

Formation of Coalition Structures as a Non-Cooperative Game

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Abstract. The paper proposes a list of requirements for a game able to describe individually motivated social interactions: be non-cooperative, able to construct multiple coalitions in an equilibrium and incorporate intra and inter coalition externalities. For this purpose the paper presents a family of non-cooperative games for coalition structure construction with an equilibrium existence theorem for a game in the family. Few examples illustrate the approach. One of the results is that efficiency is not equivalent to cooperation as an allocation in one coalition. Further papers will demonstrate other applications of the approach.

Keywords: Non-cooperative games, Nash equilibrium, cooperative games

1 Introduction

There is a conventional dichotomy in existing game theories: the cooperative game theory (CGT) versus the non-cooperative game theory (NGT). CGT deals with coalitions as elementary items, NGT deals with strategic individual behavior. Comparison of the game theories is in Table 1. The enlisted properties are important for studying non-cooperative fundamentals of individually motivated social behavior.

Table 1. Comparison of game theories

| | CGT | NGT |
|--|-------------------------|---|
| type of a mapping | a finite set to a point | a vector to a vector |
| individual action | no | always |
| individual motivation | no | always |
| explicit allocation of players over coalitions | yes | no |
| inter-coalition externalities | sometimes | vague |
| intra-coalition externalities | no | always, but with vague coalition definition |

Cooperative game theories are based on the notion that a coalition, a subset of all players, moves as a whole unit only. This approach seems somewhat

simplified. Usually economic agents are self-interested, and do not necessarily make the same move even being together. From another side non-cooperative approach is based on a notion that every player is able to make individual decisions, but there is a low interest to partitioning players into coalitions or coalition structures.¹

Studying formation of only one coalition from a non-cooperative game, was introduced by Nash, [6], and is known now as Nash Program, Serrano, [8]. Practice of social analysis and social design requires studying non-cooperative and simultaneous formation of *multiple* coalitions, or coalition structures. This makes Nash program be too restrictive. Importance of studying externalities between coalitions, a case impossible within the Nash Program, was mentioned by Maskin, [4].

The main complication for a non-cooperative coalition structure formation stems from the necessity to describe complex inter and intra coalition interactions while forming multiple coalition structures.

Cooperative game theory equilibrium concepts (also the strong Nash equilibrium) evade such issues. Consider an example with 10 members in a committee, which make the grand coalition. Let there are 5, who do not agree and deviate from some joint agreement. Do they have the same reason and deviate as one group, five different reasons and deviate independently in five groups, two reasons and deviate in some groups of two and three, etc? And every deviator may have more than one deviating strategy inside a deviating group. Deviating groups may have internal conflicts, there could be conflicts between those who stay and those who deviate, or between deviators from different groups. These questions are different from those addressed by Aumann [1] in the strong Nash equilibrium and by Bernheim et al [2] in the coalition proof equilibrium.

The central idea of the presented model is to incorporate possible cases of deviation into individual strategy sets of players. This is done by parametrizing possible coalition structures by maximum coalition size. A constructed game satisfies properties in Table 1, and has an equilibrium in mixed strategies. As a result it is possible to study deviations of more than one player. The game differs from those existing in CGT and NGT. Varying a maximum coalition size we construct a family of games.

Section 2 contains an example of a game, Section 3 presents a mechanism to construct such games. The last Section 4 shows an example with preferences over coalition structures for an equilibrium refinement. The proof is in Appendix.

2 Example

There are 2 players, $i = 1, 2$. They can make choices over a desirable coalition structure, indexed as $a = \{\{1\}, \{2\}\}$, if a player chooses to be alone, and indexed $t = \{1, 2\}$, if a player chooses to be together.² A player has two strategies NC

¹ The terms a coalition structure and a partition of players are considered as synonyms.

² An example with preferences over coalition structures in in Section 4.

and C for every coalition structure.³ A set of strategies of i is $S_i(K = 2) = \{NC_{ia}, C_{ia}, NC_{it}, C_{it}\}$, $i = 1, 2$. Table 2 presents strategies and outcomes of the game. A cell contains a payoff profile and a final coalition structure. Every strategy profile is mapped to the pair: a payoff profile and a coalition structure. In the equilibrium a player uses mixed strategies C_{ia}, C_{it} with equal probabilities. There are 4 equilibria outcomes with payoff profiles $(-2; -2)$, all being inefficient.

The example uses a unanimous rule for coalition formation, the grand coalition $\{1, 2\}$ can be formed only if both choose it. Otherwise a player obtains the singleton coalition. From twelve possible strategy profile only four result in the grand coalition. Table 2 presents a game with two types of externalities, inter-coalition $\{\{1\}, \{2\}\}$ and intra-coalition in $\{1, 2\}$. So the example satisfies all desirable properties from Table 1.

Table 2. Payoff for the unanimous coalition structure formation rule: **the same efficient payoffs (0;0) in different partitions.** What is a cooperation then?

| | $NC_{2,a}$ | $C_{2,a}$ | $NC_{2,t}$ | $C_{2,t}$ |
|------------|--|---|--|---------------------------------------|
| $NC_{1,a}$ | $(\mathbf{0;0})$ $\{\{1\}, \{2\}\}$ | $(-5;3)$ $\{\{1\}, \{2\}\}$ | $(\mathbf{0;0})$ $\{\{1\}, \{2\}\}$ | $(-5;3)$ $\{\{1\}, \{2\}\}$ |
| $N_{1,a}$ | $(3;-5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)^{*,**}$ $\{\{1\}, \{2\}\}$ | $(3;-5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)^{**}$ $\{\{1\}, \{2\}\}$ |
| $NC_{1,t}$ | $(\mathbf{0;0})$ $\{\{1\}, \{2\}\}$ | $(-5;3)$ $\{\{1\}, \{2\}\}$ | $(\mathbf{0;0})$ $\{1, 2\}$ | $(-5; 3)$ $\{1, 2\}$ |
| $C_{1,t}$ | $(3;-5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)^{**}$ $\{\{1\}, \{2\}\}$ | $(3; -5)$ $\{1, 2\}$ | $(-2; -2)^{**}$ $\{1, 2\}$ |

For the coalition structure $\{\{1\}, \{2\}\}$ with a maximum coalition size $K = 1$ every player has two equilibrium strategies. A respective equilibrium is marked with an upper star, * and it is inefficient.

For the coalition structure $\{1, 2\}$ a player has two more equilibrium strategies. Final equilibria are marked with two upper stars, **. The equilibria belong to different coalition structures, what make this game be a stochastic game, Shapley, [9]. Stochastic property appears here not from outside shock, but from strategic actions of the players.

The presented example let players think that they can deviate either alone or together. If both can deviate, then i has and anticipate another has also the following strategies: a move within the same or a different coalition structure with strategies NC or C for each case. Every such possibility is in individual strategy set S_i .

³ For simplicity of the motivating example players do not have preferences over coalition structures, used in example with introvert and extravert players further.

3 A game

A game is parametrized by a number K , $K = 1, \dots, N$, a size of maximum coalition size in any feasible coalition structure. The parameter K has two meanings: no more than K agents are required to make this largest coalition, and at the same time this coalition needs no more than the same K agents to dissolve. Every player has a set of strategies for every feasible coalition structure. A strategy profile of all players is mapped into a coalition structure (an allocation of players into coalitions) and a payoff profile for all players. Payoffs are defined separately for every feasible coalition structure.

Definition 1 (a simultaneous coalition structure formation game). *A non-cooperative game for coalition structure formation with maximum coalition size K is*

$$\Gamma(K) = \langle N, \{K, \mathcal{P}(K), \mathcal{R}(K)\}, (S_i(K), \mathcal{U}_i(K))_{i \in N} \rangle,$$

where $\{K, \mathcal{P}(K), \mathcal{R}(K)\}$ - coalition structure formation mechanism, K is maximum coalition size, $\mathcal{P}(K)$ is a family of allowed coalition structures, $\mathcal{R}(K)$ is a family of coalition structure partition rules, $(S_i(K), \mathcal{U}_i(K))_{i \in N}$ are properties of players (individual strategies and payoffs), such that:

$$\times_{i \in N} S_i(K) \xrightarrow{\mathcal{R}(K)} \{ \mathcal{P}(K), (\mathcal{U}_i(K) = \{U_i(P) : P \in \mathcal{P}(K)\})_{i \in N} \}$$

Every individual strategy set $S_i(K)$ is a topological space, and every family of utility functions $\mathcal{U}_i(K)$ is integrable and bounded over $\times_{i \in N} S_i(K)$.

Definition 2 (an equilibrium in a game $\Gamma(K)$). *For a non-cooperative game $\Gamma(K)$ a mixed strategies profile $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$ is an equilibrium if for every subset $n(k)$ from N , with a size $1 \leq k \leq K$, and for every player $i \in n(k)$ a deviation from an equilibrium, $\forall \sigma_i(K) \neq \sigma_i^*(K)$, does not generate an individual gain:*

$$EU_i^{\Gamma(K)}(\sigma_i^*(K), \sigma_{-i}^*(K)) \geq EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}^*(K)).$$

Definition of an equilibrium makes a claim only about strategy profiles, but not about resulting coalition structures.

Theorem 1 (existence). *An equilibrium for a game $\Gamma(K)$ exists for any $K = 1, \dots, N$.*

In short, a proof follows from standard properties of finite number of bounded integral operators with a final application of a fixed point theorem. If every $S_i(K)$ is a topological space, then we can always define a dual space of probability measure $\Delta_i(S_i)$ on it, $\Delta_i(S_i)$ is a set of mixed strategies, with a weak convergence. Then for every $i \in N$ expected utility operator $EU_i^{\Gamma(K)}$ is bounded, continuous and compact. Finally a fixed point theorem is applied. Complete proof is in Appendix.

An increment in K generates a family of nested games with nested components: $\mathcal{P}(\bar{K}) \subset \mathcal{P}(\bar{K} + 1)$, $S_i(\bar{K}) \subset S_i(\bar{K} + 1)$, $\mathcal{U}_i(\bar{K}) \subset \mathcal{U}_i(\bar{K} + 1)$, for $\forall i \in N$ and every $\bar{K} = 1, \dots, N - 1$.

Definition 3 (family of nested games).

A family of games $\mathcal{G} = \{\Gamma(K) : K = 1, \dots, N\}$ is *nested* if :

$$\Gamma(K = 1) \subset \dots \subset \Gamma(K) \subset \dots \subset \Gamma(K = N),$$

i.e. for any two games $\Gamma(\bar{K})$ and $\Gamma(\bar{K} + 1)$ there is $\mathcal{P}(\bar{K}) \subset \mathcal{P}(\bar{K} + 1)$, $\mathcal{R}(\bar{K}) \subset \mathcal{R}(\bar{K} + 1)$, $S_i(\bar{K}) \subset S_i(\bar{K} + 1)$, $\mathcal{U}_i(\bar{K}) \subset \mathcal{U}_i(\bar{K} + 1)$, for every $i \in N$ and every $\bar{K} = 1, \dots, N - 1$.

Clearly, every game in a family has an equilibrium in mixed strategies

3.1 Why not to use a "threat", Nash (1953)

In the paper on cooperative behavior Nash (1953) offered to use a "threat" as a basic concept for coalition formation analysis. Further development of this concept was transformed into a "blocking coalition" (Aumann, 1960).

The reason not to take them for non-cooperative coalition structure formation is the following. Consider a strategy profile from only part of players. Let this profile be a threat to someone, beyond this subset. The threatening players may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat? From an other side, there may be some other player beyond the subset of players who may obtain a bonanza from this threat. But this beneficiary may not join the group due to some intra-group negative externalities for members or from members of this group. Resulting internal complexity and recursive nature of the example was the reason not to use a threat and blocking coalition concepts to establish non-cooperative formation of coalition structures.

4 Equilibrium refinement in the example

Non-cooperative game theory extensively studied refinement of the Nash equilibrium. We can demonstrate equilibrium refinement for the game above with an introduction of individual preferences over coalition structures. Let player $i = 1$ be an "extrovert" and always prefers to be together, in $\{1, 2\}$, but player $i = 2$ is an "introvert" and always prefers to be alone $\{\{1\}, \{2\}\}$. Player $i = 1$ has an additional gain from being together (in comparison to Table 2) ϵ , $0 < \epsilon$, from being together with another. From another side player $i = 2$ has an additional gain from being alone, δ , $0 < \delta$.

Payoffs of the Table 3 are adjusted according to these assumptions and present the payoff matrix for the new game. Using the rule of unanimous agreement for coalition formation the grand coalition, $\{1, 2\}$, can never be formed, what leaves only one equilibrium payoffs $(-2; -2)$ for the game in $\{\{1\}, \{2\}\}$.

Players differ in number of equilibrium strategies. Player $i = 2$ has one equilibrium strategy $C_{2,a}$. Player $i = 1$ has two pure equilibrium strategies, $C_{1,a}$ and $C_{1,t}$ and mixes them with equal weights. Equilibrium for $K = 1$ is labeled with one star, *, for $K = 2$ with two stars, **

Table 3. Payoff for a game when player 1 is "extrovert" and player 2 is "introvert".

| | $NC_{2,a}$ | $C_{2,a}$ | $NC_{2,t}$ | $C_{2,t}$ |
|------------|--|--|--|---|
| $NC_{1,a}$ | $(0; 0 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$ | $(0; 0 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$ |
| $C_{1,a}$ | $(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-2; -2 + \delta)^{*,**}$ $\{\{1\}, \{2\}\}$ | $(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-2; -2 + \delta)$ $\{\{1\}, \{2\}\}$ |
| $NC_{1,t}$ | $(0; 0 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$ | $(0 + \epsilon; 0)$ $\{1, 2\}$ | $(-5 + \epsilon;)$ $\{1, 2\}$ |
| $C_{1,t}$ | $(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$ | $(-2; -2 + \delta)^{*,**}$ $\{\{1\}, \{2\}\}$ | $(3 + \epsilon; -5)$ $\{1, 2\}$ | $(-2 + \epsilon; -2)$ $\{1, 2\}$ |

If both players are extroverters with a premium $\epsilon > 0$ for being together in $\{1, 2\}$, then a change in a game from $K = 1$ to $K = 2$ changes the equilibrium strategy profile, and a resulting equilibrium coalition structure as well. Table 4 contains payoffs for this game. The additional payoff makes players form only the grand coalition $\{1, 2\}$ with a unique equilibrium, but a resulting payoff is still inefficient.

Table 4. Payoff for two extrovert players who prefer to be together. Uniqueness of an equilibrium is fixed.

| | $L_{2,a}$ | $H_{2,a}$ | $L_{2,t}$ | $H_{2,t}$ |
|-----------|---------------------------------|------------------------------------|---|---|
| $L_{1,a}$ | $(0; 0)$ $\{\{1\}, \{2\}\}$ | $(-5; 3)$ $\{\{1\}, \{2\}\}$ | $(0; 0)$ $\{\{1\}, \{2\}\}$ | $(-5; 3)$ $\{\{1\}, \{2\}\}$ |
| $H_{1,a}$ | $(3; -5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)^*$ $\{\{1\}, \{2\}\}$ | $(3; -5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)$ $\{\{1\}, \{2\}\}$ |
| $L_{1,t}$ | $(0; 0)$ $\{\{1\}, \{2\}\}$ | $(-5; 3)$ $\{\{1\}, \{2\}\}$ | $(0 + \epsilon; 0 + \epsilon)$ $\{1, 2\}$ | $(-5 + \epsilon; 3 + \epsilon)$ $\{1, 2\}$ |
| $H_{1,t}$ | $(3; -5)$ $\{\{1\}, \{2\}\}$ | $(-2; -2)$ $\{\{1\}, \{2\}\}$ | $(3 + \epsilon; -5 + \epsilon)$ $\{1, 2\}$ | $(-2 + \epsilon; -2 + \epsilon)^{**}$ $\{1, 2\}$ |

5 Conclusion

The paper suggest a new family of simultaneous non-cooperative games which satisfy the following criteria: be non-cooperative, be able to form multiple-

coalition structures and to contain two types of externalities, intra and inter-coalition. Further application of the proposed mechanism to follow in the next papers. Earlier version of the paper with more applications was published at SSRN.

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Appendix. Proofs

Theorem 1. Let K be a maximum size of a coalition. For fixed K a game $\Gamma(K)$ has the Nash equilibrium in mixed strategies.

The parameter K is omitted everywhere below for simplicity, besides expected utility.

Step 1 Preliminary step.

Let S_i be a topological space for every $i \in N$. Let $\mathcal{B}(S_i)$ be σ -algebra on S_i , and the pair $(S_i, \mathcal{B}(S_i))$ be a measurable set. Let a non-negative measure Δ_i on S_i be a mapping $\Delta_i: \mathcal{B}(S_i) \rightarrow [0, 1]$, and $(S_i, \mathcal{B}(S_i), \Delta_i(S_i))$ be a measure space of mixed strategies with $\Delta_i(S_i) = 1$. By theorem 5 from [7] the space Δ_i can be uniquely assigned. The space $(S_i, \mathcal{B}(S_i), \Delta_i)$ has weak convergence, what we do not prove here.

Lemma 1. *The space $(S_i, \mathcal{B}(S_i), \Delta_i(S_i))$ is complete.*

Proof. Let σ_i , be a mixed strategy, $\sigma_i \in \Delta_i$, and $\{\sigma_i^{(k)}\}_{k=1}^{\infty}$ be a sequence of mixed strategies of i . All the sequences in $(S_i, \mathcal{B}(S_i), \Delta_i)$ are bounded. By Bolzano-Weirstrass theorem every bounded sequence has a least one limit point. Let $\sigma_i^{(0)}$ be a limit point of a sequence $\{\sigma_i^{(k)}\}_{k=1}^{\infty}$, and

$$\int_{S_i} \sigma_i^{(0)}(s_i) ds_i = \lim_{k \rightarrow \infty} \int_{S_i} \sigma_i^{(k)}(s_i) ds_i.$$

Existence of a limit point in the weak convergence sense follows from the weak compactness property of the set Δ_i , by Prohorov's theorem, [7]. From this immediately follows completeness of the set of mixed strategies Δ_i .

A set of probability measures $\Delta_{-i} := (\Delta_j)_{j \neq i}$ of all other players besides i is bounded. Let σ_{-i} be a mixed strategy of all other players, $\sigma_{-i} \in \Delta_{-i}$.

Step 2 Expected utility of one player.

Let $U_i: \times_{i \in N} S_i \rightarrow \mathbb{R}_+$ be a payoff function of i , continuous on $S_i \times S_{-i}$ and $\sqrt{\int_{S_i \times S_{-i}} (U_i(s_i, s_{-i}))^2 ds_i ds_{-i}} < \infty$.

We will use the word operator to emphasize, that player i controls only own strategies, variables indexed with i , but does not control variables indexed with $-i$ controlled by other players. Expected utility operator of i in a game $\Gamma(K)$ is a Lebegue integral

$$EU_i^K(\sigma_i, \sigma_{-i}) := EU_i^{\Gamma(K)}(\sigma_i, \sigma_{-i}) = \int_{S_i \times S_{-i}} U_i(s_i, s_{-i}) \sigma_i(s_i) \sigma_{-i}(s_{-i}) d(s_i) d(s_{-i}).$$

Probability measures for all players and each partition-contingent payoffs $U_i(s_i, s_{-i})$ were defined above, so expected utility $EU_i^K(\sigma_i, \sigma_{-i})$ is well defined.

Theorem 2. *Expected utility operator $EU_i^K(\sigma_i, \sigma_{-i})$ maps a weakly converging sequence $\{\sigma_i^{(k)}\}_{k=1}^{\infty}$ into a bounded converging sequence $\{EU_i^K(\sigma_i^{(k)}, \sigma_{-i})\}_{k=1}^{\infty}$.*

Proof. Let $\lim_{k \rightarrow \infty} \int_{S_i} \sigma_i^{(k)}(s_i) ds_i = \int_{S_i} \sigma_i^{(0)}(s_i) ds_i$, where $\{\sigma_i^{(k)}\}_{k=1}^{\infty}$ is a bounded weakly converging sequence from Δ_i . Then

$$\begin{aligned} \lim_{k \rightarrow \infty} EU_i^K(\sigma_i^{(k)}, \sigma_{-i}) &= \lim_{k \rightarrow \infty} \int_{S_i \times S_{-i}} U_i(s_i, s_{-i}) \sigma_i^{(k)}(s_i) \sigma_{-i}(s_{-i}) d(s_i) d(s_{-i}) \\ &= \int_{S_i \times S_{-i}} U_i(s_i, s_{-i}) \sigma_i^{(0)}(s_i) \sigma_{-i}(s_{-i}) d(s_i) d(s_{-i}) = EU_i^K(\sigma_i^{(0)}, \sigma_{-i}), \end{aligned}$$

Expected utility $EU_i^K(\sigma_i, \sigma_{-i})$ inherits compactness and weak convergence over $\Delta_i(S_i)$.

Lemma 2. The sequence $\left\{EU_i^K(\sigma_i^{(k)}, \sigma_{-i})\right\}_{k=1}^\infty$ is continuous on $\Delta_i, \forall \sigma_{-i} \in \Delta_{-i}$.

Proof.

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left(EU_i^K(\sigma_i^{(k)}, \sigma_{-i}) - EU_i^K(\sigma_i^{(0_i)}, \sigma_{-i}) \right) \\ &= \lim_{k \rightarrow \infty} \left(\int_{S_i \times S_{-i}} U_i(s_i, s_{-i}) \left(\sigma_i^{(k)}(s_i) - \sigma_i^{(0_i)}(s_i) \right) \sigma_{-i}(s_{-i}) ds_i ds_{-i} \right) \\ &\leq \int_{S_i \times S_{-i}} U_i(s_i, s_{-i}) ds_i ds_{-i} \lim_{k \rightarrow \infty} \left(\int_{S_i \times S_{-i}} \left(\sigma_i^{(k)}(s_i) - \sigma_i^{(0_i)}(s_i) \right) \sigma_{-i}(s_{-i}) ds_i ds_{-i} \right) = 0 \end{aligned} \tag{1}$$

Let $F_i(\sigma_{-i}) = \arg \max_{\sigma_i \in \Delta_i} EU_i^K(\sigma_i, \sigma_{-i})$, $F_i : \Delta_i \rightarrow \Delta_{-i}$. Operator F_i is a best response operator, it is bounded, compact and continuous.

Best response correspondence F_i determines the best mixed strategies of i to maximize expected utility given mixed strategy profile from other players. Expected utility operator $EU_i^K(\sigma_i, \sigma_{-i})$ maps a convex, closed, compact set $\Delta_i, F_i \subset \Delta_i$, to a convex, closed, compact set.

Theorem 3. Let Δ_i be a set of probability measures of $i \in N$, and $X_i = EU_i^K(\Delta_i, \Delta_{-i})$ be a bounded and compact set of i 's expected utility values. Then the set $F_i(X_{-i}) = \arg \max_{\sigma_i \in \Delta_i(s_i)} EU_i^K(\sigma_i, \sigma_{-i})$ is also compact.

Proof. The set X_{-i} is a compact set. Let $\left\{\sigma_i^{(k_i)}\right\}_{k_i=1}^\infty$ be a sequence from Δ_i . For every point $\sigma_i^{(k)}$ we take one of the pre-images $X_{-i}^{(k)}, F_i(X_{-i}^{(k)}) = \sigma_i^{(k)}$. The set X_i is compact, then from the sequence $\left\{X_{-i}^{(k)}\right\}_{k=1}^\infty \in X_{-i}$ we can extract a converging sub-sequence $\left\{X_{-i}^{(k(r))}\right\}_{k(r)=1}^\infty$, which converges to $X_{-i}^{(0_i)} \in X_{-i}$. By continuity of $F_i(X_{-i})$ on X_i at $X_{-i}^{(0_i)}$ there is $\sigma_i^{(k(r))} = F_i(X_{-i}^{(k(r))}) \rightarrow F_i(X_{-i}^{(0_i)}) = \sigma_i^{(0_i)} \in \Delta_i$.

Step 3 Fixed point argument for the game

To apply fixed point theorem we construct a system of N equations and demonstrate existence of a fixed point for the whole system. Let a vector operator $EU^K(\sigma) = (EU_i^K(\sigma_i, \sigma_{-i}))_{i \in N}$ be a list of expected utility operators of all players. By the Tykhonov theorem the set of expected payoff values $EU^K(\Delta)$, s.t. $\Delta = \Delta_i \times \Delta_{-i}$, is compact, $\Delta_{-i} = \times_{j \neq i} \Delta_j$. Thus the set $EU^K(\Delta)$ is also bounded and continuous over the set of probability measures, as every component of $EU_i^K(\Delta_i, \Delta_{-i})$ has this property. Let k be a number of a probability measure (mixed strategy) of i in a sequence of probability measures $\left\{\left\{\sigma_i^{(k)}\right\}_{k=1}^\infty\right\}_{i \in N}$ in the sense of the weak convergence,

s.t. $\sigma_i^{(k)} \xrightarrow{weak} \sigma_i^{(0)}$. We want to construct a finite collection of converging expected utility operators.

For every i exists an argmax operator:

$$F_i(\sigma_{-i}): \Delta_{-i} \mapsto \Delta_i,$$

i.e. the best response operator, that allows to choose the best mixed strategy from Δ_i in response to σ_{-i} from Δ_{-i} , such that

$$F_i(\sigma_{-i}) = \arg \max_{\sigma_i \in \Delta_i} EU_i^K(\sigma_i, \sigma_{-i}),$$

where Δ_i is a bounded, closed, convex set with weak convergence. Let

$$\left\{ F_i^{(k)} \right\} = \left\{ \arg \max_{\sigma_i^{(k)} \in \Delta} EU_i^K(\sigma_i^{(k)}, \sigma_{-i}) \right\}_{k=1}^{\infty}$$

be a sequence of vectors of operators.

Let

$$\left(F_1^{(k)}, \dots, F_N^{(k)} \right)_{k=1}^{\infty} = \left(\arg \max_{\sigma_1^{(k)} \in \Delta_1} EU_1^K(\sigma_1^{(k)}, \sigma_{-1}), \dots, \arg \max_{\sigma_N^{(k)} \in \Delta_N} EU_N^K(\sigma_N^{(k)}, \sigma_{-N}) \right)_{k=1}^{\infty} \quad (2)$$

be a sequence of best response operators.

Every vector operator $F^{(k)} = \left(F_i^{(k)} \right)_{i \in N}$ is a finite vector, and by Tykhonov theorem it is also a compact and closed set. Every mapping $F^{(k)}$ transforms a convex, closed, compact set $\Delta = (\Delta_i)_{i \in N}$ into itself.

Lemma 3 (Schauder). *Any operator F compact on $(\Delta_i)_{i \in N}$ has a uniform limit on $\Delta = (\Delta_i)_{i \in N}$ as a sequence of continuous finite dimensional operators $\left(F_1^{(k)}, \dots, F_N^{(k)} \right)_{k=1}^{\infty}$, and maps $(\Delta_i(S_i), \Delta_{-i}(S_{-i})) \rightarrow (\Delta_i(S_i), \Delta_{-i}(S_{-i}))$.*

The proof follows Sobolev and Ljusternik, (1976).

Proof. Operator F is a compact operator, hence $F(\Delta)$ is a compact set. Let there is a sequence of positive numbers $\{\epsilon_1, \epsilon_2, \epsilon_k, \dots\}$, which converges to zero. Let $y \in F(\Delta)$ be an element of Δ . For every small positive ϵ_k we construct $L^{(k)} = \{y_1^{(k)}, \dots, y_N^{(k)}\}$, $y^{(k)} = (y_1^{(k)}, \dots, y_N^{(k)}) \in F(\Delta)$, where for every i there is $y_i^{(k)} \in \{\sigma_i^{(k)}\}_{i \in N}$. All $L^{(k)}$ are continuous on a mixed strategy profile $\left(\sigma_i^{(k)} \right)_{i \in N}$ in a sense of the weak convergence, and denote $\|y - y^{(k)}\| = \max_{i \in N} \int_{S_i \times S_{-i}} |y_i^{(k)}(s_i) - y_i(s_i)| y_{-i}(s_{-i}) ds_i ds_{-i}$. Define on $F(\Delta)$ an operator $J^{(k)}$ such that for any $y \in F(\Delta)$ there is

$$J^{(k)}(y) = \frac{\sum_{i=1}^N \theta_i^{(k)}(y) y_i^{(k)}}{\sum_{i=1}^N \theta_i^{(k)}(y)},$$

where

$$\theta_i^{(k)}(y) = \begin{cases} \epsilon_k - \|y - y_i^{(k)}\|, & \|y - y_i^{(k)}\| < \epsilon_k \\ 0, & \|y - y_i^{(k)}\| > \epsilon_k \end{cases}.$$

Operator $J^{(k)}(y)$ is well defined for every $y \in F(\Delta)$ as $\theta_i^{(k)} \geq 0$ and $\theta_i^{(k)} > 0$ at least for one j . Operator $J^{(k)}(y)$ is continuous on $F(\Delta)$. This follows from that all $\theta_i^{(k)}(y)$ are continuous over y , $\sum_{i=1}^N \theta_i^{(k)}(y)$ is continuous over y . For every $y \in F(\Delta)$ there is $\sum_{i=1}^N \theta_i^{(k)}(y) > 0$, from where follows continuity of $\frac{\theta_i^{(k)}(y)}{\sum_{i=1}^N \theta_i^{(k)}(y)}$. Then

$$\begin{aligned} \|y - J^{(k)}(y)\| &= \left\| y - \frac{\sum_{i=1}^N \theta_i^{(k)}(y) y_i^{(k)}}{\sum_{i=1}^N \theta_i^{(k)}(y)} \right\| = \left\| \frac{\sum_{i=1}^N \theta_i^{(k)}(y) (y - y_i^{(k)})}{\sum_{i=1}^N \theta_i^{(k)}(y)} \right\| \leq \\ &\leq \frac{\sum_{i=1}^N \theta_i^{(k)}(y) \|y - y_i^{(k)}\|}{\sum_{i=1}^N \theta_i^{(k)}(y)} \leq \epsilon_k \frac{\sum_{i=1}^N \theta_i^{(k)}(y)}{\sum_{i=1}^N \theta_i^{(k)}(y)} = \epsilon_k. \end{aligned}$$

If there is $\|y - y_i^{(k)}\| \geq \epsilon_k$ then a corresponding coefficient $\theta_i^{(k)}(y) = 0$. Let $F^{(k)}\sigma = J^{(k)}(F\sigma)$, then for any $\sigma \in \Delta$ there is a sequence of operators $\{F^{(k)}\}_{k=1}^\infty$ such that

$$\|F\sigma - F^{(k)}\sigma\| = \|F\sigma - F^{(k)}(F\sigma)\| \leq \epsilon,$$

for any $\sigma \in \Delta$.

For every $y^{(k)}$ there is $y^{(k)} \in F(\Delta)$, so the values of the constructed operators belong to a convex closure of $F(\Delta)$.

Lemma 4. *Let a sequence of operators $\{F^{(k)}\}_{k=1}^\infty$ be continuous on Δ and it uniformly converges to an operator F on Δ . Let $L^{(k)} = F^{(k)}(\Delta)$, $k = 1, 2, \dots$. Then the set $L = \cup_{k=1}^\infty L^{(k)}$ is compact.*

The proof follows Sobolev and Ljusternik, (1976).

Proof. According to the Lemma above an operator $F^{(0)}$ is compact, as there is a sequence of operators, $\{F^{(k)}\}_{k=1}^\infty$, which converges to $F^{(0)}$. Then for any $\epsilon_k > 0$ and when $\bar{k} \geq k$, for every $y \in L^{(k)}$ there is $y^{(0)} \in L^{(0)}$ such that $\|y - y^{(0)}\| < \epsilon_k$. This is possible as y is an element from $L^{(k)}$ and σ is one of pre-images of y under the mapping $F^{(k)}$. So we can take $y^{(0)} = F^{(0)}\sigma$. Now we construct the set $\cup_{k=1}^{\bar{k}} L^{(k)}$. This set is compact. We need to show that it is an ϵ -net for L . Let $y \in \bar{K}$. If $y \in \cup_{k=1}^{\bar{k}} L^{(k)}$, then the result of the Lemma is trivial. If $y \in L^{(k)}$ for $\bar{k} > k$, then there is $y^{(0)} \in L^{(0)}$, such that $\|y - y^{(0)}\| < \epsilon_k$. Thus $\cup_{k=\bar{k}+1}^\infty L^{(k)}$ is compact for ϵ_k from set L , and thus L is compact.

Theorem 4 (Schauder fixed point theorem). *If a compact operator F maps a bounded closed convex set Δ into itself, then the mapping has a fixed point, $\sigma \in \Delta$, $F\sigma = \sigma$.*

The proof follows Sobolev and Ljusternik, [3].

Proof. Take a sequence of positive $\{\epsilon^{(k)}\}_{k=1}^{\infty}$, converging to zero, and as above construct a sequence of continuous finite-dimension operators $\{F^{(k)}\}_{k=1}^{\infty}$, which uniformly converges on Δ to F .

The infinite set of all probability distributions Δ is convex, $\{F^{(k)}\}_{k=1}^{\infty} = \{\sigma^{(k)}\}_{k=1}^{\infty} \subset \Delta$ for every $\sigma \in \Delta$. Let $X^{(k)}$ be a finite dimension subspace with the set $F^{(k)}(\Delta)$, $F^{(k)}(\Delta) \subset X^{(k)}$. Take the operator $F^{(k)}$ on the subset $\Delta^{(k)} = \Delta \cap F^{(k)}$ from $X^{(k)}$. The set $\Delta^{(k)}$ is also a closed convex set. As $F^{(k)}(\Delta) \subset \Delta$ and $F^{(k)}(\Delta) \subset \Delta^{(k)}$, then $F^{(k)}(\Delta) \subset \Delta^{(k)}$, and therefore $F^{(k)}(\Delta^{(k)}) \subset \Delta$.

Thus the operator $F^{(k)}$ in $\Delta^{(k)}$, $\Delta^{(k)} \subset \Delta$, maps a closed convex set $\Delta^{(k)}$. By the Bauer fixed-point theorem there is a fixed point of this mapping, i.e. $F^{(k)}\sigma^{(k)} = \sigma^{(k)}$. From another side $\Delta^{(k)} \subset \Delta$, thus $\sigma^{(k)}$ is a fixed point also for the mapping $F^{(k)}(\Delta)$. As the point $\sigma^{(k)} \in F^{(k)}(\Delta)$, and the sequence $\{\sigma^{(k)}\}_{k=1}^{\infty}$ belongs to the set $\tilde{\Delta} = \cup_{k=1}^{\infty} F^{(k)}(\Delta) \subset \Delta$. Thus the set $\tilde{\Delta}$ is compact. Then from the sequence $\{\sigma^{(k)}\}_{k=1}^{\infty}$ we can extract a converging subsequence $\{\sigma^{(k_r)}\}_{k_r=1}^{\infty}$ and a limit of this subsequence is in Δ , as Δ is closed.

We can define $\|F^{(k_r)}\sigma^{(k_r)}, \sigma^{(0)}\| = \max_{i \in N} \int_{S_i \times S_{-i}} \left(F_i^{(k_r)} \sigma_{-i}^{(k_r)} - \sigma_i^{(0)} \right) \sigma_{-i}^{(0)} ds_i ds_{-i}$, $F_i^{(k_r)} \sigma_{-i}^{(k_r)} = \sigma_i^{(k_r)}$. We need to demonstrate that $\sigma^{(0)}$ is a fixed point. This follows from

$$\begin{aligned} \|F\sigma^{(0)}, \sigma^{(0)}\| &\leq \|F\sigma^{(0)}, F\sigma^{(k_r)}\| + \|F\sigma^{(k_r)}, F^{(k_r)}\sigma^{(k_r)}\| + \|F^{(k_r)}\sigma^{(k_r)}, \sigma^{(0)}\| = \\ &= \|F\sigma^{(0)}, F\sigma^{(k_r)}\| + \|F\sigma^{(k_r)}, F^{(k_r)}\sigma^{(k_r)}\| + \|\sigma^{(k_r)}, \sigma^{(0)}\|. \end{aligned}$$

For any fixed $\epsilon_{\bar{k}} > 0$ we choose k' be big enough that for any $\bar{k} > k'$ there is $\|\sigma^{(\bar{k})}, \sigma^{(0)}\| < \epsilon_{\bar{k}}/3$ and $\|F\sigma^{(0)}, F\sigma^{\bar{k}}\| < \epsilon_{\bar{k}}/3$.

Then we choose k'' be big enough that for $\bar{k} \geq k''$ there is $\|F\sigma, F\sigma^{(\bar{k})}\| < \epsilon_{\bar{k}}/3$ is uniform on Δ for all $\{\sigma^{(\bar{k})}\}_{\bar{k}=\bar{k}}^{\infty}$. Then for some $\bar{k} \geq \max\{k', k''\}$ there is $\|F\sigma^{(0)}, \sigma^{(0)}\| < \epsilon_{\bar{k}}/3$. As $\epsilon_{\bar{k}} > 0$, then this is possible only if $F\sigma^{(0)} = \sigma^{(0)}$. Then $\sigma^{(0)}$ is a fixed point.

The proof does not depend on K , thus every game in the family \mathcal{G} has an equilibrium in mixed strategies.