs or MPVs for a given level of system availability. Based on the MCVs or MPVs several importance measures introduced in this paper can be computed. The measures represent generalization of the FVI, which has been developed originally for BSSs. The proposed generalizations of the FVI allows quantifying contribution of a degradation (improvement) of a given state of a given system component to system unavailability (availability) or total contribution of a given component to system unavailability (availability).

One of the most important parts of the approach presented in this paper is identification of MCVs or MPVs. For this purpose, the algorithm based on IDPLDs [16], [17] can be used. According to the experiments performed in [17], this algorithm can be used for small and medium MSSs that are composed of 15 to 20 components (depending on the number of components states). However, the main benefit of this algorithm is that it can be applied to systems with complicated structure because, as has been shown in [17], its time complexity does not depend on the number of MCVs or MPVs, which closely relate to internal structure of the system.

Acknowledgment
This work was supported by grants VEGA 1/0038/16 and VEGA 1/0354/17.

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