

# Detuning and dipole-dipole interaction effects on the entanglement of two qubits interacting with quantum fields of resonators

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## Abstract

We investigate the entanglement dynamics between two dipole-coupled qubits interacting with vacuum or thermal fields of lossless resonators. Double Jaynes-Cummings model and two-atom Jaynes-Cummings model are considered taking into account detuning and direct dipole-dipole interaction. Using the dressed-states technique we derive the exact solutions for models under consideration. The computer modeling of the time dependence of qubit-qubit negativity is carried out for different strength of the dipole-dipole interaction and detuning. Results show that dipole-dipole interaction and detuning may be used for entanglement operating and controlling.

*Keywords:* Entanglement, Superconducting qubits, Detuning, Dipole-dipole interaction, Vacuum field, Thermal field

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## 1. Introduction

Quantum computers are devices that store information on quantum variables such as spins, photons, and atoms, and that process that information by making those variables interact in a way that preserves quantum coherence. To perform a quantum computation, one must be able to prepare qubits in a desired initial state, coherently manipulate superpositions of a qubits two states, couple qubits together, measure their state, and keep them relatively free from interactions that induce noise and decoherence [1]. Qubits have been physically implemented in a variety of systems, including cavity quantum electrodynamics, superconducting qubits, atoms and ions in traps, quantum dots, spins and hybrid systems [2]. The connection between qubits can be arranged through their interaction with quantum fields of resonators. Basic protocols of quantum physics calculations are based on the use of entangled states [1]. Therefore, great efforts have been made to investigate entanglement characterization, entanglement control, and entanglement production in different systems. It is well known that the Jaynes-Cummings model (JCM) [3] is the simplest possible physical model that describes the interaction of a natural or artificial two-level atom (qubit) with a single-mode cavity [2], and has been used to understand a wide variety of phenomena in quantum optics and condensed matter systems, such as superconducting circuits, spins, quantum dots, atoms or ions in a cavity [2]. In order to explore a wider range of phenomena caused by the interaction of the qubits with the quantum fields in resonators the numerous generalizations of the JCM have been investigated in recent years (see references in [4]-[8]). Yöncü et al. [9] have proposed the so-called double JCM (DJCM), consisting of two two-level atoms and two resonator modes, provided that each atom interacts only with one field of the resonator, and investigated the pairwise entanglement dynamics of this model. Recently, the DJCM have been extensively investigated [10]-[17].

The direct dipole-dipole interaction between the qubits is the natural mechanism of entanglement producing and controlling. It's very important that the effective dipole-dipole interaction for superconducting Josephson qubits may be much greater than the coupling between the qubit and cavity field [18, 19]. The numerous references to the theoretical papers devoted to investigation of entanglement in two-qubit systems taking into account the dipole-dipole interaction are cited in our works [20]-[24]. In this paper, we considered two two-atom Jaynes-Cummings models taking into account the direct dipole-dipole interaction between qubits. We concerned our attention on two-atom double JCM and two-atom JCM with common cavity field. We investigated the entanglement between qubits, and

discussed the dependence of the entanglement on the parameters of the considered systems, such as the intensity of dipole-dipole interaction and the detuning between the atomic transition frequency and the cavity field frequencies.

## 2. Double Jaynes-Cummings model

In this section we consider two identical superconducting qubits labeled A and B, and two cavity modes of coplanar resonators labeled a and b. Qubit A not-resonantly interacts with a single-mode cavity field a, and qubit B not-resonantly interacts with a single-mode cavity field b. Due to the randomness of the qubits positions in the cavity, it is very difficult to control the couplings between different atom-cavity systems to be the same. Therefore the coupling constants between the atoms and cavities are assumed to be unequal. For superconducting qubits interacting with microwave coplanar resonators or LC superconducting circuits the intensity of effective dipole-dipole interaction can be compared with the atom-cavity coupling constant. In this case the dipole-dipole interaction should be included in the model Hamiltonian. Therefore the Hamiltonian for the system under rotating wave approximation can be written as

$$H = (\hbar\omega_0/2) \sigma_A^z + (\hbar\omega_0/2) \sigma_B^z + \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\gamma_a (\sigma_A^+ a + a^\dagger \sigma_A^-) + \gamma_b (\sigma_B^+ b + b^\dagger \sigma_B^-) + \hbar J (\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+), \quad (1)$$

where  $(1/2)\sigma_i^z$  is the inversion operator for the  $i$ th qubit ( $i = A, B$ ),  $\sigma_i^+ = |+\rangle_{ii}\langle-|$ , and  $\sigma_i^- = |-\rangle_{ii}\langle+|$  are the transition operators between the excited  $|+\rangle_i$  and the ground  $|-\rangle_i$  states in the  $i$ th qubit,  $a^\dagger$  and  $a$  are the creation and the annihilation operators of photons of the cavity mode a,  $b^\dagger$  and  $b$  are the creation and the annihilation operators of photons of the cavity mode b,  $\gamma_a$  is the coupling constant between qubit A and the cavity field a and  $\gamma_b$  is the coupling constant between qubit A and the cavity field a,  $\delta_a = \omega_a - \omega_0$  and  $\delta_b = \omega_b - \omega_0$  are the detunings for mode a and b and  $J$  is the coupling constant of the dipole interaction between the qubits A and B. Here  $\omega_0$  is the qubit frequency and  $\omega_a$  and  $\omega_b$  are the frequencies of the cavities modes.

Firstly we take two qubits initially in the Bell-like pure state of the following form

$$|\Psi(0)\rangle_A = \cos\theta|+, -\rangle + \sin\theta|-, +\rangle, \quad (2)$$

where  $0 \leq \theta \leq \pi$  and the cavity fields initially are in vacuum states  $|0, 0\rangle = |0\rangle \otimes |0\rangle$ . We take into account that optimal temperature at which the superconducting qubits are used for quantum computing is mK. For such temperature the influence of thermal photons of the microwave cavity field on the dynamics of qubits can be neglected.

Then the full initial state is

$$|\Psi(0)\rangle = (\cos\theta|+, -\rangle + \sin\theta|-, +\rangle) \otimes |0, 0\rangle. \quad (3)$$

The evolution of the system is confined in the subspace  $|-, -, 0, 1\rangle, |-, -, 1, 0\rangle, |-, +, 0, 0\rangle, |+, -, 0, 0\rangle$ . To obtain the time-dependent wave function of considered model one can use the so-called dressed states or eigenvectors of the Hamiltonian (1). We have obtained these for general case when parameters of the Hamiltonian (1) take the arbitrary values. But the general expressions for eigenvectors are too cumbersome to display here. Therefore, we present below the eigenvectors and eigenvalues of the Hamiltonian (1) for special case when  $\delta_a = -\delta_b = \delta$  and  $\gamma_a = \gamma_b = \gamma$ .

In this case the eigenvectors of the Hamiltonian (1) in a frame rotating with the qubit frequency  $\omega_0$  can be written as

$$|\Phi_i\rangle = \xi_i (X_{i1}|-, -, 0, 1\rangle + X_{i2}|-, -, 1, 0\rangle + X_{i3}|-, +, 0, 0\rangle + X_{i4}|+, -, 0, 0\rangle) \quad (i = 1, 2, 3, 4),$$

where

$$\xi_i = 1 / \sqrt{|X_{i1}|^2 + |X_{i2}|^2 + |X_{i3}|^2 + |X_{i4}|^2}$$

and

$$X_{11} = \frac{2\alpha}{\alpha^2 + \Delta^2 - B + \sqrt{2}\Delta\sqrt{A-B}}, \quad X_{12} = \frac{\sqrt{2}}{\Delta\sqrt{2} - \sqrt{A-B}}, \quad X_{13} = \frac{-\alpha^2 - \Delta^2 + B + \sqrt{2}\Delta\sqrt{A-B}}{\alpha(-2\Delta + \sqrt{2}\sqrt{A-B})}, \quad X_{14} = 1;$$

$$X_{21} = \frac{2\alpha}{\alpha^2 + \Delta^2 - B - \sqrt{2}\Delta\sqrt{A-B}}, \quad X_{22} = \frac{\sqrt{2}}{\Delta\sqrt{2} + \sqrt{A-B}}, \quad X_{23} = \frac{\alpha^2 + \Delta^2 - B + \sqrt{2}\Delta\sqrt{A-B}}{\alpha(2\Delta + \sqrt{2}\sqrt{A-B})}, \quad X_{24} = 1,$$

$$\begin{aligned}
 X_{31} &= \frac{2\alpha}{\alpha^2 + \Delta^2 + B + \sqrt{2}\Delta\sqrt{A+B}}, & X_{32} &= \frac{\sqrt{2}}{\Delta\sqrt{2} - \sqrt{A+B}}, & X_{33} &= \frac{\alpha^2 + \Delta^2 + B - \sqrt{2}\Delta\sqrt{A+B}}{2\alpha\Delta - \sqrt{2}\alpha\sqrt{A+B}}, & X_{34} &= 1, \\
 X_{41} &= \frac{2\alpha}{\alpha^2 + \Delta^2 + B - \sqrt{2}\Delta\sqrt{A+B}}, & X_{42} &= \frac{\sqrt{2}}{\Delta\sqrt{2} + \sqrt{A+B}}, & X_{43} &= \frac{\alpha^2 + \Delta^2 + B + \sqrt{2}\Delta\sqrt{A+B}}{\alpha(2\Delta + \sqrt{2}\sqrt{A+B})}, & X_{44} &= 1,
 \end{aligned}$$

where  $\Delta = \delta/\gamma$ ,  $\alpha = J/\gamma$  and  $A = 2 + \alpha^2 + \Delta^2$ ,  $B = \sqrt{\alpha^4 + 4\Delta^2 + \Delta^4 - 2\alpha^2(-2 + \Delta^2)}$ .

The corresponding eigenvalues are

$$E_1 = -\hbar\gamma\sqrt{A-B}/\sqrt{2}, \quad E_2 = \gamma\hbar\sqrt{A-B}/\sqrt{2}, \quad E_3 = -\hbar\gamma\sqrt{A+B}/\sqrt{2}, \quad E_4 = \hbar\gamma\sqrt{A+B}/\sqrt{2}.$$

For entanglement modeling we can obtain the time dependent wave function

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle. \quad (4)$$

Using the eigenvalues and eigenvectors of Hamiltonian (1) and the initial state (3) we can derive from (4)

$$|\Psi(t)\rangle = C_1^{(1)}(t)|-, -, 0, 1\rangle + C_2^{(1)}(t)|-, -, 1, 0\rangle + C_3^{(1)}(t)|-, +, 0, 0\rangle + C_4^{(1)}(t)|+, -, 0, 0\rangle, \quad (5)$$

where

$$\begin{aligned}
 C_1^{(1)} &= \cos\theta Z_{11} + \sin\theta Z_{12}, & C_2^{(1)} &= \cos\theta Z_{21} + \sin\theta Z_{22}, \\
 C_3^{(1)} &= \cos\theta Z_{31} + \sin\theta Z_{32}, & C_4^{(1)} &= \cos\theta Z_{41} + \sin\theta Z_{42}
 \end{aligned}$$

and

$$\begin{aligned}
 Z_{11} &= e^{-iE_1t/\hbar} \xi_1 Y_{41} X_{11} + e^{-iE_2t/\hbar} \xi_2 Y_{42} X_{21} + e^{-iE_3t/\hbar} \xi_3 Y_{4n} X_{31} + e^{-iE_4t/\hbar} \xi_4 Y_{44} X_{41}, \\
 Z_{12} &= e^{-iE_1t/\hbar} \xi_1 Y_{31} X_{11} + e^{-iE_2t/\hbar} \xi_2 Y_{3n} X_{21} + e^{-iE_3t/\hbar} \xi_3 Y_{33} X_{31} + e^{-iE_4t/\hbar} \xi_4 Y_{34} X_{41}, \\
 Z_{21} &= e^{-iE_1t/\hbar} \xi_1 Y_{41} X_{12} + e^{-iE_2t/\hbar} \xi_2 Y_{42} X_{22} + e^{-iE_3t/\hbar} \xi_3 Y_{43} X_{32} + e^{-iE_4t/\hbar} \xi_4 Y_{44} X_{42}, \\
 Z_{22} &= e^{-iE_1t/\hbar} \xi_1 Y_{31} X_{12} + e^{-iE_2t/\hbar} \xi_2 Y_{32} X_{22} + e^{-iE_3t/\hbar} \xi_3 Y_{33} X_{32} + e^{-iE_4t/\hbar} \xi_4 Y_{34} X_{42}, \\
 Z_{31} &= e^{-iE_1t/\hbar} \xi_1 Y_{41} X_{13} + e^{-iE_2t/\hbar} \xi_2 Y_{42} X_{23} + e^{-iE_3t/\hbar} \xi_3 Y_{43} X_{33} + e^{-iE_4t/\hbar} \xi_4 Y_{44} X_{43}, \\
 Z_{32} &= e^{-iE_1t/\hbar} \xi_1 Y_{31} X_{13} + e^{-iE_2t/\hbar} \xi_2 Y_{32} X_{23} + e^{-iE_3t/\hbar} \xi_3 Y_{33} X_{33} + e^{-iE_4t/\hbar} \xi_4 Y_{34} X_{43}, \\
 Z_{41} &= e^{-iE_1t/\hbar} \xi_1 Y_{41} X_{14} + e^{-iE_2t/\hbar} \xi_2 Y_{42} X_{24} + e^{-iE_3t/\hbar} \xi_3 Y_{43} X_{34} + e^{-iE_4t/\hbar} \xi_4 Y_{44} X_{44}, \\
 Z_{42} &= e^{-iE_1t/\hbar} \xi_1 Y_{31} X_{14} + e^{-iE_2t/\hbar} \xi_2 Y_{32} X_{24} + e^{-iE_3t/\hbar} \xi_3 Y_{33} X_{34} + e^{-iE_4t/\hbar} \xi_4 Y_{34} X_{44},
 \end{aligned}$$

where  $Y_{ij} = \xi_j X_{ji}^*$ .

We also can consider an another type of Bell-like pure initial state of two qubits

$$|\Psi(0)\rangle_A = \cos\theta|+, +\rangle + \sin\theta|-, -\rangle.$$

For this initial atomic state and vacuum cavities fields the full initial state of the system is

$$|\Psi(0)\rangle = (\cos\theta|+, +\rangle + \sin\theta|-, -\rangle) \otimes |0, 0\rangle. \quad (6)$$

For initial state (6) the time-dependent wave function can be written in the form

$$\begin{aligned}
 |\Psi(t)\rangle &= C_1^{(2)}(t)|+, +, 0, 0\rangle + C_2^{(2)}(t)|+, -, 0, 1\rangle + C_3^{(2)}(t)|-, +, 1, 0\rangle + C_4^{(2)}(t)|+, -, 1, 0\rangle + \\
 &+ C_5^{(2)}(t)|-, +, 0, 1\rangle + C_6^{(2)}(t)|-, -, 2, 0\rangle + C_7^{(2)}(t)|-, -, 0, 2\rangle + C_8^{(2)}(t)|-, -, 1, 1\rangle + C_9^{(2)}(t)|-, -, 0, 0\rangle. \quad (7)
 \end{aligned}$$

The coefficients  $C_i(t)$  may be obtained by using the way which is described in previous case. But these are too cumbersome. Therefore, we will use below the numerical results for coefficients under consideration.

For two-qubit system described by the reduced density operator  $\rho_A(t)$ , a measure of entanglement or negativity can be defined in terms of the negative eigenvalues  $\mu_i^-$  of partial transpose of the reduced atomic density matrix  $\rho_A^{T_1}$  [26, 27]

$$\varepsilon = -2 \sum \mu_i^- \quad (8)$$

Using the wave functions (5) or (7) one can obtain the density operator for the whole system as

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)| \quad (9)$$

Taking a partial trace over the field variable one can obtain from (9) the reduced atomic density operator in the two-qubit basis  $|+, +\rangle$ ,  $|+, -\rangle$ ,  $|-, +\rangle$ ,  $|-, -\rangle$  for initial state (3) in the form

$$\rho_A(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & V(t) & H(t) & 0 \\ 0 & H(t)^* & W(t) & 0 \\ 0 & 0 & 0 & R(t) \end{pmatrix} \quad (10)$$

The matrix elements of (10) are

$$V(t) = |C_4^{(1)}(t)|^2, \quad W(t) = |C_3^{(1)}(t)|^2, \quad R(t) = |C_1^{(1)}(t)|^2 + |C_2^{(1)}(t)|^2, \quad H(t) = C_4^{(1)}(t)C_3^{(1)}(t)^*.$$

The partial transpose of the reduced atomic density matrix (10) is

$$\rho_A^{T_1}(t) = \begin{pmatrix} 0 & 0 & 0 & H(t)^* \\ 0 & V(t) & 0 & 0 \\ 0 & 0 & W(t) & 0 \\ H(t) & 0 & 0 & R(t) \end{pmatrix} \quad (11)$$

From equation (11), we obtain four eigenvalues. Three of them are always positive. The eigenvalue  $\mu_4^- = 1/2(R - \sqrt{4H^2 + R^2})$  is always negative. As a result, the negativity can be written as

$$\varepsilon(t) = \sqrt{R(t)^2 + 4|H(t)|^2} - R(t) \quad (12)$$

The partial transpose of the reduced atomic density matrix  $\rho_A^{T_1}$  for initial state (6) has the form

$$\rho_A^{T_1}(t) = \begin{pmatrix} U_1(t) & 0 & 0 & \tilde{H}_1(t)^* \\ 0 & V_1(t) & H_1(t)^* & 0 \\ 0 & H_1(t) & W_1(t) & 0 \\ \tilde{H}_1(t) & 0 & 0 & R_1(t) \end{pmatrix} \quad (13)$$

where one can obtain with using (7)

$$U_1(t) = |C_1^{(2)}(t)|^2, \quad H_1(t) = C_1^{(2)}(t)C_9^{(2)}(t)^*, \quad \tilde{H}_1(t) = C_2^{(2)}(t)C_5^{(2)}(t)^* + C_4^{(2)}(t)C_3^{(2)}(t)^*,$$

$$V_1(t) = |C_2^{(2)}(t)|^2 + |C_4^{(2)}(t)|^2, \quad W_1(t) = |C_3^{(2)}(t)|^2 + |C_5^{(2)}(t)|^2, \quad R_1(t) = |C_6^{(2)}(t)|^2 + |C_7^{(2)}(t)|^2 + |C_8^{(2)}(t)|^2 + |C_9^{(2)}(t)|^2.$$

Two eigenvalues of matrix (13) may be negative. Then, the negativity can be written as a superposition of two terms

$$\varepsilon(t) = \sqrt{(U_1(t) - R_1(t))^2 + 4|\tilde{H}_1(t)|^2} - U_1(t) - R_1(t) + \sqrt{(V_1(t) - W_1(t))^2 + 4|H_1(t)|^2} - V_1(t) - W_1(t) \quad (14)$$

The first term is taken into account if and only if  $|\tilde{H}_1|^2 > U_1 R_1$  and the second term is taken into account if and only if  $|H_1|^2 > V_1 W_1$ .

### 3. Two-atom Jaynes-Cummings model

In this section we consider two-atom JCM with common thermal cavity field. We have two identical qubits A and B (spins, quantum dots etc.) non-resonantly interacting with common one-mode quantum electromagnetic field of resonator. As in a previous case we assume that the direct dipole-dipole interaction between qubits takes place. But in contrast with previous case we investigated the entanglement induced by a thermal field. In a frame rotating with the field frequency, the Hamiltonian for the system under rotating wave approximation can be written as

$$H = \hbar\delta\sigma_A^z + \hbar\delta\sigma_B^z + \hbar\gamma \sum_{i=A}^B (\sigma_i^+ a + a^+ \sigma_i^-) + \hbar J(\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+). \quad (15)$$

Here  $\sigma_A^z$  and  $\sigma_B^z$  are the inversion operators for qubit A and B respectively,  $\delta = \omega - \omega_0$  is detuning, where  $\omega$  is the cavity field frequency and  $\omega_0$  is the atom transition frequency. The other notations are similar to these used in Section 2. Let us note that concurrence dynamics for system with Hamiltonian (15) without dipole-dipole interaction has been earlier investigated by Zhang [25].

We consider two type of initial atomic states: separable state  $|+, -\rangle$  (or  $|-, +\rangle$ ) and entangled state (2). The initial cavity mode state are assumed to be the thermal one-mode state  $\rho_F(0) = \sum_n p_n |n\rangle\langle n|$ , where the weight functions are  $p_n = \bar{n}^n / (1 + \bar{n})^{n+1}$ . Here  $\bar{n}$  is the mean photon number in a cavity mode,  $\bar{n} = (\exp[\hbar\omega_i/k_B T] - 1)^{-1}$ ,  $k_B$  is the Boltzmann constant and  $T$  is the equilibrium cavity temperature.

Before considering the interaction between two qubits and thermal field, it is straightforward to first study two qubits simultaneously interacting with Fock state. Suppose that the excitation number of the atom-field system is  $n$  ( $n \geq 0$ ). The evolution of the system is confined in the subspace

$$|-, -, n+2\rangle, \quad |+, -, n+1\rangle, \quad |-, +, n+1\rangle, \quad |+, +, n\rangle.$$

On this basis, the eigenfunctions of the Hamiltonian (15) can be written as

$$|\Phi_{in}\rangle = \xi_{in}(X_{i1n}|-, -, n+2\rangle + X_{i2n}|+, -, n+1\rangle + X_{i3n}|-, +, n+1\rangle + X_{i4n}|+, +, n\rangle) \quad (i = 1, 2, 3, 4),$$

where

$$\begin{aligned} X_{11n} &= 0, & X_{12n} &= -1, & X_{13n} &= 1, & X_{14n} &= 0, \\ X_{i1n} &= -\frac{2\sqrt{1+n}\sqrt{2+n}}{4+2n+2\Delta+E_{in}-2\Delta E_{in}-E_{in}^2}, & X_{i2n} &= -\frac{\sqrt{1+n}(2\Delta+E_{in})}{4+2n+2\Delta+E_{in}-2\Delta E_{in}-E_{in}^2}, \\ X_{i3n} &= -\frac{\sqrt{1+n}(2\Delta+E_{in})}{4+2n+2\Delta+E_{in}-2\Delta E_{in}-E_{in}^2}, & X_{i4n} &= 1 \quad (i = 2, 3, 4). \end{aligned}$$

The corresponding eigenvalues are

$$\begin{aligned} E_{1n} &= -\hbar\gamma \alpha, & E_{2n} &= (1/3) \hbar\gamma (\alpha + A_n/B_n + B_n), \\ E_{3n} &= (1/6) \hbar\gamma \operatorname{Re} [2\alpha - (1+i\sqrt{3})A_n/B_n + i(i+\sqrt{3})B_n], \\ E_{4n} &= (1/6) \hbar\gamma \operatorname{Re} [2\alpha + i(i+\sqrt{3})A_n/B_n - (1+i\sqrt{3})B_n]. \end{aligned}$$

Here

$$\begin{aligned} A_n &= 18 + 12n + \alpha^2 + 12\Delta^2, \\ B_n &= \left( \alpha^3 - 54\Delta + 9\alpha(3+2n-4\Delta^2) + \frac{1}{2} \sqrt{-4(18+12n+\alpha^2+12\Delta^2)^3 + 4(\alpha^3-54\Delta+9\alpha(3+2n-4\Delta^2))^2} \right)^{1/3}. \end{aligned}$$

To derive the full dynamics of our model one can consider also the basis states  $|-, -, 1\rangle, |+, -, 0\rangle, |-, +, 0\rangle$ .

Assume that the system is initially prepared in the state  $|+, -, n\rangle$  ( $n \geq 0$ ), then at time  $t$ , the whole system will evolve to

$$|\Psi(t)\rangle = Z_{12,n}|-, -, n+2\rangle + Z_{22,n}|+, -, n+1\rangle + Z_{32,n}|-, +, n+1\rangle + Z_{42,n}|+, +, n\rangle. \quad (16)$$

Here

$$\begin{aligned}
 Z_{12,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{21n} X_{11n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{22n} X_{21n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{23n} X_{31n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{24n} X_{41n}, \\
 Z_{22,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{21n} X_{12n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{22n} X_{22n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{23n} X_{32n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{24n} X_{42n}, \\
 Z_{32,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{21n} X_{13n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{22n} X_{23n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{23n} X_{33n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{24n} X_{43n}, \\
 Z_{42,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{21n} X_{14n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{22n} X_{24n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{23n} X_{34n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{24n} X_{44n},
 \end{aligned}$$

where  $Y_{ijn} = \xi_{jn} X_{jin}^*$ .

If the initial state of our system is  $|+, -, 0\rangle$ , the time dependent wave function takes the form

$$|\Psi(t)\rangle = Z_{12}|-, -, 1\rangle + Z_{22}|+, -, 0\rangle + Z_{32}|-, +, 0\rangle, \quad (17)$$

where

$$\begin{aligned}
 Z_{12} &= -2ie^{-(\alpha-2\Delta)t/2} \sin(\Omega t/2)/\Omega, \quad Z_{22} = e^{-i(\alpha-2\Delta)t/2} \left( e^{i(3\alpha-2\Delta)t/2} + \Omega \cos(\Omega t/2) - 2t \sin(\Omega t/2) \right) / (2\Omega), \\
 Z_{32} &= e^{-i(\alpha-2\Delta)t/2} \left( -e^{i(3\alpha-2\Delta)t/2} + \Omega \cos(\Omega t/2) - 2t \sin(\Omega t/2) \right) / (2\Omega)
 \end{aligned}$$

and  $\Omega = \sqrt{8 + (\alpha + 2\Delta)^2}$ .

For initial state  $|-, +, n+1\rangle$  ( $n \geq 0$ ) the time-dependent wave function is

$$|\Psi(t)\rangle = Z_{13,n}|-, -, n+2\rangle + Z_{23,n}|+, -, n+1\rangle + Z_{33,n}|-, +, n+1\rangle + Z_{43,n}|+, +, n\rangle. \quad (18)$$

Here

$$\begin{aligned}
 Z_{13,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{31n} X_{11n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{32n} X_{21n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{33n} X_{31n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{34n} X_{41n}, \\
 Z_{23,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{31n} X_{12n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{32n} X_{22n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{33n} X_{32n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{34n} X_{42n}, \\
 Z_{32,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{31n} X_{13n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{32n} X_{23n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{33n} X_{33n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{34n} X_{43n}, \\
 Z_{42,n} &= e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{31n} X_{14n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{32n} X_{24n} + e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{33n} X_{34n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{34n} X_{44n}.
 \end{aligned}$$

If the initial state is  $|-, +, 0\rangle$ , the time dependent wave function takes the form

$$|\Psi(t)\rangle = Z_{13}|-, -, 1\rangle + Z_{23}|+, -, 0\rangle + Z_{33}|-, +, 0\rangle, \quad (19)$$

where  $Z_{13} = Z_{12}$ ,  $Z_{23} = Z_{22}$ ,  $Z_{33} = Z_{32}$ .

Now we go back to the theme of this Section. Using the equations (16)-(19) one can obtain the density operator for the whole system. Taking a partial trace over the field variables one can obtain the reduced atomic density operator and partial transpose of the reduced atomic density matrix  $\rho_A^{T_1}$ . For initial atomic state  $|+, -\rangle$  the partial transpose of the reduced atomic density operator has the form

$$\rho_A^{T_1}(t) = \begin{pmatrix} U_2(t) & 0 & 0 & H_2(t)^* \\ 0 & V_2(t) & 0 & 0 \\ 0 & 0 & W_2(t) & 0 \\ H_2(t) & 0 & 0 & R_2(t) \end{pmatrix}. \quad (20)$$

where

$$\begin{aligned}
 U_2(t) &= \sum_{n=0}^{\infty} p_n |Z_{42,n}(t)|^2, & V_2(t) &= \sum_{n=1}^{\infty} p_n |Z_{22,n-1}(t)|^2 + p_0 |Z_{22}(t)|^2, \\
 W_2(t) &= \sum_{n=1}^{\infty} p_n |Z_{32,n-1}(t)|^2 + p_0 |Z_{32}(t)|^2, & R_2(t) &= \sum_{n=1}^{\infty} p_n |Z_{12,n-1}(t)|^2 + p_0 |Z_{12}(t)|^2, \\
 H_2(t) &= \sum_{n=1}^{\infty} p_n Z_{22,n-1}(t) Z_{32,n-1}(t)^* + p_0 Z_{22}(t) Z_{32}(t)^*.
 \end{aligned} \tag{21}$$

Only one of the eigenvalues of matrix (20) may be negative. Therefore the negativity can be written in the form

$$\epsilon(t) = \sqrt{(|R_2(t)| - |U_2(t)|)^2 + 4|H_2(t)|^2} - |R_2(t)| - |U_2(t)|.$$

For initial atomic state  $|-, +\rangle$  the partial transpose of the reduced atomic density operator has the form (20). Its matrix elements may be obtained from (21) by replacing the coefficients  $Z_{i2n}$  with  $Z_{i3n}$ , where  $i = 1, 2, 3, 4$ . For entangled initial atomic state (2) the partial transpose of the reduced matrix also has the form (20). The elements of this matrix may be obtained by combining the elements of two partial transpose matrix for initial states  $|+, -\rangle$  and  $|-, +\rangle$ .

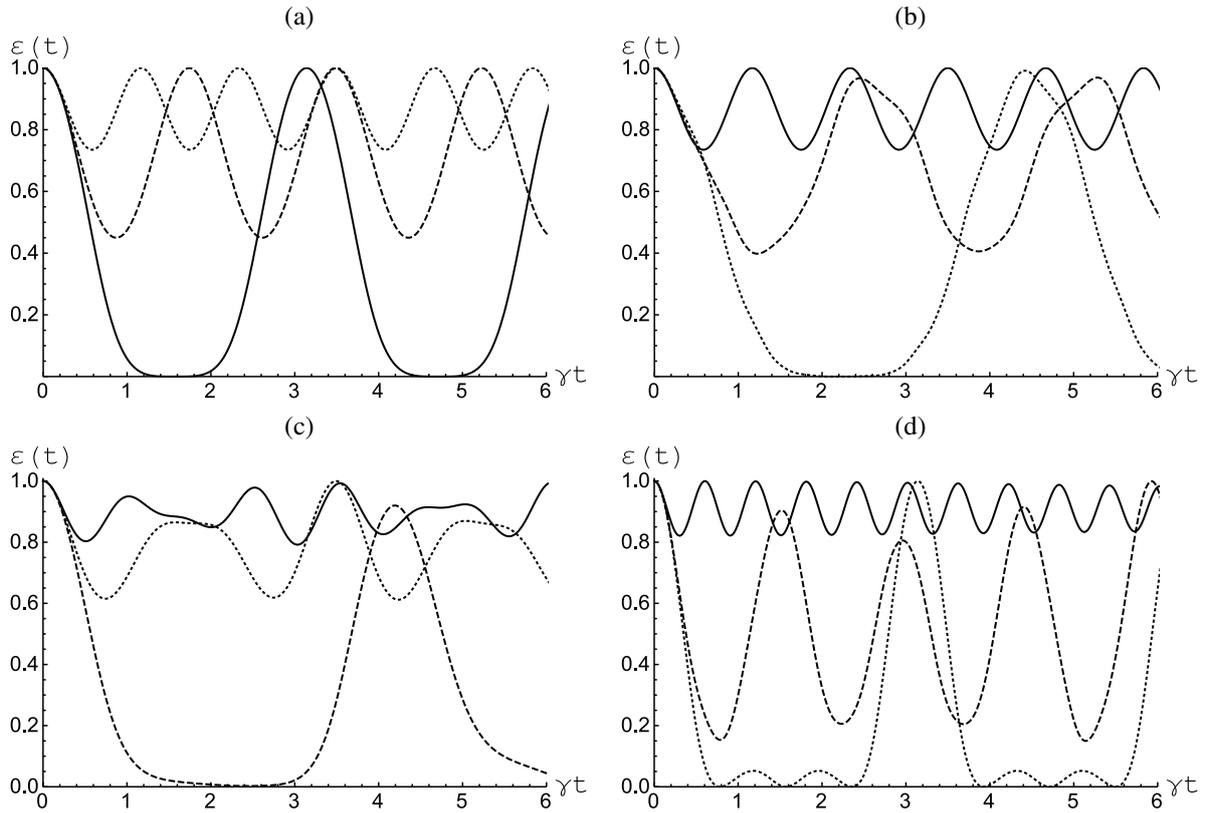


Figure 1: The negativity as a function of  $\gamma t$  for double JCM and initial state (3) with  $\theta = \pi/4$ . Parameters  $\delta_a = \delta_b = 0$ ,  $\gamma_b = \gamma_a$  (a),  $\delta_a = -\delta_b = 5$ ,  $\gamma_b = \gamma_a$  (b),  $\delta_a = \delta_b = 5$ ,  $\gamma_b = \gamma_a$  (c) and  $\delta_a = \delta_b = 0$ ,  $\gamma_a = 2\gamma_b$  (d). The strength of dipole interaction  $\alpha = 0$  (dotted),  $\alpha = 3$  (dashed) and  $\alpha = 5$  (solid).

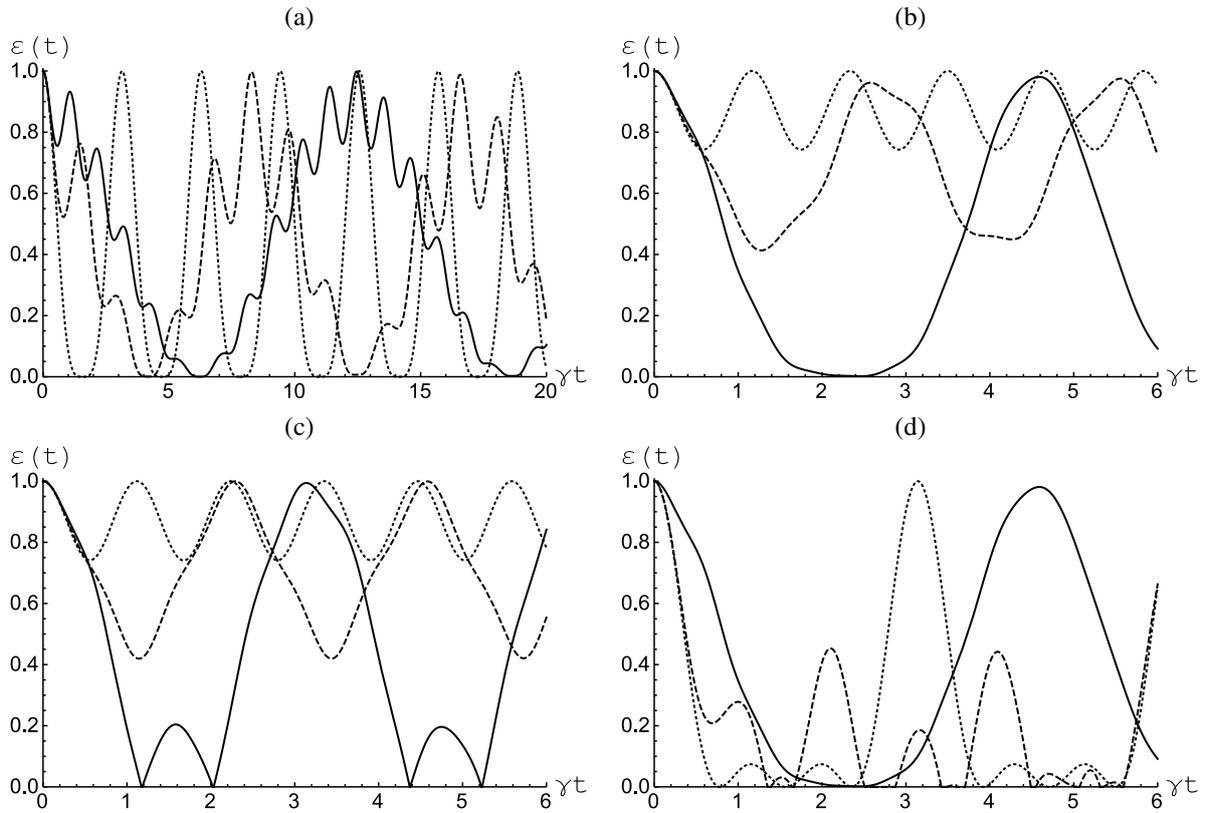


Figure 2: The negativity as a function of  $\gamma t$  for double JCM and initial state (6) with  $\theta = \pi/4$ . Parameters  $\delta_a = \delta_b = 0$ ,  $\gamma_b = \gamma_a$  (a),  $\delta_a = -\delta_b = 5$ ,  $\gamma_b = \gamma_a$  (b),  $\delta_a = \delta_b = 5$ ,  $\gamma_b = \gamma_a$  (c) and  $\delta_a = \delta_b = 0$ ,  $\gamma_a = 2\gamma_b$  (d). The strength of dipole interaction  $\alpha = 0$  (dotted),  $\alpha = 3$  (dashed) and  $\alpha = 5$  (solid).

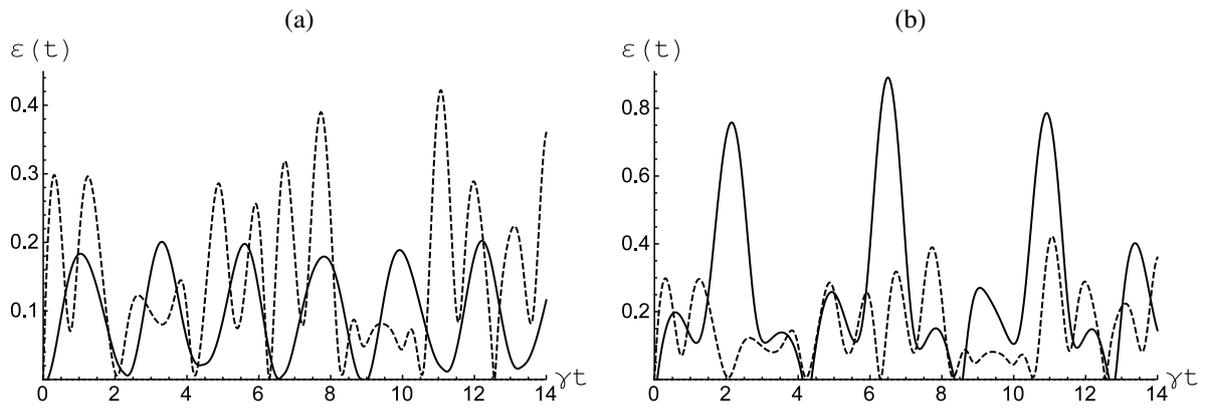


Figure 3: The negativity as a function of  $\gamma t$  for two-atom JCM with common field and initial atomic state  $|+, -\rangle$ . The strength of dipole interaction  $\delta = 0$  (a) and  $\delta = 0.5$  (b). The detuning  $\delta = 0$  (solid) and  $\delta = 1$  (dashed). Mean photon number  $\bar{n} = 0.1$ .

#### 4. Modeling of qubits entanglement dynamics

The results of calculations of entanglement parameter (12) for double JCM and initial state (3) are shown in Figs. 1(a)-(d). Results of calculations of entanglement parameter (14) for initial state (6) are displayed in Figs. 2(a-

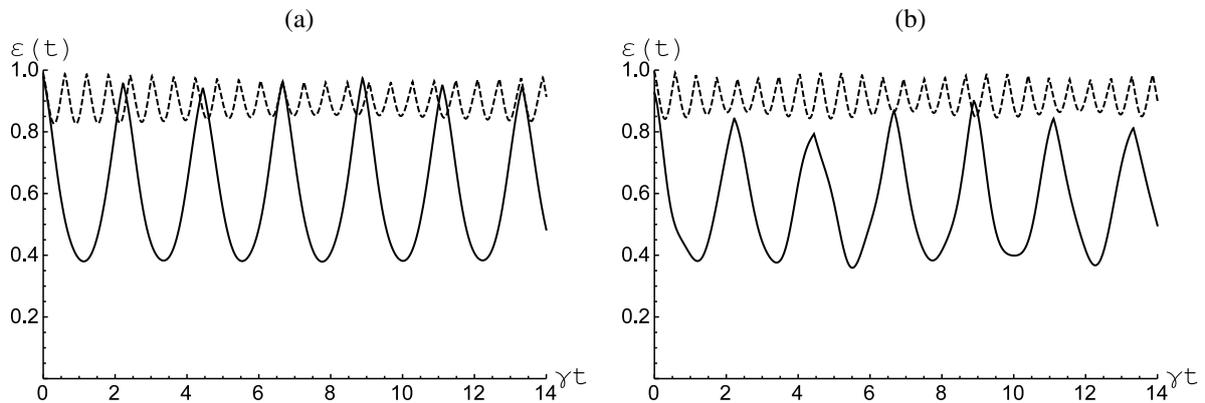


Figure 4: The negativity as a function of  $\gamma t$  for two-atom JCM with common field and entangled initial atomic state (2) with  $\theta = \pi/4$ . The strength of dipole interaction  $\alpha = 0$  (a) and  $\alpha = 0.5$  (b). The detuning  $\delta = 0$  (solid) and  $\delta = 5$  (dashed). Mean photon number  $\bar{n} = 0.1$ .

d). Fig. 1(a) shows that for exact resonance the negativity evolves periodically between 0 and 1, but the period is affected by the coupling constant. For resonance interaction the inclusion of the dipole-dipole interaction leads to a stabilization of entanglement behavior. Figs. 1(b)-1(d) show the effect of dipole-dipole interaction on negativity for non-resonant interaction and different couplings. When qubits A and B interact with a single-mode cavity fields via non-zero detuning the presence of dipole-dipole interaction with intermediate strength leads to increasing of the amplitudes of the negativity oscillations. But for large values of dipole-dipole interaction strength one can see the stabilization of entanglement oscillations as in the case of exact resonance. Figs. 2(a)-(d) show the time dependence of negativity for initial state (6) and different strength of dipole-dipole interaction. Fig. 2(a) gives the entanglement behavior for exact resonance. This behavior is different from that obtained for initial state (3) in resonance regime. The dipole-dipole interaction does not lead to stabilization of the entanglement, but has only an effect on the periods and amplitudes of the oscillations of entanglement. However, for non-resonant interaction between dipole-coupled qubits and fields the reverse behavior of atom-atom entanglement is true. For large values of the dipole-dipole interaction strength we have to deal with the stabilization of entanglement. Figs. 3 and 4 show the influence of detuning and dipole-dipole strength on atom-atom entanglement for two atoms interacting with common thermal field of resonator. Fig. 3(a) shows the entanglement time behavior for different couplings and separable atomic state  $|+, -\rangle$  ignored the dipole-dipole interaction. One can easily find that as the detuning increases, higher entanglement is obtainable. Zhang [25] earlier discovered such behavior and noted that when the atom-field detuning is large enough, the atoms tend to exchange energy with each other instead of with the field, and the field, which acts as a medium, is virtually excited during the atom-atom coupling process. Fig. 3(b) shows the negativity behavior for dipole-coupled qubits. For this case the reverse behavior of the entanglement is true. It seems like the negativity for qubits decreases as the detuning increases. We can also consider the negativity behavior for entangled initial state (2). In this case the inclusion of the detuning leads to a stabilization of entanglement behavior both to the model with dipole-dipole interaction and to the model without such interaction.

## 5. Conclusion

In this paper, we investigated the entanglement between two qubits interacting with fields of resonators in the framework of two types of JCM: double JCM with different coupling constants and detunings and two-qubit JCM with common cavity field taking into account the direct dipole-dipole interaction. For double JCM we discussed the influence of dipole-dipole interaction on qubit-qubit entanglement for resonance and non-resonance interactions. The results showed that these parameters have great impact on the amplitude and the period of the atom-atom entanglement evolution. In addition, the presence of sufficiently large dipole-dipole interaction leads to stabilization of entanglement for all Bell-type initial qubit states and different couplings and detuning. For two-qubit JCM with common field we

investigated the entanglement dynamics taking into account the dipole-dipole interaction for separable and entangled initial qubits states and thermal cavity field. For dipole coupled qubits prepared in a separable state the entanglement decreases as the detuning increases. For dipole uncoupled qubits the reverse behavior of the entanglement is true. For entangled initial states the inclusion of the detuning leads to a stabilization of entanglement behavior.

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