Reasoning for A Fuzzy Description Logic with Comparison Expressions *

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Abstract

The fuzzy extensions of description logics support representation and reasoning for fuzzy knowledge. But the current fuzzy description logics do not support the expression of comparisons between fuzzy membership degrees. The paper proposes $\mathcal{ALC}_{fc}$, a fuzzy extension of description logic $\mathcal{ALC}$ with comparison expressions. $\mathcal{ALC}_{fc}$ defines comparison cut concepts to express complex comparisons, and integrate them with fuzzy $\mathcal{ALC}$. The challenges of reasoning in $\mathcal{ALC}_{fc}$ is discussed, and a tableau algorithm is presented. It enables representation and reasoning for expressive fuzzy knowledge about comparisons.

1 Introduction

Description logics (DLs) [1] are widely used in the semantic web. Fuzzy extensions of description logics import the fuzzy theory to enable the capability of dealing with fuzzy knowledge [5]. In fuzzy DLs, an individual is an instance of a fuzzy concept only to a certain degree (called *fuzzy membership degree*); e.g. "Tom is tall to a degree greater than 0.8," where *Tall* is a fuzzy concept. It is also a familiar description that "Tom is taller than Mike," which can be seemed as a comparison between two degrees without any restrictions on their exact

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values. Such comparison is important in many fuzzy information systems. For example, people are often interested in "the cheapest goods with the highest quality," disregarding the exact prices and qualities. There are even more complex comparison expressions, e.g. "no close friend of Tom is taller or stronger than him." However, the current fuzzy DLs do not support the expression of comparisons between fuzzy membership degrees. This paper defines comparison cut concepts to express complex comparisons, then presents $\mathcal{ALC}_{fc}$, and provides its reasoning algorithm. It enables representation and reasoning for expressive comparisons between fuzzy membership degrees.

2 Comparison expressions

In DLs, an assertion $a : C$ is either true or false (0 or 1). But in fuzzy DLs, any assertion is true only to a certain degree in $[0, 1]$. A *fuzzy assertion* is of the form $\langle a : C \bowtie n\rangle$, where $n \in [0, 1]$ is a constant and $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$. In order to express a comparison that "Tom is taller than Mike," the most intuitionistic solution is $\langle \mathit{Tom} : \mathit{Tall} > \mathit{Mike} : \mathit{Tall}\rangle$. However, such fuzzy assertion is not allowed in current fuzzy DLs [3, 5].

There are more complex comparisons, like "Tom is either taller *or* stronger than Mike." We define *comparison cut concepts* (or *cuts* for short) to express such complex comparisons. The idea is as follows: individuals can be divided into different classes based on basic comparisons; the classes can be represented in basic cuts; and we can use constructors with such cuts to express more complex comparisons. Three kinds of basic cuts are defined:

numerical cuts $[C \bowtie n]$ represents a set of individuals s such that $\langle s : C \bowtie n\rangle$. For example, $[\mathit{Tall} < 0.9]$ refers to any person that is not very tall.

comparative cuts $[C \bowtie D]$ represents s such that $\langle s : C \bowtie s : D\rangle$. $[\mathit{Absolutist} < \mathit{Liberalist}]$ refers to people who prefer liberalism to absolutism.

relative cuts $[C \bowtie D^{\uparrow}]$ represents any s such that $\langle s : C \bowtie t : D\rangle$ with respect to an individual t. $[C \bowtie]$ is an abbreviation of $[C \bowtie C^{\uparrow}]$. They describe the comparisons between individuals.

The idea of cuts is not completely innovate. [3] and [6] defined concepts similar to the numerical cuts. But they do not consider other cuts.

Complex cuts are constructed from basic cuts: if P, Q are cuts, then $\neg P$, $P \sqcap Q$ and $P \sqcup Q$ are also cuts. More expressive fuzzy concepts can be defined by importing cuts: for any cuts P, Q and a fuzzy role R, $\exists R.P$ and $\forall R.P$ are fuzzy concepts. For example, $\langle \mathit{Tom} : \forall \mathit{friend}.[\mathit{Tall} \leq] \geq 0.8\rangle$ means Tom is tallest among his close friends. Such fuzzy concepts can be used in cuts again. We use $\langle a : P(b)\rangle$ to assert that a is in a cut P with respect to b. Then $\langle \mathit{Tom} : \mathit{Tall} \geq 0.8\rangle$ and $\langle \mathit{Tom} : \mathit{Tall} > \mathit{Mike} : \mathit{Tall}\rangle$ can be seemed as aliases of $\langle \mathit{Tom} : [\mathit{Tall} \geq 0.8]\rangle$ and $\langle \mathit{Tom} : [\mathit{Tall} >](\mathit{Mike})\rangle$.

3 The fuzzy description logic $\mathcal{ALC}_{fc}$

Definition 1. *Let N_I, N_C and N_R be three disjoint sets of names of individuals, fuzzy concepts and fuzzy roles respectively. $\mathcal{ALC}_{fc}$-*concepts *are defined as*

- *$\top$, $\bot$ and A are concepts, where $A \in N_C$;*
- *if C, D are concepts, $R \in N_R$, and P is a cut, then $\neg C$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$, $\forall R.C$, $\exists R.P$ and $\forall R.P$ are concepts.*

A fuzzy interpretation *$\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ consists a nonempty set $\Delta^{\mathcal{I}}$, and a function $\cdot^{\mathcal{I}}$ maps every $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, maps every $A \in N_C$ to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, and maps every $R \in N_R$ to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. For $\mathcal{ALC}_{fc}$, $\cdot^{\mathcal{I}}$ also maps every concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ such that for any $s \in \Delta^{\mathcal{I}}$,*

$$\begin{aligned}
&\top^{\mathcal{I}}(s) = 1; \bot^{\mathcal{I}}(s) = 0; && (\exists R.C)^{\mathcal{I}}(s) = \sup_{t \in \Delta^{\mathcal{I}}}\{\min(R^{\mathcal{I}}(s,t), C^{\mathcal{I}}(t))\};\\
&(\neg C)^{\mathcal{I}}(s) = 1 - C^{\mathcal{I}}(s); && (\forall R.C)^{\mathcal{I}}(s) = \inf_{t \in \Delta^{\mathcal{I}}}\{\max(1 - R^{\mathcal{I}}(s,t), C^{\mathcal{I}}(t))\};\\
&(C \sqcap D)^{\mathcal{I}}(s) = \max(C^{\mathcal{I}}(s), D^{\mathcal{I}}(s)); && (\exists R.P)^{\mathcal{I}}(s) = \sup_{t \in P^{\mathcal{I}}(s)} R^{\mathcal{I}}(s,t);\\
&(C \sqcup D)^{\mathcal{I}}(s) = \min(C^{\mathcal{I}}(s), D^{\mathcal{I}}(s)); && (\forall R.P)^{\mathcal{I}}(s) = \inf_{t \in (\neg P)^{\mathcal{I}}(s)}(1 - R^{\mathcal{I}}(s,t)).
\end{aligned}$$

Definition 2. *The* comparison cut concepts *are defined as:*

- *if C, D are concepts, $n \in [0,1]$ and $\bowtie \in \{=, \neq, >, \geq, <, \leq\}$, then $[C \bowtie n]$, $[C \bowtie D]$ and $[C \bowtie D^{\uparrow}]$ are cuts;*
- *if P, Q are cuts, then $\neg P$, $P \sqcap Q$ and $P \sqcup Q$ are cuts.*

The interpretation function $\cdot^{\mathcal{I}}$ of a fuzzy interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ maps, additionally, every cut P into a function $P^{\mathcal{I}} : \Delta^{\mathcal{I}} \to 2^{\Delta^{\mathcal{I}}}$:

$$\begin{aligned}
&[C \bowtie n]^{\mathcal{I}}(s) = \{t | C^{\mathcal{I}}(t) \bowtie n\}; && (\neg P)^{\mathcal{I}}(s) = \Delta^{\mathcal{I}} \backslash P^{\mathcal{I}}(s);\\
&[C \bowtie D]^{\mathcal{I}}(s) = \{t | C^{\mathcal{I}}(t) \bowtie D^{\mathcal{I}}(t)\}; && (P \sqcap Q)^{\mathcal{I}}(s) = P^{\mathcal{I}}(s) \cap Q^{\mathcal{I}}(s);\\
&[C \bowtie D^{\uparrow}]^{\mathcal{I}}_{\bowtie}(s) = \{t | C^{\mathcal{I}}(t) \bowtie D^{\mathcal{I}}(s)\}; && (P \sqcup Q)^{\mathcal{I}}(s) = P^{\mathcal{I}}(s) \cup Q^{\mathcal{I}}(s).
\end{aligned}$$

For any cut P and $a \in N_I$, $P(a)$ is called an absolute cut, and $(P(a))^{\mathcal{I}} = P^{\mathcal{I}}(a^{\mathcal{I}})$. If a cut P contains no $\uparrow$, then P itself is an absolute cut, and for any a, we do not distinguish $P(a)$ and P, since they are equivalent in semantics.

Definition 3. *An $\mathcal{ALC}_{fc}$* knowledge base *is composed of a TBox and an ABox:*

A TBox *is a finite set of axioms of the form $C \sqsubseteq D$ or $C \sqsubset D$. An interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ or $C \sqsubset D$ iff for any $s \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(s) \leq D^{\mathcal{I}}(s)$ or $C^{\mathcal{I}}(s) < D^{\mathcal{I}}(s)$. $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ iff $\mathcal{I}$ satisfies every axiom in $\mathcal{T}$.*

An ABox *is a finite set of assertions of the form $\langle (a,b) : R \bowtie n \rangle$ or $\langle a : P(b) \rangle$, where $n \in [0,1]$, C is a concept, R is a role, $a, b \in N_I$, and P is a cut. An interpretation $\mathcal{I}$ satisfies $\langle (a,b) : R \bowtie n \rangle$ iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie n$, satisfies $\langle a : P(b) \rangle$ iff $a^{\mathcal{I}} \in P^{\mathcal{I}}(b^{\mathcal{I}})$. $\mathcal{I}$ is a model of an ABox $\mathcal{A}$ iff $\mathcal{I}$ satisfies every assertion in $\mathcal{A}$.*

The basic inference problem of $\mathcal{ALC}_{fc}$ is consistency of ABoxes*: an ABox $\mathcal{A}$ is consistent w.r.t. a TBox $\mathcal{T}$, iff there exists a model $\mathcal{I}$ of both $\mathcal{A}$ and $\mathcal{T}$.*

4 Reasoning issues

4.1 Challenges

In the current reasoning algorithms for fuzzy DLs [4, 5], they always assume that for an element s in a model $\mathcal{I}$ with $(\exists R.C)^{\mathcal{I}}(s) \geq n$, there is t such that $\min(R^{\mathcal{I}}(s,t), C^{\mathcal{I}}(t)) \geq n$. In [2], the author pointed out that this is an unessential error. However, it yields a serious problem in $\mathcal{ALC}_{fc}$.

Example 4. *Let* $\mathcal{A} = \{\langle a : \exists R.C \geq 1\rangle, \langle a : \exists R.[c \geq 1] \leq 0\rangle\}$. *For any element* s *in a model* $\mathcal{I}$ *with* $(\exists R.[c \geq 1])^{\mathcal{I}}(s) \leq 0$, *there cannot be any* t *such that* $\min(R^{\mathcal{I}}(s,t), C^{\mathcal{I}}(t)) \geq 1$. *Nevertheless,* $\mathcal{A}$ *is consistent: there can be infinite elements* $t_1, t_2, \ldots$ *such that* $\sup_{i=1,2,\ldots}\{\min(R^{\mathcal{I}}(s,t_i), C^{\mathcal{I}}(t_i))\} = 1$, *and for any* $i = 1, 2, \ldots$, $\min(R^{\mathcal{I}}(s,t_i), C^{\mathcal{I}}(t_i)) < 1$.

Another challenge is that current algorithms only consider a finite set of constants in $[0, 1]$; but there can be infinite different degrees in $\mathcal{ALC}_{fc}$.

Example 5. *Assume* $\mathcal{T} = \{C \sqsubset \exists R.C\}$, $\mathcal{A} = \{s_0 : C > n\}$. *In a model* $\mathcal{I}$ *of* $\mathcal{T}$ *and* $\mathcal{A}$, *it holds* $n < C^{\mathcal{I}}(s_0) < (\exists R.C)^{\mathcal{I}}(s_0)$. *For any element* s_i, $C^{\mathcal{I}}(s_i) < (\exists R.C)^{\mathcal{I}}(s_i)$ *and there is* s_{i+1} *with* $C^{\mathcal{I}}(s_{i+1}) > C^{\mathcal{I}}(s_i)$. *It also follows* $C^{\mathcal{I}}(s_{i+1}) < (\exists R.C)^{\mathcal{I}}(s_{i+1})$. *So there must be an infinite path of elements* $s_0, s_1, s_2, \ldots$ *such that* $n < C^{\mathcal{I}}(s_0) < C^{\mathcal{I}}(s_1) < C^{\mathcal{I}}(s_2) < \cdots$.

Because of the above challenges, the current reasoning algorithms for fuzzy DLs are not capable for $\mathcal{ALC}_{fc}$. The following parts firstly define a novel fuzzy tableau for $\mathcal{ALC}_{fc}$. The essential difference from the tableau in [4] is **P4** and **P6**. Then presents a tableau algorithm to overcome the challenges.

Definition 6. *Let* $\mathcal{A}$ *be an ABox,* $\mathcal{T}$ *be a TBox, a* fuzzy tableau $\boldsymbol{T}$ *of* $\mathcal{A}$ *w.r.t.* $\mathcal{T}$ *is a quadruple* $\boldsymbol{T} = \langle \mathbf{S}, \mathcal{L}, \mathcal{E}, \mathcal{V}\rangle$ *such that*

- $\mathbf{S}$ *is a non-empty set of elements;*
- $\mathcal{L} : \mathbf{S} \to \mathbf{AC}$ *maps each element of* $\mathbf{S}$ *to a set of absolute cuts;*
- $\mathcal{E} : N_R \times [0, 1] \to 2^{\mathbf{S}\times\mathbf{S}}$ *maps each pair of a role name and a number in* $[0, 1]$ *to a set of pairs of elements;*
- $\mathcal{V} : I_{\mathcal{A}} \to \mathbf{S}$ *maps individual names occurring in* $\mathcal{A}$ *to elements in* $\mathbf{S}$.

P1 *If* $[\neg C = n] \in \mathcal{L}(s)$, *then* $[C = 1 - n] \in \mathcal{L}(s)$.
P2 *If* $[C \sqcap D = n] \in \mathcal{L}(s)$, *then* $\{[C = m_1], [D = m_2]\} \subseteq \mathcal{L}(s)$ *and* $\min(m_1, m_2) = n$.
P3 *If* $[C \sqcup D = n] \in \mathcal{L}(s)$, *then* $[\neg C \sqcap \neg D = 1 - n] \in \mathcal{L}(s)$
P4 *If* $[\exists R.C = n] \in \mathcal{L}(s)$, *then for any* t *with* $(s,t) \in \mathcal{E}(R, m_1)$ *and* $m_1 > n$, *it holds* $[C = m_2] \in \mathcal{L}(t)$ *and* $m_2 \leq n$, *and there are two possible cases:*

- *there is* $t \in \mathbf{S}$ *with* $(s,t) \in \mathcal{E}(R, m_1)$, $[C = m_2] \in \mathcal{L}(t)$, $\min(m_1, m_2) = n$;
- *there are* $t_1, t_2, \cdots \in \mathbf{S}$ *with for any* $i = 1, 2, \ldots$, $(s, t_i) \in \mathcal{E}(R, n_i)$, $[C = m_i] \in \mathcal{L}(t_i)$, $\min(n_i, m_i) < n$, *and* $\sup_{i=1,2,\ldots}\{\min(n_i, m_i)\} = n$.

P5 *If* $[\forall R.C = n] \in \mathcal{L}(s)$*, then* $[\exists R.\neg C = 1 - n] \in \mathcal{L}(s)$*.*
P6 *If* $[\exists R.P = n] \in \mathcal{L}(s)$*, then for any* t *with* $(s,t) \in \mathcal{E}(R,m)$ *that* $m > n$*, it holds* $\neg P(s) \in \mathcal{L}(t)$*, and there are two possible cases:*

- *there is* $t \in \mathbf{S}$ *with* $(s,t) \in \mathcal{E}(R,n)$*,* $P(s) \in \mathcal{L}(t)$*;*
- *there are* $t_1, t_2, \cdots \in \mathbf{S}$ *with for any* $i = 1, 2, \ldots,$ $(s, t_i) \in \mathcal{E}(R, n_i)$*,* $P(s) \in \mathcal{L}(t_i)$*,* $n_i < n$*, and* $\sup_{i=1,2,\ldots}\{n_i\} = n$*.*

P7 *If* $[\forall R.P = n] \in \mathcal{L}(s)$*, then* $[\exists R.\neg P = 1 - n] \in \mathcal{L}(s)$*.*
P8 *If* $[C \bowtie n] \in \mathcal{L}(s)$*, then* $[C = m] \in \mathcal{L}(s)$ *and* $m \bowtie n$*.*
P9 *If* $[C \bowtie D] \in \mathcal{L}(s)$*, then* $\{[C = m], [D = n]\} \subseteq \mathcal{L}(s)$ *and* $m \bowtie n$*.*
P10 *If* $[C \bowtie D^{\uparrow}](t) \in \mathcal{L}(s)$*, then* $[C = m] \in \mathcal{L}(s)$*,* $[D = n] \in \mathcal{L}(t)$ *and* $m \bowtie n$*.*
P11 *If* $(P \sqcap Q)(t) \in \mathcal{L}(s)$*, then* $\{P(t), Q(t)\} \subseteq \mathcal{L}(s)$*.*
P12 *If* $(P \sqcup Q)(t) \in \mathcal{L}(s)$*, then* $P(t) \in \mathcal{L}(s)$ *or* $Q(t) \in \mathcal{L}(s)$*.*
P13 *If* $\langle (a,b) : R \bowtie n \rangle \in \mathcal{A}$*, then* $(\mathcal{V}(a), \mathcal{V}(b)) \in \mathcal{E}(R,m)$ *and* $m \bowtie n$*.*
P14 *If* $\langle a : P(b) \rangle \in \mathcal{A}$*, then* $P(\mathcal{V}(b)) \in \mathcal{L}(\mathcal{V}(a))$*.*
P15 *If* $C \sqsubseteq D \in \mathcal{T}$*, then for any* $s \in \mathbf{S}$*,* $\{[C = m], [D = n]\} \subseteq \mathcal{L}(s)$ *and* $m \leq n$*.*
P16 *If* $C \sqsubset D \in \mathcal{T}$*, then for any* $s \in \mathbf{S}$*,* $\{[C = m], [D = n]\} \subseteq \mathcal{L}(s)$ *and* $m < n$*.*
P17 *If* $[c = n] \in \mathcal{L}(s)$*, then there is no* $[c = m] \in \mathcal{L}(s)$ *such that* $n \neq m$*.*
P18 *If* $(s,t) \in \mathcal{E}(R,n)$*, then there is no* $(s,t) \in \mathcal{E}(R,m)$ *such that* $n \neq m$*.*

All concepts and cuts are transformed into *negation normal form*, where $\neg$ can only occur in front of a concept name, by pushing negations inwards: $\neg[C \bowtie W] \rightarrow [C \bowtie^{\ominus} W]$; $\neg(X \sqcup Y) \rightarrow \neg X \sqcap \neg Y$; $\neg(X \sqcap Y) \rightarrow \neg X \sqcup \neg Y$; $\neg(\exists R.X) \rightarrow \forall R.\neg X$; $\neg(\forall R.X) \rightarrow \exists R.\neg X$; where $W \in \{n, D, D^{\uparrow}\}$, X, Y are concepts or cuts, and $>^{\ominus} = \leq$, $<^{\ominus} = \geq$, $\geq^{\ominus} = <$, $\leq^{\ominus} = >$. All elements satisfy $[\top = 1]$ and $[\bot = 0]$. So $\top, \bot, \neg P$ are not considered here. It is easy to prove that the existence of a fuzzy tableau is equivalent with the existence of a model: $\Delta^{\mathcal{I}} = \mathcal{S}$, $A^{\mathcal{I}}(s) = n$ iff $[A = n] \in \mathcal{L}(s)$, and $R^{\mathcal{I}}(s,t) = n$ iff $(s,t) \in \mathcal{E}(R,n)$.

Theorem 7. *An* $\mathcal{ALC}_{fc}$ *ABox* $\mathcal{A}$ *is consistent w.r.t. a TBox* $\mathcal{T}$*, iff there is a fuzzy tableau* $\boldsymbol{T}$ *of* $\mathcal{A}$ *w.r.t.* $\mathcal{T}$*.*

4.2 A tableau algorithm

Here presents an algorithm to decide the existence of a fuzzy tableau by constructing a completion graph. There are three key points in the algorithm:

1. All degrees such as $C(x)$ and $R(x,y)$ are not constants but variables. The relations between degrees are recorded in a set δ, which is a partially ordered set $(V, \leq, \neq)$. The set of degrees V can be easily mapping to $[0,1]$.
2. In **P4** and **P6**, there can be infinite elements. We just generate one node to represent them with a new relation $\lessdot$ added into δ by **R5** and **R8**.
3. It is clear that the construction of a completion graph may not terminate if $\mathcal{T} \neq \emptyset$. A cycle detection technique called *blocking* is employed to unravel a finite completion graph into an infinite fuzzy tableau.

Definition 8. *A* completion graph *is* $T = \langle S, E, L, \delta \rangle$, *where*

- S *is a set of nodes in the graph.*
- E *is a set of edges (pairs of nodes) in the graph.*
- L *is a function: for every node* $x \in S$, $L(x)$ *is a set of concepts or absolute cuts; for every edge* $(x, y) \in E$, $L(x, y)$ *is a set of roles.*
- δ *is a set of formulas of the form* $X \leq Y$, $X \neq Y$ *or* $X \lessdot Y$, *where* $X, Y ::= n|C(x)|R(x, y)|1 - X$ *such that* $n \in [0, 1]$, C *is a concept,* R *is a role,* $x, y \in S$, *and for any* X, $1 - (1 - X) = X$.

Several abbreviations are defined below:

$$X \leq_\delta Y =_{def} X \leq Y \in \delta, \text{ or } X \leq_\delta Y, Y \leq_\delta Z, \text{ or } 1 - Y \leq_\delta 1 - X$$
$$\min(X, Y) =_\delta Z =_{def} Z \leq_\delta X, Z \leq_\delta Y, W \leq_\delta Z \text{ for some } W \in \{X, Y\};$$
$$X \neq_\delta Y =_{def} X \neq Y \in \delta; \qquad X \lessdot_\delta Y =_{def} X \lessdot Y \in \delta;$$
$$X \geq_\delta Y =_{def} Y \leq_\delta X; \qquad X =_\delta Y =_{def} X \leq_\delta Y, Y \leq_\delta X;$$
$$X <_\delta Y =_{def} X \leq_\delta Y, X \neq_\delta Y; \quad X_\delta > Y =_{def} Y \leq_\delta X, X \neq_\delta Y.$$

The completion graph T of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$ initializes with: $S = \{a \in N_I | a \text{ occurs in } \mathcal{A}\}$; for any $\langle a : P(b) \rangle \in \mathcal{A}$, $P(b) \in L(a)$; for any $\langle (a, b) : R \bowtie n \rangle \in \mathcal{A}$, $R \in L(a, b)$ and $R \bowtie_\delta n$. Let $V_0 = \{v_1, v_2, \ldots, v_k\} = \{0, 1, 0.5\} \cup \{n \in [0, 1] | n \text{ or } 1 - n \text{ occurs in } \mathcal{A} \text{ or } \mathcal{T}\}$, where $0 = v_1 < v_2 < \cdots < v_k = 1$. For any $v_i < v_j$, let $v_i <_\delta v_j$.

Then the graph grows up by applying the *expansion rules* showed in Fig. 1. If a rule applied to x creates a new node y, then y is a *successor* of x. Let *descendant* be transitive closure of successor.

The $\lessdot_\delta$ relation is used to simulate the infinite supreme. For any $a \in N_I$, $\text{lev}(a) = 1$. If $\text{lev}(x) = i$, y is a successor of x by updating $\lessdot_\delta$, then $\text{lev}(y) = i+1$. For any X of the form $C(x)$, $1 - C(x)$, $R(y, x)$, or $1 - R(y, x)$, if $\text{lev}(x) = i$, then $X \in V_i$. If $X \lessdot_\delta Y$ and $Y \in V_i$, then for any $Z \in V_j$ such that $j \leq i$, $Z <_\delta Y \rightarrow Z <_\delta X$ and $Z >_\delta X \rightarrow Z \geq_\delta Y$. So $X \lessdot_\delta Y$ means X is greater than any $Z < Y$ such that $Z \in V_0 \cup V_1 \cup \cdots \cup V_i$ and $Y \in V_i$. It ensures that for any given constant ε, we can assign values to the variables in V such that $X - Y < \varepsilon$ without inducing any conflict.

Since there are variables, the blocking condition in $\mathcal{ALC}_{fc}$ is different from classical DLs. It has to consider the comparisons between degrees.

Definition 9. *For any* x, *let* $\delta(x) = \{X \bowtie Y | X \bowtie_\delta Y$, X, Y *are of the form* $C(x)$, $1 - C(x)$, *or* v_i. *A node* x *is* blocked *by* y, *iff* x *is an descendant of* y, *and* $\delta(x) = [x/y]\delta(y)$, *where* $[x/y]\delta(y)$ *means to replace any* y *in* $\delta(y)$ *by* x. *Then we call* y blocks x. *When* x *is blocked, all descendants of* x *is also blocked.*

No rules in Fig. 1 can be applied to blocked nodes. T is said to contain a *clash* if $\{X \neq_\delta Y, X =_\delta Y\} \subseteq \delta$, or $X >_\delta 1$, or $X <_\delta 0$. T is said to be *clash-free*

R1	if	$\neg C \in L(x)$, and not $C(x) =_\delta 1 - (\neg C)(x)$
	then	$L(x) \rightarrow L(x) \cup \{C\}$, and $C(x) =_\delta 1 - (\neg C)(x)$
R2	if	$C \sqcap D \in L(x)$, and not $\min(C(x), D(x)) =_\delta (C \sqcap D)(x)$
	then	$L(x) \rightarrow L(x) \cup \{C, D\}$, and $\min(C(x), D(x)) =_\delta (C \sqcap D)(x)$
R3	if	$C \sqcup D \in L(x)$, and not $(C \sqcup D)(x) =_\delta 1 - (\neg C \sqcap \neg D)(x)$
	then	$L(x) \rightarrow L(x) \cup \{C \sqcup D\}$, and $(C \sqcup D)(x) =_\delta 1 - (\neg C \sqcap \neg D)(x)$
R4	if	$\exists R.C \in L(x)$, and there is y with $R \in L(x, y)$
		but not $X \leq_\delta (\exists R.C)(x)$ for some $X \in \{R(x, y), C(x)\}$
	then	$L(y) \rightarrow L(y) \cup \{C\}$, and $X \leq_\delta (\exists R.C)(x)$
R5	if	$\exists R.C \in L(x)$, and there is no y with $X =_\delta (\exists R.C)(x)$
		or $X \lessdot_\delta (\exists R.C)(x)$, for some $X \in \{R(x, y), C(x)\}$
	then	add a new node y with $L(x, y) = \{R\}, L(y) = \{C\}$,
		and $X =_\delta (\exists R.C)(x)$ or $X \lessdot_\delta (\exists R.C)(x)$
R6	if	$\forall R.C \in L(x)$, and not $(\forall R.C)(x)_\delta = 1 - (\exists R.\neg C)(x)$
	then	$L(x) \rightarrow L(x) \cup \{\exists R.\neg C\}$, and $(\forall R.C)(x) =_\delta 1 - (\exists R.\neg C)(x)$
R7	if	$\exists R.P \in L(x)$, and there is y with $R \in L(x, y)$
		but not $R(x, y)_\delta \leq (\exists R.C)(x)$ nor $\neg P(x)_\delta \in L(y)$
	then	$R(x, y) \leq_\delta (\exists R.C)(x)$, or $L(y) \rightarrow L(y) \cup \{\neg P(x)\}$
R8	if	$\exists R.P \in L(x)$, and there is no y with $P(x) \in L(y)$,
		$R(x, y) =_\delta (\exists R.C)(x)$ or $R(x, y) \lessdot_\delta (\exists R.C)(x)$
	then	add a new node y with $L(x, y) = \{R\}, L(y) = \{P(x)\}$,
		and $R(x, y) =_\delta (\exists R.C)(x)$ or $R(x, y) \lessdot_\delta (\exists R.C)(x)$
R9	if	$\forall R.P \in L(x)$, and not $(\forall R.P)(x) =_\delta 1 - (\exists R.\neg P)(x)$
	then	$L(x) \rightarrow L(x) \cup \{\exists R.\neg P\}$, and $(\forall R.P)(x) =_\delta 1 - (\exists R.\neg P)(x)$
R10	if	$[C \bowtie n] \in L(x)$, and not $C(x) \bowtie_\delta n$
	then	$L(x) \rightarrow L(x) \cup \{C\}$, and $C(x) \bowtie_\delta n$
R11	if	$[C \bowtie D] \in L(x)$, and not $C(x) \bowtie_\delta D(x)$
	then	$L(x) \rightarrow L(x) \cup \{C, D\}$, and $C(x) \bowtie_\delta D(x)$
R12	if	$[C \bowtie D^\uparrow](y) \in L(x)$, and not $C(x) \bowtie_\delta D(y)$
	then	$L(x) \rightarrow L(x) \cup \{C\}$, $L(y) \rightarrow L(y) \cup \{D\}$, and $C(x) \bowtie_\delta D(y)$
R13	if	$(P \sqcap Q)(y) \in L(x)$, and not $\{P(y), Q(y)\} \subseteq L(x)$
	then	$L(x) \rightarrow L(x) \cup \{P(y), Q(y)\}$
R14	if	$(P \sqcup Q)(y) \in L(x)$, and $\{P(y), Q(y)\} \cap L(x) = \emptyset$
	then	$L(x) \rightarrow L(x) \cup \{X\}$ for some $X \in \{P(y), Q(y)\}$
R15	if	$C \sqsubseteq D \in \mathcal{T}$, and there is x with no $C(x) \leq_\delta D(x)$
	then	$L(x) \rightarrow L(x) \cup \{C, D\}$, and $C(x) \leq_\delta D(x)$
R16	if	$C \sqsubset D \in \mathcal{T}$, and there is x with no $C(x) <_\delta D(x)$
	then	$L(x) \rightarrow L(x) \cup \{C, D\}$, and $C(x) <_\delta D(x)$
R17	if	$C \in L(x)$ or $R \in L(x, y)$, and let $X = C(x)$ or $R(x, y)$
		there is no i such that $v_i <_\delta X <_\delta v_{i+1}$, or $X =_\delta v_i$
	then	$v_i <_\delta X <_\delta v_{i+1}$ for some v_i, v_{i+1}, or $X =_\delta v_i$ for some v_i

Figure 1: Expansion rules for $\mathcal{ALC}_{fc}$

if it contains no clash. If none of the expansion rules can be applied to T, then T is said to be *complete*.

From the blocking condition and the number of concepts in any $L(x)$ is finite, the algorithm terminates. There is a fuzzy tableau $\boldsymbol{T}$ of $\mathcal{A}$ w.r.t. $\mathcal{T}$, iff a complete and clash-free completion graph can be constructed from $\mathcal{A}$ w.r.t. $\mathcal{T}$. So the above algorithm is a decision procedure for consistency of $\mathcal{ALC}_{fc}$ ABoxes w.r.t. empty TBoxes.

5 Conclusion

This paper presents $\mathcal{ALC}_{fc}$, a fuzzy extension of description logic $\mathcal{ALC}$ with comparison expressions. New challenges of reasoning within $\mathcal{ALC}_{fc}$ are discussed and a reasoning algorithm for $\mathcal{ALC}_{fc}$ is proposed to overcome the challenges. It enables representation and reasoning for expressive fuzzy knowledge about comparisons. The future work is to extend comparison expressions in more expressive fuzzy description logics and design their reasoning algorithms.

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