Combining Inductive Generalization and Factual Abduction

Mathieu Beirlaen

Ruhr University Bochum
Heinrich Heine University Düsseldorf
mathieubeirlaen@gmail.com

Abstract. The aim of this paper is to outline a first-order model for ampliative reasoning that fruitfully combines the inference patterns of inductive generalization and factual abduction. The pattern of inductive generalization is the archetype pattern of inductive inference by which we arrive at a universally quantified statement (All Ps are Q) given one or more instances (Some Ps are Q). In factual abduction, we reason from a universally quantified statement (All Ps are Q) and an instance of its consequent (object a is Q) to an instance of its antecedent (object a is P). It is shown how these patterns can be combined in such a way that inductively inferred generalizations can be used as premises in abductive inferences, and that conclusions of abductive inferences in turn can be used to inductively infer new generalizations. This process is formally explicated within the adaptive logics framework in terms of a preferential model semantics.

Keywords: induction, abduction, non-monotonic logic, ampliative reasoning, adaptive logics

1 Introduction

This is an exploratory investigation into combinations of ampliative reasoning patterns. Ampliative reasoning occurs whenever we draw inferences the conclusions of which cannot be deduced from the available premises by means of one's preferred standard of deduction. Examples of ampliative reasoning patterns include inductive generalization, abduction or inference to the best explanation, causal discovery, and reasoning by analogy. The study of these patterns is of interest to philosophers investigating the foundations of defeasible reasoning, to logicians investigating the formalization of defeasible reasoning, to computer scientists investigating the automation of defeasible reasoning, and to psychologists investigating defeasible reasoning in the wild.

The focus of this paper is on the formalization of two specific patterns of ampliative reasoning and their combination. The first is that of *inductive generalization*, the archetype pattern of inductive inference by which we reason to a universally quantified statement ("All Ps are Q") given one or more instances of it. The second pattern is that of *factual abduction*, by which we reason from a universally quantified statement ("All Ps are Q") and an instance of its consequent ("a is Q") to an instance of its antecedent ("a is P"). The inference patterns studied here are sub-patterns of the larger classes of inductive inferences and abductive inferences. For a comprehensive taxonomy of patterns of inductive inference, see [17]. For a comprehensive taxonomy of patterns of abductive inference, see [19]. The pattern of factual abduction is also known as *simple abduction* [23] or *plain abduction* [1].

The technical implementation and combination of inductive generalization and factual abduction is realized within the adaptive logics framework for modelling patterns of defeasible reasoning. There are two main reasons for choosing this framework. The first is that both inductive generalization and factual abduction are well-studied within this framework – see [8, 6, 7, 9, 12, 14, 16]. The second is that different means are available for combining adaptive logics – see [24, 25, 21].

Section 3 provides a short introduction to the adaptive logics framework, tailored to the aim of this paper. In Section 4 the logic for inductive generalization $\mathbf{LI^r}$ from [4, 6, 8] is presented and illustrated. In Section 5 the logic for factual abduction $\mathbf{FA^r}$ is introduced. The latter system is a close cousin of an adaptive logic for factual abduction defined within the framework of [7] (see footnote 9 below). The logics presented in Sections 4 and 5 are then sequentially combined (Section 6), resulting in the system $\mathbf{SIA^r}$.

The modest contribution of this paper is that it provides a full formal explication of how inductive generalization and factual abduction can be combined within a single system, and that this combination is fruitful in the following sense: inductively obtained conclusions can be used as premises in abductive inferences, and vice versa. From this, no conclusions should be drawn yet regarding the normative or descriptive adequacy of this system: more work remains to be done. For instance, an adequate formalization of these inference patterns and their combination requires a detailed study of their alternative logical characterizations, and a richer formal language. Some of these alternatives and enrichments are discussed in Section 7, alongside a number of design choices which are best motivated after defining $\mathbf{SIA^r}$.

2 Notational Conventions

Let $\mathcal{L}$ be a first-order language built using a set $\mathcal{P}$ of unary predicates, a set $\mathcal{C}$ of individual constants, a set $\mathcal{V}$ of individual variables, and the logical symbols $\top, \bot, \neg, \vee, \wedge, \supset, \equiv, \exists, \forall$. In what follows, **CL** refers to first-order classical logic without identity, and restricted to $\mathcal{L}$ (no n-ary predicates for $n > 1$, no function symbols).

Upper case letters P, Q, R, etc., lower case letters a, b, c, etc., respectively lower case letters x, y, z, denote members of $\mathcal{P}, \mathcal{C}$, respectively $\mathcal{V}$. For all $\alpha \in \mathcal{C} \cup \mathcal{V}$, $\mathcal{L}^\alpha = \{\pi\alpha, \neg\pi\alpha \mid \pi \in \mathcal{P}\}$, and $\mathcal{F}^\alpha$ is the set of truth-functions of formulas in $\mathcal{L}^\alpha$. For instance, $Pa \in \mathcal{L}^a$ and $\neg Px \vee (Qx \supset Rx) \in \mathcal{F}^x$. Where $\alpha \in \mathcal{C} \cup \mathcal{V}$, $A(\alpha), B(\alpha)$, etc. denote members of $\mathcal{F}^\alpha$, unless further specified.

Where M is a **CL**-model, $A \in \mathcal{L}$ and $\Gamma \subseteq \mathcal{L}$, $M \Vdash A$ means that M verifies A; M is a model of Γ iff $M \Vdash A$ for all $A \in \Gamma$. Relative to a logic **L**, $\mathcal{M}_{\mathbf{L}}(\Gamma)$ denotes the set of **L**-models of Γ, and $\Gamma \vDash_{\mathbf{L}} A$ means that A is verified by all $M \in \mathcal{M}_{\mathbf{L}}(\Gamma)$. $Cn_{\mathbf{L}}(\Gamma)$ is the set of **L**-consequences of Γ.

3 Adaptive logics

Adaptive logics are tools for explicating defeasible reasoning patterns. They were originally developed by Batens, who also defined a *standard format* for adaptive logics [3, 5, 7]. Systems defined within this format are equipped with a dynamic proof theory and a selection semantics in the vein of Shoham's preferred models [20], KLM's preferential models [13], or Makinson's default valuations [15, Ch. 3].[1] For conciseness of presentation, the adaptive logics presented here are defined only from a semantic point of view.

Adaptive logics strengthen a core logic called the *lower limit logic*. The adaptive semantics is a mechanism for selecting a preferred subset among the models of the lower limit logic relative to a premise set. The selected set contains models that are minimal with respect to a *set of abnormalities*: a set of formulas characterized by some logical form. The exact way in which an adaptive logic minimizes abnormalities verified by its lower limit models varies with the *adaptive strategy* used. Depending on the strategy used in the minimization process, different sets of lower limit models may be selected relative to a premise set, giving rise to possibly different sets of logical consequences. An adaptive logic defined within the standard format is fully characterized in terms of three elements: a lower limit logic, a set of abnormalities, and an adaptive strategy.

Below two adaptive logics will be presented: the logic for inductive generalization $\mathbf{LI^r}$, and the logic of factual abduction $\mathbf{FA^r}$. These logics have **CL** as their lower limit logic. They differ with respect to their respective sets of abnormalities. The superscript **r** is the first letter of the adaptive strategy used by these logics: the *reliability* strategy.

Adaptive logics provide a flexible framework for studying different types of defeasible reasoning patterns and their combinations. This makes them very suitable for the present exploration of combining inductive generalization and factual abduction. The format for combination used here is that of *sequential superposition* [21, Ch. 3], [22].

Given a premise set $\Gamma \subseteq \mathcal{L}$, the logics $\mathbf{LI^r}$ and $\mathbf{FA^r}$ are sequentially combined in the following way:

$$\ldots Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma))))\ldots \quad (1)$$

[1] For the sake of historical accuracy: the semantics for adaptive logics – first presented in [2] for the minimal abnormality strategy (cfr. infra) – was developed independently of the accounts of Shoham, KLM, and Makinson.

In a first step, $\mathbf{LI^r}$ is applied to check which generalizations can be inferred from the premise set Γ. Next, $\mathbf{FA^r}$ is applied to infer new predictions via factual abduction. These new predictions can in turn be used to check for new generalizations by means of $\mathbf{LI^r}$, and so on.

4 Inductive Generalization

The adaptive logic $\mathbf{LI^r}$ strengthens its lower limit logic, **CL**, by interpreting the world 'as uniformly as possible'. It does so by taking as its set of abnormalities a set of falsified universally quantified statements, so that its least abnormal models are those in which these universally quantified statements hold true. The set Ω_i of $\mathbf{LI^r}$-abnormalities is defined as follows:[2]

$$\Omega_i = \{\neg\forall\alpha(A_1(\alpha) \vee \ldots \vee A_n(\alpha)) \mid \alpha \in \mathcal{V}, A_1(\alpha), \ldots, A_n(\alpha) \in \mathcal{L}^\alpha\}$$

In the remainder the term *generalization* refers to formulas of the form $\forall\alpha(A_1(\alpha)\vee \ldots \vee A_n(\alpha))$, so that Ω_i is the set of negated generalizations.

To complete the characterization of $\mathbf{LI^r}$, a mechanism is needed for selecting a 'preferred' subset of the **CL**-models of a given premise set relative to the set Ω_i. This mechanism is provided by the reliability strategy, which selects a set $\mathcal{M}^r_i(\Gamma)$ of i-reliable models of $\Gamma \subseteq \mathcal{L}$. The characterization of this set requires some more terminology. 'Dab' is an acronym for 'disjunction of abnormalities'. Where $\Delta \subseteq \Omega_i$, $Dab_i(\Delta) = \bigvee\Delta$.[3] $Dab_i(\Delta)$ is a *Dab_i-consequence* of Γ iff $\Gamma \vDash_{\mathbf{CL}} Dab_i(\Delta)$, and $Dab_i(\Delta)$ is a *minimal Dab_i-consequence* of Γ iff $Dab_i(\Delta)$ is a Dab_i-consequence of Γ and there is no $\Delta' \subset \Delta$ such that $Dab_i(\Delta')$ is a Dab_i-consequence of Γ. Where $Dab_i(\Delta_1), Dab_i(\Delta_2), \ldots$ are the minimal Dab_i-consequences of Γ, $U_i(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ is the set of *i-unreliable formulas* of Γ. Where $Ab_i(M) = \{A \in \Omega_i \mid M \Vdash A\}$:

$$\mathcal{M}^r_i(\Gamma) = \{M \in \mathcal{M}_{\mathbf{CL}}(\Gamma) \mid Ab_i(M) \subseteq U_i(\Gamma)\}$$

Definition 1. *$\Gamma \vDash_{\mathbf{LI^r}} A$ iff $M \Vdash A$ for all $M \in \mathcal{M}^r_i(\Gamma)$.*

As an illustration of the workings of $\mathbf{LI^r}$, consider the premise set $\Gamma_1 = \{Pa \wedge Qa \wedge \neg Ra \wedge \neg Sa, Qb \wedge Rb, Pb \supset \neg Sb, \neg Pc \wedge \neg Qc \wedge \neg Rc \wedge Sc, \neg Pd \wedge Qd \wedge \neg Rd \wedge Sd, \neg Pe \wedge \neg Qe \wedge \neg Re \wedge \neg Se, Pf \wedge Qf \wedge \neg Rf \wedge Sf, Pg \wedge Qg \wedge Rg, \neg Ph \wedge \neg Qh \wedge Rh, \neg Pi \wedge Qi \wedge Ri, \neg Pj \wedge Qj \wedge \neg Rj \wedge \neg Sj\}$.

For future reference, it is convenient to list all i-abnormalities that can be formed using only the four predicates occurring in Γ_1 (see Table 1).

The set of Dab_i-consequences of Γ_1 contains, amongst others, *all* disjunctions between formulas listed in Table 1 that are **CL**-derivable from Γ_1, including 'single-disjunct' disjunctions. The minimal Dab_i-consequences of Γ_1 are all minimal such disjunctions. They include

[2] In [26, Sec. 4.2.2] it is shown that the same logic is obtained if Ω_i is defined as the set of formulas of the form $\neg\forall\alpha A(\alpha)$, where $\alpha \in \mathcal{V}$ and $A(\alpha) \in \mathcal{F}^\alpha$.

[3] If Δ is a singleton $\{A\}$, $Dab_i(\Delta) = A$.

1. $\neg\forall x(Px)$	28. $\neg\forall x(\neg Qx \vee \neg Sx)$	55. $\neg\forall x(\neg Px \vee \neg Rx \vee Sx)$
2. $\neg\forall x(\neg Px)$	29. $\neg\forall x(Rx \vee Sx)$	56. $\neg\forall x(\neg Px \vee \neg Rx \vee \neg Sx)$
3. $\neg\forall x(Qx)$	30. $\neg\forall x(Rx \vee \neg Sx)$	57. $\neg\forall x(Qx \vee Rx \vee Sx)$
4. $\neg\forall x(\neg Qx)$	31. $\neg\forall x(\neg Rx \vee Sx)$	58. $\neg\forall x(Qx \vee Rx \vee \neg Sx)$
5. $\neg\forall x(Rx)$	32. $\neg\forall x(\neg Rx \vee \neg Sx)$	59. $\neg\forall x(Qx \vee \neg Rx \vee Sx)$
6. $\neg\forall x(\neg Rx)$	33. $\neg\forall x(Px \vee Qx \vee Rx)$	60. $\neg\forall x(Qx \vee \neg Rx \vee \neg Sx)$
7. $\neg\forall x(Sx)$	34. $\neg\forall x(Px \vee Qx \vee \neg Rx)$	61. $\neg\forall x(\neg Qx \vee Rx \vee Sx)$
8. $\neg\forall x(\neg Sx)$	35. $\neg\forall x(Px \vee \neg Qx \vee Rx)$	62. $\neg\forall x(\neg Qx \vee Rx \vee \neg Sx)$
9. $\neg\forall x(Px \vee Qx)$	36. $\neg\forall x(Px \vee \neg Qx \vee \neg Rx)$	63. $\neg\forall x(\neg Qx \vee \neg Rx \vee Sx)$
10. $\neg\forall x(Px \vee \neg Qx)$	37. $\neg\forall x(\neg Px \vee Qx \vee Rx)$	64. $\neg\forall x(\neg Qx \vee \neg Rx \vee \neg Sx)$
11. $\neg\forall x(\neg Px \vee Qx)$	38. $\neg\forall x(\neg Px \vee Qx \vee \neg Rx)$	65. $\neg\forall x(Px \vee Qx \vee Rx \vee Sx)$
12. $\neg\forall x(\neg Px \vee \neg Qx)$	39. $\neg\forall x(\neg Px \vee \neg Qx \vee Rx)$	66. $\neg\forall x(Px \vee Qx \vee Rx \vee \neg Sx)$
13. $\neg\forall x(Px \vee Rx)$	40. $\neg\forall x(\neg Px \vee \neg Qx \vee \neg Rx)$	67. $\neg\forall x(Px \vee Qx \vee \neg Rx \vee Sx)$
14. $\neg\forall x(Px \vee \neg Rx)$	41. $\neg\forall x(Px \vee Qx \vee Sx)$	68. $\neg\forall x(Px \vee Qx \vee \neg Rx \vee \neg Sx)$
15. $\neg\forall x(\neg Px \vee Rx)$	42. $\neg\forall x(Px \vee Qx \vee \neg Sx)$	69. $\neg\forall x(Px \vee \neg Qx \vee Rx \vee Sx)$
16. $\neg\forall x(\neg Px \vee \neg Rx)$	43. $\neg\forall x(Px \vee \neg Qx \vee Sx)$	70. $\neg\forall x(Px \vee \neg Qx \vee Rx \vee \neg Sx)$
17. $\neg\forall x(Px \vee Sx)$	44. $\neg\forall x(Px \vee \neg Qx \vee \neg Sx)$	71. $\neg\forall x(Px \vee \neg Qx \vee \neg Rx \vee Sx)$
18. $\neg\forall x(Px \vee \neg Sx)$	45. $\neg\forall x(\neg Px \vee Qx \vee Sx)$	72. $\neg\forall x(Px \vee \neg Qx \vee \neg Rx \vee \neg Sx)$
19. $\neg\forall x(\neg Px \vee Sx)$	46. $\neg\forall x(\neg Px \vee Qx \vee \neg Sx)$	73. $\neg\forall x(\neg Px \vee Qx \vee Rx \vee Sx)$
20. $\neg\forall x(\neg Px \vee \neg Sx)$	47. $\neg\forall x(\neg Px \vee \neg Qx \vee Sx)$	74. $\neg\forall x(\neg Px \vee Qx \vee Rx \vee \neg Sx)$
21. $\neg\forall x(Qx \vee Rx)$	48. $\neg\forall x(\neg Px \vee \neg Qx \vee \neg Sx)$	75. $\neg\forall x(\neg Px \vee Qx \vee \neg Rx \vee Sx)$
22. $\neg\forall x(Qx \vee \neg Rx)$	49. $\neg\forall x(Px \vee Rx \vee Sx)$	76. $\neg\forall x(\neg Px \vee Qx \vee \neg Rx \vee \neg Sx)$
23. $\neg\forall x(\neg Qx \vee Rx)$	50. $\neg\forall x(Px \vee Rx \vee \neg Sx)$	77. $\neg\forall x(\neg Px \vee \neg Qx \vee Rx \vee Sx)$
24. $\neg\forall x(\neg Qx \vee \neg Rx)$	51. $\neg\forall x(Px \vee \neg Rx \vee Sx)$	78. $\neg\forall x(\neg Px \vee \neg Qx \vee Rx \vee \neg Sx)$
25. $\neg\forall x(Qx \vee Sx)$	52. $\neg\forall x(Px \vee \neg Rx \vee \neg Sx)$	79. $\neg\forall x(\neg Px \vee \neg Qx \vee \neg Rx \vee Sx)$
26. $\neg\forall x(Qx \vee \neg Sx)$	53. $\neg\forall x(\neg Px \vee Rx \vee Sx)$	80. $\neg\forall x(\neg Px \vee \neg Qx \vee \neg Rx \vee \neg Sx)$
27. $\neg\forall x(\neg Qx \vee Sx)$	54. $\neg\forall x(\neg Px \vee Rx \vee \neg Sx)$	

Table 1. i-abnormalities for the predicates P, Q, R, S.

- the abnormalities 1–10, 12–30, 33–36, 39–44, 47–50, 53, 54, 57, 58, 61, 62, 65, 66, 69, 70, 77, and 78 from Table 1, and
- the disjunctions listed in Table 2.[4]

31∨32	31∨64	32∨51	32∨67	51∨60	52∨59	55∨56	56∨79	63∨64	64∨79
31∨52	31∨68	32∨55	32∨71	51∨64	52∨63	55∨64	59∨60	63∨72	67∨68
31∨56	31∨72	32∨59	32∨79	51∨68	52∨67	55∨80	59∨68	63∨80	71∨72
31∨60	31∨80	32∨63	51∨52	51∨72	52∨71	56∨63	60∨67	64∨71	79∨80

Table 2. Two-disjunct minimal Dab_i-consequences of Γ_1. Numbers refer to the corresponding abnormalities in Table 1.

Importantly, the abnormalities 11, 37, 38, 45, 46, 73–76 do not occur as disjuncts in any minimal Dab_i-consequence of Γ_1. Indeed, for any Dab_i-consequence of Γ_1 containing one of these abnormalities as one of its disjuncts, there is a strictly shorter disjunction which is a *minimal* Dab_i-consequence of Γ_1 and which does *not* contain the abnormality in question as one of its disjuncts. Thus the set $U_i(\Gamma_1)$ of i-abnormalities that behave unreliably with respect to Γ_1 contains all abnormalities in Table 1 *except* for 11, 37, 38, 45, 46, 73–76. This means that the set $\mathcal{M}_i^r(\Gamma_1)$ of i-reliable models of Γ_1 contains no models which verify any of these abnormalities. So the negations of 11, 37, 38, 45, 46, 73–76 hold true in all i-reliable models of Γ_1. By Definition 1:

$$\Gamma_1 \vDash_{\mathbf{LI^r}} \forall x(\neg Px \vee Qx) \tag{2}$$

Clearly, the negations of 37, 38, 45, 46, 73–76, which are **CL**-consequences of $\forall x(\neg Px \vee Qx)$, are also among the $\mathbf{LI^r}$-consequences of Γ_1.

The logic $\mathbf{LI^r}$, like all adaptive logics defined within the standard format, inherits a number of meta-theoretical properties such as

- $Cn_{\mathbf{CL}}(Cn_{\mathbf{LI^r}}(\Gamma)) = Cn_{\mathbf{LI^r}}(\Gamma)$ (**CL**-closure)
- $Cn_{\mathbf{LI^r}}(Cn_{\mathbf{LI^r}}(\Gamma)) = Cn_{\mathbf{LI^r}}(\Gamma)$ (fixed point)
- If $M \in \mathcal{M}_{\mathbf{CL}}(\Gamma) \setminus \mathcal{M}_i^r(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{LI^r}}(\Gamma)$ such that $Ab_i(M') \subset Ab_i(M)$ (smoothness)

For the generic proofs of these properties for adaptive logics in standard format, see [5, Sec. 6-8]. For a slower-paced introduction to $\mathbf{LI^r}$, and for more illustrations of its workings, see [4, 8].

5 Factual Abduction

The inference pattern of factual abduction is a defeasible version of the *backward modus ponens* (BMP) rule. Where $\alpha \in \mathcal{V}$, $\beta \in \mathcal{C}$, $A(\alpha), B(\alpha) \in \mathcal{F}^\alpha$, $A(\beta), B(\beta) \in$

[4]The tedious exercise of verifying that Γ_1 has no minimal Dab_i-consequences of three or more disjuncts is safely left to the interested reader.

$\mathcal{F}^{\beta}$:

$$\forall\alpha(A(\alpha) \supset B(\alpha)), B(\beta)/A(\beta) \qquad \text{(BMP)}$$

In order to prevent that the adaptive logic $\mathbf{FA}^{\mathbf{r}}$ over- or undergenerates abductive consequences, a number of further technical requirements must be imposed on inferences of the form (BMP), as the following examples illustrate.[5]

Example 1. Let $\Gamma_2 = \{\forall x(Sx \supset Qx), \forall x(Rx \supset Px), Pa, Qa\}$. Sa is derivable by (BMP) applied to $\forall x(Sx \supset Qx)$ and Qa. But $\forall x(Rx \supset Px) \vDash_{\mathbf{CL}} \forall x((Rx \wedge \neg Sx) \supset Px)$, so by the same token (BMP) can be applied to $\forall x((Rx \wedge \neg Sx) \supset Px)$ and Pa so as to infer $Ra \wedge \neg Sa$, which contradicts the earlier inference of Sa.
This example motivates a restriction according to which (BMP) is not applicable to universally quantified conditionals the antecedents of which have been strengthened, such as $\forall x((Rx \wedge \neg Sx) \supset Px)$.

Example 2. Let $\Gamma_3 = \{\forall x(Px \supset Qx), Ra\}$. One would not expect Pa to be derivable via (BMP). But $\forall x(Px \supset Qx) \vDash_{\mathbf{CL}} \forall x(Px \supset (Qx \vee Rx))$ and $Ra \vDash_{\mathbf{CL}} Qa \vee Ra$. So Pa can be inferred by applying (BMP) to $\forall x(Px \supset (Qx \vee Rx))$ and $Qa \vee Ra$. The resulting logic overgenerates.
This example motivates a restriction according to which (BMP) is not applicable to universally quantified conditionals the consequents of which have been weakened, such as $\forall x(Px \supset (Qx \vee Rx))$.

A single technical requirement suffices to ensure that the problems in Examples 1 and 2 are avoided. Note that the universally quantified conditional in arguments of the form (BMP) can be expressed equivalently as a (conjunction of) universally quantified disjunction(s). For instance, $\forall x(Px \supset Qx)$, respectively $\forall x((Px \vee Rx) \supset Qx$, $\forall x(Px \supset (Qx \wedge Rx))$ are equivalent to $\forall x(\neg Px \vee Qx)$, respectively $\forall x(\neg Px \vee Qx) \wedge \forall x(\neg Rx \vee Qx)$, $\forall x(\neg Px \vee Qx) \wedge \forall x(\neg Px \vee Rx)$. If we do the same in Examples 1 and 2, it is immediate that in the undesirable applications of factual abduction the universally quantified premise results from weakening a logically stronger generalization. In Example 1, the generalization $\forall x(\neg Rx \vee Px)$ was weakened to $\forall x(\neg Rx \vee Sx \vee Px)$. In Example 2, the generalization $\forall x(\neg Px \vee Qx)$ was weakened to $\forall x(\neg Px \vee Qx \vee Rx)$. These weakened generalizations or their conditional equivalents cause trouble when used as premises in abductive inferences. This motivates a restriction of applications of factual abduction to generalizations from which no disjuncts can be removed. Such generalizations will be called *starred* generalizations. They make use of a starred quantifier '$\overset{*}{\forall}$', expressing that the generalization in question cannot be shortened. Where $\alpha \in \mathcal{V}$ and $A_1, \ldots, A_n, B_1, \ldots, B_k \in \mathcal{L}^{\alpha}$:

$$\overset{*}{\forall}\alpha(A_1 \vee \ldots \vee A_n) = \forall\alpha(A_1 \vee \ldots \vee A_n) \wedge \neg \bigvee\{\forall\alpha(B_1 \vee \ldots \vee B_k) \mid \emptyset \neq \{B_1, \ldots, B_k\} \subset \{A_1, \ldots, A_n\}\}$$

[5]Both examples presuppose **CL** in the background. The first example is adopted from the technical appendix in [9]. The second example is by Frederik Van De Putte (personal communication).

The logic $\mathbf{FA^r}$ allows for the defeasible application of the factual abduction pattern to starred generalizations. More precisely, it implements a defeasible version of the 'backward disjunctive syllogism' rule obtained by replacing $\forall\alpha(A(\alpha) \supset B(\alpha))$ with $\overset{*}{\forall}\alpha(\neg A(\alpha) \vee B(\alpha))$ in (BMP).

The lower limit logic of $\mathbf{FA^r}$ is $\mathbf{CL}$. Its set of abnormalities is the set Ω_a. Where $\alpha \in \mathcal{V}, \beta \in \mathcal{C}, A_1(\alpha), \ldots, A_i(\alpha), B_1(\alpha), \ldots, B_j(\alpha) \in \mathcal{L}^\alpha, i \geq 1, j \geq 1$:

$$\Omega_a = \{\overset{*}{\forall}\alpha(A_1(\alpha) \vee \ldots \vee A_i(\alpha) \vee B_1(\alpha) \vee \ldots \vee B_j(\alpha)) \wedge (A_1(\beta) \vee \ldots \vee A_i(\beta)) \wedge (B_1(\beta) \vee \ldots \vee B_j(\beta))\}$$

Given a premise set Γ, a $\mathbf{CL}$-model M of Γ, and a set $\Delta \subseteq \Omega_a$, the sets $\mathcal{M}^r_a(\Gamma)$, $Dab_a(\Delta)$, the set of (minimal) Dab_a-consequences of Γ, and the set $U_a(\Gamma)$ of a-unreliable formulas of Γ are defined exactly like their inductive counterparts: just replace subscripts 'i' with 'a' in their respective counterpart definitions in Section 4.

Definition 2. *$\Gamma \vDash_{\mathbf{FA^r}} A$ iff $M \Vdash A$ for all $M \in \mathcal{M}^r_a(\Gamma)$.*

Given a premise set Γ, $\mathbf{FA^r}$ selects the $\mathbf{CL}$-models of Γ which verify no a-abnormalities except for those in $U_a(\Gamma)$, just like $\mathbf{LI^r}$ would select the $\mathbf{CL}$-models of Γ which verify no i-abnormalities except for those in $U_i(\Gamma)$. By way of illustration, let $\Gamma_4 = Cn_{\mathbf{LI^r}}(\Gamma_1)$. Recall that $\forall x(\neg Px \vee Qx) \in \Gamma_4$, and note that $\Gamma_4 \vDash_{\mathbf{CL}} \neg\forall x \neg Px \wedge \neg\forall x Qx$. Thus $\Gamma_4 \vDash_{\mathbf{CL}} \overset{*}{\forall}x(\neg Px \vee Qx)$. In fact, $\overset{*}{\forall}x(\neg Px \vee Qx)$ is the *only* starred generalization which is $\mathbf{CL}$-derivable from Γ_4: all other generalizations are either not in the set of $\mathbf{LI^r}$-consequences of Γ_4, or they are logically weaker than $\forall x(\neg Px \vee Qx)$. Generalizations which are not $\mathbf{LI^r}$-consequences of Γ_4 include the negations of all i-abnormalities which are $\mathbf{CL}$-derivable from Γ_1, as well as the negations of all i-abnormalities occurring as a disjunct in Table 2. Generalizations which are $\mathbf{CL}$-equivalent to a generalization which is logically weaker than $\forall x(\neg Px \vee Qx)$ include the negations of abnormalities 37, 38, 45, 46, 73–76 in Table 1.

Γ_4 has three minimal Dab_a-consequences:

$$\overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pd \wedge Qd \tag{3}$$

$$\overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pi \wedge Qi \tag{4}$$

$$\overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pj \wedge Qj \tag{5}$$

Thus, $U_a(\Gamma_4) = \{\overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pd \wedge Qd, \overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pi \wedge Qi, \overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pj \wedge Qj\}$. Reliable $\mathbf{CL}$-models of Γ_4 – members of $\mathcal{M}^r_a(\Gamma_4)$ – verify no further a-abnormalities. So they falsify the abnormality $\overset{*}{\forall}x(\neg Px \vee Qx) \wedge \neg Pb \wedge Qb$. Since they verify both $\overset{*}{\forall}x(\neg Px \vee Qx)$ and Qb, they must falsify $\neg Pb$, so that:

$$\Gamma_4 \vDash_{\mathbf{FA^r}} Pb \tag{6}$$

6 Iteration

Let $\Gamma_5 = Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$. If $\mathbf{LI^r}$ were applied to this premise set, would that deliver new consequences on top of the members of Γ_5? Note that since $Pb \in \Gamma_5$, the i-abnormalities 31, 55, 63, and 79 are **CL**-consequences of Γ_5. Thus, a number of disjunctions in Table 2 are no longer minimal with respect to Γ_5. In particular, the disjunctions 31∨56, 31∨80, 55∨56, 55∨80, 56∨63, 56∨79, and 79∨80 are no longer minimal. As a result, the abnormalities 56 and 80 no longer occur as disjuncts in minimal Dab-consequences of Γ_5. Because of this, they do not belong to $U_i(\Gamma_5)$, and they are falsified by all reliable models of Γ_5. As a result:

$$\Gamma_5 \vDash_{\mathbf{LI^r}} \forall x(\neg Px \vee \neg Rx \vee \neg Sx) \tag{7}$$

$\forall x(\neg Px \vee \neg Rx \vee \neg Sx) \notin \Gamma_5$, so a new generalization becomes derivable upon applying $\mathbf{LI^r}$ to $Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$.

So far, new information was obtained at each 'round' of application of the logics $\mathbf{LI^r}$ and $\mathbf{FA^r}$: $\forall x(\neg Px \vee Qx) \in Cn_{\mathbf{LI^r}}(\Gamma_1)$ while $\forall x(\neg Px \vee Qx) \notin Cn_{\mathbf{CL}}(\Gamma_1)$, $Pb \in Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$ while $Pb \notin Cn_{\mathbf{LI^r}}(\Gamma_1)$, and $\forall x(\neg Px \vee \neg Rx \vee \neg Sx) \in Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1)))$ while $\forall x(\neg Px \vee \neg Rx \vee \neg Sx) \notin Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$. What if $\mathbf{FA^r}$ was applied to $Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1)))$? Can new information be abduced still? No. The inference pattern of factual abduction is only applicable to *starred* generalizations. The only new generalization obtained in the previous round was $\forall x(\neg Px \vee \neg Rx \vee \neg Sx)$, so the only way to obtain new information by factual abduction is via the use of this generalization. But we cannot infer its starred version.

$$\overset{*}{\forall} x(\neg Px \vee \neg Rx \vee \neg Sx) \notin Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))) \tag{8}$$

The reason is that we cannot infer $\neg\forall x(\neg Rx \vee \neg Sx)$. Indeed, neither this i-abnormality nor its negation is a member of $Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1)))$.[6] Since we cannot infer any new starred generalizations from $Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1)))$, nothing new can be abduced.

The iterative process of applying inductive generalization and factual abduction can be repeated *ad infinitum*. The consequence operation $Cn_{\mathbf{SIA^r}}$ is defined as follows:

$$Cn_{\mathbf{SIA^r}}(\Gamma) = \ldots Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma)))) \ldots \tag{9}$$

Alternatively, this operation can be described as follows. Given a premise set Γ, first select the **CL**-models of Γ (level 0). Next select the $\mathbf{LI^r}$-models of the resulting set (level 1). Next, select the $\mathbf{FA^r}$-models (level 2), then again select via $\mathbf{LI^r}$ (level 3), and so on.

[6] This i-abnormality is number 32 in Table 1. It is a member of $U_i(Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1)))$ in view of the following minimal Dab_i-consequences of $Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$: $32 \vee 51$, $32 \vee 59$, $32 \vee 67$, and $32 \vee 71$. In view of this, some but not all reliable models of $Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma_1))$ verify $\forall x(\neg Rx \vee \neg Sx)$, while others falsify this generalization.

Definition 3. *Where $j \geq 1$:*

$$\mathcal{M}_0(\Gamma) = \mathcal{M}_{\mathbf{CL}}(\Gamma)$$

$$\mathcal{M}_j(\Gamma) = \begin{cases} \{M \in \mathcal{M}_{j-1}(\Gamma) \mid Ab_i(M) \subseteq U_i(\{A \mid M' \vDash A \\ \qquad \text{for all } M' \in \mathcal{M}_{j-1}(\Gamma)\})\} \text{ if } j \text{ is odd,} \\ \{M \in \mathcal{M}_{j-1}(\Gamma) \mid Ab_a(M) \subseteq U_a(\{A \mid M' \vDash A \\ \qquad \text{for all } M' \in \mathcal{M}_{j-1}(\Gamma)\})\} \text{ if } j \text{ is even.} \end{cases}$$

Definition 4. *Where $j \in \mathbb{N}$, $\Gamma \vDash_{\mathbf{SIA^r_j}} A$ iff $M \Vdash A$ for all $M \in \mathcal{M}_j(\Gamma)$.*

It was shown generically (for adaptive logics using the reliability strategy) in [21, Sec. 3.2.1] that, at each step in the construction, the resulting logics are semantically adequate with respect to the sequence in (9): $\Gamma \vDash_{\mathbf{SIA^r_1}} A$ iff $A \in Cn_{\mathbf{LI^r}}(\Gamma)$, $\Gamma \vDash_{\mathbf{SIA^r_2}} A$ iff $A \in Cn_{\mathbf{FA^r}}(Cn_{\mathbf{LI^r}}(\Gamma))$, and so on. Next, we turn to the limiting case.

$$\mathcal{M}_\infty(\Gamma) = \liminf_{j\to\infty} \mathcal{M}_j(\Gamma) = \bigcap_{j\in\mathbb{N}} \mathcal{M}_j(\Gamma) \tag{10}$$

Definition 5. *$\Gamma \vDash_{\mathbf{SIA^r}} A$ iff $M \Vdash A$ for all $M \in \mathcal{M}_\infty(\Gamma)$.*

In [24, Sec. 3.3.2] the generic semantic adequacy result from [21] is extended to the infinite case. Applied to the present setting, (11) follows immediately:

$$\Gamma \vDash_{\mathbf{SIA^r}} A \text{ iff } A \in Cn_{\mathbf{SIA^r}}(\Gamma) \tag{11}$$

For languages with a finite signature the logic $\mathbf{SIA^r}$ is decidable. It remains an open question whether this is also the case for languages of infinite signature. Another open issue is that of determining the computational complexity of $\mathbf{SIA^r}$. In [18] it was shown that for adaptive logics defined within the standard format – such as $\mathbf{LI^r}$ and $\mathbf{FA^r}$ – the complexity upper bound in the arithmetical hierarchy is Σ^0_3. There are currently no published results on the computational complexity of sequentially combined adaptive logics such as $\mathbf{SIA^r}$.

7 Discussion and Variation

Here is a different way of writing the outcomes obtained for Γ_1 in Sections 4-6:

$$\Gamma_1 \vDash_{\mathbf{SIA^r_1}} \forall x(\neg Px \vee Qx) \tag{12}$$

$$\Gamma_1 \vDash_{\mathbf{SIA^r_2}} Pb \tag{13}$$

$$\Gamma_1 \vDash_{\mathbf{SIA^r_3}} \forall x(\neg Px \vee \neg Rx \vee \neg Sx) \tag{14}$$

The example shows how information obtained via factual abduction can in turn serve to inductively infer generalizations not previously derivable from the premise set. Stretching things a bit, this example logically explicates and confirms the view – revived by Douglas in [10] – that part of what makes (abduced) explanations useful is their help in generating new predictions (in this case, via inductive generalization). The 'stretch' here concerns the use of the term 'explanation' for referring to formulas inferred via factual abduction. Arguably, conclusions drawn via factual abduction classify at best as mere *potential* explanations, and a richer formalism is needed to adequately represent their epistemic status as opposed to e.g. observations in the premise set, cfr. infra. In this respect, the logic $\mathbf{SIA^r}$ oversimplifies matters.

Besides factual abduction, the logic $\mathbf{SIA^r}$ goes some way towards explicating another 'pattern' of abductive inference, namely the pattern of *law-abduction*, which has the following logical form [19]:

$\forall\alpha(A(\alpha) \supset B(\alpha))$	(Explanandum)
$\forall\alpha(C(\alpha) \supset B(\alpha))$	(Background law)
$\forall\alpha(A(\alpha) \supset C(\alpha))$	(Explanatory hypothesis)

Following an illustration given in [19], let P, Q, R denote respectively 'contains sugar', 'tastes sweet', and 'is a pineapple'. Our background knowledge includes $\forall x(Px \supset Qx)$. Some things contain sugar while others don't, and some things taste sweet while others don't, so $\overset{*}{\forall} x(\neg Px \vee Qx)$. The aim is to explain $\forall x(Rx \supset Qx)$ – which we obtained by inductive generalization from a number of instances $Ra \wedge Qa$, $Rb \wedge Qb$, etc. Via factual abduction applied to $\overset{*}{\forall} x(\neg Px \vee Qx)$ and $Qa, Qb, \ldots$, the formulas $Pa, Pb, \ldots$ can be inferred. And by inductive generalization applied to $Ra \wedge Pa, Rb \wedge Pb$, etc., we obtain $\forall x(Rx \supset Px)$. When asked why pineapples taste sweet, we can now answer by telling that pineapples contain sugar.[7]

An important design choice in the construction of $\mathbf{SIA^r}$ is the preference for a *sequential* combination of the patterns of inductive generalization and factual abduction. As is clear from the characterization of $\mathbf{SIA^r}$-consequence in Definitions 3 and 4, $\mathbf{SIA^r}$-models are selected sequentially or stepwise relative to either U_i or U_a. At each step in the sequence we select either exclusively with respect to i-unreliable formulas, or we select exclusively with respect to a-unreliable formulas.

A different, 'parallel' rather than sequential, combination strategy would be to look at *both* i-unreliable formulas *and* a-unreliable formulas in one single step. To this end, we could define a unique set of abnormalities $\Omega_{ia} = \Omega_i \cup \Omega_a$. The sets $\mathcal{M}_{ia}$, $Dab_{ia}(\Delta)$, U_{ia}, etc. are then redefined accordingly in terms of Ω_{ia}. In the resulting logic, (minimal) Dab_{ia}-consequences may consist of disjunctions between one or more members of Ω_i and/or one or more members of Ω_a. For

[7] Flach & Kakas thought of law-abduction as a hybrid inference pattern combining inductive generalization and factual abduction [11, pp. 21-22]. This view was criticized by Schurz on the grounds that this decomposition of law-abduction is "somewhat artificial. Law-abductions are usually performed in one single conjectural step" [19, p. 212]. For an adaptive logic explicating the latter view, see [12].

instance, the disjunction

$$\neg\forall x(\neg Px \vee Qx) \vee (\overset{*}{\forall} x(\neg Px \vee Qx) \wedge Qd \wedge \neg Pd) \tag{15}$$

is a minimal Dab_{ia}-consequence of Γ_1, since it is a **CL**-consequence of Γ_1 and neither of its disjuncts is a **CL**-consequence of Γ_1. As a result, $\neg\forall x(\neg Px \vee Qx)$ is a member of $U_{ia}(\Gamma_1)$, and $\forall x(\neg Px \vee Qx)$ is not a logical consequence in the resulting logic, so the resulting logic clearly differs from $\mathbf{SIA^r}$.

The disjunction in (15) serves to illustrate that the 'parallel' combination of inductive generalization and factual abduction is problematic. To see why, note that this disjunction is a **CL**-consequence of $\{Pa, Qa, \neg Pc, \neg Qc, \neg Pd, Qd\}$, which is a proper subset of $Cn(\Gamma_1)$. The instances a, c, and d all confirm the generalization $\forall x(\neg Px \vee Qx)$. Still, we can infer (15) as a minimal Dab_{ia}-consequence of this premise set, effectively blocking the derivation of the confirmed generalization $\forall x(\neg Px \vee Qx)$.

In $\mathbf{SIA^r}$ the logic $\mathbf{LI^r}$ is applied first in the sequence in (9). Alternatively, a logic could be defined which applies $\mathbf{FA^r}$ in the first step of the sequence. In the absence of quantifiers in the premise set, both approaches – 'inductive generalization first' vs. 'factual abduction first' – would lead to the same consequence set, since we need generalizations (and so we need to apply $\mathbf{LI^r}$) before we can abduce further facts. If generalizations are already present in the premises, the two approaches may lead to a different set of consequences, since in this case a generalization step may be incompatible with an abductive step at the beginning of the sequence. The premise set may contain, for instance, Pa, Ra, and $\overset{*}{\forall} x(\neg Qx \vee Px)$ amongst its **CL**-consequences, so that the i-abnormality $\neg\forall x(\neg Rx \vee \neg Qx)$ is true if Qa holds, while the a-abnormality $\overset{*}{\forall} x(\neg Qx \vee Px) \wedge Pa \wedge \neg Qa$ is true if $\neg Qa$ holds. An 'inductive generalization first' approach then prefers the falsity of $\neg\forall x(\neg Rx \vee \neg Qx)$ (and the truth of $\neg Qa$) while a 'factual abduction first' approach prefers the falsity of $\overset{*}{\forall} x(\neg Qx \vee Px) \wedge Pa \wedge \neg Qa$ (and the truth of Qa).

There is a 'chicken or egg' reason in favor of the 'inductive generalization first' approach. Inductive generalization has priority over factual abduction in the sense that we need to generalize before we can even start abducing (every application of factual abduction requires a generalization among its premises). A 'factual abduction first' approach would require some explanation as to how generalizations are attained prior to abduction, if not by inductive generalization. No such explanation is required in an 'inductive generalization first' approach of the kind adopted here.

The logic $\mathbf{SIA^r}$ is instructive in explicating what a combination of the inference patterns of inductive generalization and factual abduction could (and could not) look like. It was used to show how these patterns of ampliative reasoning can be fruitfully combined to infer new predictions and generalizations, and how they can shed light on a different pattern, law-abduction. Still, it is too early to make bold claims regarding the adequacy of $\mathbf{SIA^r}$ in capturing these patterns, for at least two reasons. First, there are many alternative ways to model these ampliative inferences. And second, a fully adequate model requires additional expressive resources.

In [6] Batens considers a number of alternative ways of modeling inductive generalization via an adaptive logic. Various roads for variation are open here. A first is to change the adaptive strategy.[8] A second is to vary the set of abnormalities. Instead of taking negated generalizations such as $\neg\forall x(Px \vee Qx)$ as members of Ω_i, one could take, for example, conjunctions of instances and counterinstances of a generalization, such as $\exists x(Px \vee Qx) \wedge \exists x \neg(Px \vee Qx)$. As shown in [6], this gives rise to a slightly different logic. A third, unexplored, road for variation is to change the lower limit logic from **CL** to some non-classical logic. More complex variations still can be obtained by coupling these roads, or even by moving to a combined adaptive logic for inductive generalization – see [6] for some examples.

The inference pattern of factual abduction too can be modeled in a variety of ways. The technical issues discussed in in Examples 1 and 2 can be avoided by means other than the restriction of applications of (BMP) to starred generalizations.[9] More generally, a richer framework with more expressive power is required for suitably representing factual abduction. Inferred explanations do not generally have the same epistemic status as observations in our premise set, and generalizations used as premises in an application of factual abduction often have a law-like status which separates them from mere regularities in the explanatory framework. These distinctions are too subtle to make in the first-order language used in this paper. One of the main open research questions for the present investigation is how we can enrich this formal language with additional expressive resources while preserving the fruitful sequential application of inductive generalization and factual abduction.

Acknowledgments Research for this article was sponsored by a Humboldt fellowship for postdoctoral researchers, and by a Sofja Kovalevskaja award, both granted by the Alexander von Humboldt Foundation, funded by the German Ministry for Education and Research. I am indebted to Frederik Van De Putte, Diderik Batens, and two anonymous referees for valuable comments on an earlier draft of this article.

[8] Two strategies are currently defined within the standard format for adaptive logics: reliability and minimal abnormality. Using the minimal abnormality strategy for sequential combinations of adaptive logics has the disadvantage that semantic adequacy results as in (11) are not guaranteed – see [24, Sec. 3.3.3] for the details.

[9] As mentioned in Section 1, $\mathbf{FA^r}$ is closely related to the system $\mathbf{AAL^r}$ defined in [7]. In the latter logic, the technical issues discussed in Examples 1 and 2 are likewise avoided by admitting only a restricted set of generalizations as candidate premises for abductive inference: $\mathbf{FA^r}$ admits only 'starred' generalizations, while $\mathbf{AAL^r}$ admits only universally quantified conditionals the antecedent [consequent] of which has a restricted conjunctive [disjunctive] normal form. In $\mathbf{FA^r}$ conclusions of abductive inferences are members of $\mathcal{L}^\alpha$ for some $\alpha \in \mathcal{C}$. In $\mathbf{AAL^r}$ conclusions of abductive inferences are formulas of the form $\pi(\alpha) \rhd \pi'(\alpha)$ where $\pi(\alpha), \pi'(\alpha) \in \mathcal{L}^\alpha$ for some $\alpha \in \mathcal{C}$. $\pi(\alpha) \rhd \pi'(\alpha)$ denotes that $\pi(\alpha)$ is a 'potential explanation' for $\pi'(\alpha)$.

References

1. A. Aliseda. *Abductive Reasoning. Logical Investigations into Discovery and Explanations.* Springer, Synthese Library Vol. 330, 2006.
2. D. Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.
3. D. Batens. A general characterization of adaptive logics. *Logique & Analyse*, 173–175:45–68, 2001.
4. D. Batens. On a logic of induction. *Logic and Philosophy of Science*, 4(1):3–32, 2006.
5. D. Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.
6. D. Batens. Logics for qualitative inductive generalization. *Studia Logica*, 97:61–80, 2011.
7. D. Batens. Abduction logics illustrating pitfalls of defeasible methods. In R. Urbaniak and G. Payette, editors, *Applications of Formal Philosophy. The Road Less Travelled*. Springer International Publishing, In print.
8. D. Batens and L. Haesaert. On classical adaptive logics of induction. *Logique et Analyse*, 173-175:255–290, 2001 (appeared 2003).
9. M. Beirlaen and A. Aliseda. A conditional logic for abduction. *Synthese*, 191(15):3733–3758, 2014.
10. H. Douglas. Reintroducing prediction to explanation. *Philosophy of Science*, 76:444–463, 2009.
11. P.A. Flach and A.C. Kakas. Abductive and inductive reasoning: background and issues. In P.A. Flach and A.C. Kakas, editors, *Abduction and Induction: Essays on their Relation and Integration*, pages 1–27. Springer, 2000.
12. T. Gauderis and F. Van De Putte. Abduction of generalizations. *Theoria*, 75:345–363, 2012.
13. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.
14. H. Lycke. A formal explication of the search for explanations: the adaptive logics approach to abductive reasoning. *Logic Journal of IGPL*, 20(2):497–516, 2012.
15. D. Makinson. *Bridges from Classical to Nonmonotonic Logic.* King's College Publications, London, 2005.
16. J. Meheus and D. Batens. A formal logic for abductive reasoning. *Logic Journal of The IGPL*, 14:221–236, 2006.
17. J. Norton. A little survey of induction. In P. Achinstein, editor, *Scientific Evidence*, pages 9–34. Baltimore: John Hopkins University Press, 2005.
18. S. Odintsov and S. Speranski. Computability issues for adaptive logics in multi-consequence standard format. *Studia Logica*, 101(6):1237–1262, 2013.
19. G. Schurz. Patterns of abduction. *Synthese*, 164:201–234, 2008.
20. Y. Shoham. A semantical approach to nonmonotonic logics. In Matthew L. Ginsberg, editor, *Readings in Nonmonotonic Reasoning*, pages 227–250. Morgan Kaufmann Publishers, 1987.
21. C. Straßer. *Adaptive Logics for Defeasible Reasoning.* Trends in Logic Vol. 38, Springer, 2014.
22. C. Straßer and F. Van De Putte. Proof theories for superpositions of adaptive logics. *Logique & Analyse*, 58, 2015.
23. P. Thagard. *Computational Philosophy of Science.* MIT Press, 1988.

24. F. Van De Putte. *Generic Formats for Prioritized Adaptive Logics. With Applications in Deontic Logic, Abduction and Belief Revision.* PhD thesis, Ghent University, 2012.
25. F. Van De Putte and C. Straßer. Three formats of prioritized adaptive logics: a comparative study. *Logic Journal of the IGPL*, 21:127–159, 2013.
26. F. Van De Putte and C. Straßer. Adaptive logics: a parametric approach. *Logic Journal of the IGPL*, 22(6):905–932, 2014.