Estimation of Attraction Domains for Multi-Input Affine Systems with Constrained Controls

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Abstract

Multi-input nonlinear affine systems represented in a canonical (normal) form are considered. The controls are assumed to be constrained. The application of feedback linearization results in a closed-loop system that is decomposed into an aggregate of independent linear subsystems in a neighborhood of the origin and is nonlinear when controls reach saturation. For the closed-loop system obtained, the problem of estimating the attraction domain is set. A method for constructing an estimate of the attraction domain that is based on results of absolute stability theory is suggested. An estimate is sought as a Cartesian product of invariant ellipsoids each of which is found by solving a system of linear matrix inequalities. An optimization problem of finding the best estimate is posed. The discussion is illustrated by numerical examples.

1 Introduction

Boundedness of control resources in practicable automatic control systems leads to saturation in actuators when the feedback signal exceeds a certain limit value. Functioning of the system designed without regard to this circumstance in a saturation mode may result in considerable reduction of regulation efficiency and, quite often, in loss of stability. Thus, it is required, on the one hand, to design a controller with regard to the possibility of the actuator saturation and, on the other hand, to have an estimate of the attraction domain for the system with the saturated control.

Ways to overcome negative effects associated with saturation in the actuators are discussed in numerous publications (see, e.g., [Tarbouriech, 2011, Tarbouriech, 2009, Turner, 2007, Blanchini, 2008, Formal'skii, 1974] and references therein). In the majority of publications, linear control systems are considered, and the authors pose either the problem of synthesizing a controller that takes into account the possibility of actuator saturation or the problem of the development of a separate controller component, in addition to a linear controller designed without regard to control constraints, that is activated in the case of saturation. Taking into account control constraints in nonlinear systems is a much more complicated task, and the number of publications on this subject is not too great. In the majority of them, an attempt is made to adapt methods designed for linear systems to the nonlinear case.

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In this paper, we consider n-dimensional nonlinear affine systems with m inputs that can be represented in a canonical (normal) form [Isidori, 1995]. The controls are assumed to be constrained. However, the problem of designing a controller that takes into account constraints imposed on the control signal is not posed. It is assumed that the system is closed by a linearizing feedback the coefficients of which are specified in advance, so that the problem of determining the feedback coefficients is not considered. The control constraints are satisfied by applying the saturation function to the feedback. For the nonlinear closed-loop system obtained, we set the problem of finding an estimate of its attraction domain, which is sought as a Cartesian product of invariant ellipsoids composing the system under study. We also pose the problem of finding the "best" (in one or another sense) estimate of the considered class and reduce its solution to solving a conventional constrained optimization problem.

The approach to construction of the attraction domain proposed in this paper relies on results of absolute stability theory and linear matrix inequalities (LMIs) and extends the approach proposed in [Pesterev, 2017] for the case of a scalar control to affine systems with m constrained inputs. The latter approach, in turn, is a generalization of that used in [Rapoport, 2006] for estimating the attraction domain of a nonlinear second-order system closed by a linearizing feedback that arises in the path following problem for a wheeled robot.

2 Problem Statement

We consider multidimensional affine control systems with vector inputs consisting of m connected subsystems of the form

$$\dot{x}_1^i = x_2^i, \dots, \dot{x}_{r_i-1}^i = x_{r_i}^i, \dot{x}_{r_i}^i = f_i(x) + g_i(x)u_i, \; i = 1, \dots, m. \tag{1}$$

Here, x is an n-dimensional state vector, $x \equiv (x^1, \dots, x^m) \in D_x \subseteq R^n$, $x^i \in R^{r_i}$, m is the number of controls,

$$r_1 + \cdots + r_m = n, \tag{2}$$

u_i are continuous constrained controls, and $f_i(x)$ and $g_i(x) \neq 0$ are continuous functions. It is required to find a feedback that stabilizes the zero solution $x = 0$ of system (1) and to construct an attraction domain of the closed-loop system obtained.

System (1) is a normal (canonical) form of affine control systems with m inputs and m outputs that have vector relative degree $\{r_1, \dots, r_m\}$ satisfying condition (2) [Isidori, 1995]. The canonical representation (1) is convenient in that, in the case of unconstrained controls, it is linearized by the feedback

$$u_i(x) = -(\sigma_i(x^i) + f_i(x))/g_i(x), \tag{3}$$

where $\sigma_i(x^i) = c_i^{\mathrm{T}} x^i$, $c_i^{\mathrm{T}} = [c_{i1}, \dots, c_{ir_i}]$, $c_{ij} > 0$, the application of which turns the system to an aggregate of m independent linear subsystems $\dot{x}_1^i = x_2^i, \dots, \dot{x}_{r_i-1}^i = x_{r_i}^i, \dot{x}_{r_i}^i = -\sigma_i(x^i)$, or, in the matrix form, $\dot{x}^i = A_i x^i$, $i = 1, \dots, m$, where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_{i1} & -c_{i2} & -c_{i3} & \cdots & -c_{ir_i} \end{bmatrix}. \tag{4}$$

It can easily be seen that solution $x = 0$ of system (1) closed by feedback (3) is globally asymptotically stable if and only if matrices A_i of all subsystems are Hurwitz ones. In what follows, we assume that this condition is fulfilled.

However, in the general case, system (1) cannot be linearized by means of feedback (3) in the entire domain of its definition in view of the boundedness of the control resources. For simplicity, let us assume that the upper and lower bounds of the ith control are equal to one another, i.e., the controls satisfy the two-sided constraints

$$-\bar{u}_i \leq u_i \leq \bar{u}_i. \tag{5}$$

To meet them, we apply the saturation function to the right-hand side of formula (3):

$$u_i(x) = -\mathrm{sat}_{\bar{u}_i}[(\sigma_i(x^i) + f_i(x))/g_i(x)]. \tag{6}$$

Closing system (1) by feedback (6), we obtain the m nonlinear subsystems

$$\dot{x}_1^i = x_2^i, \ldots, \dot{x}_{r_i-1}^i = x_{r_i}, \dot{x}_{r_i} = -\Phi_i(\sigma_i, x),\ i = 1, \ldots, m, \tag{7}$$

connected with one another through the right-hand sides $\Phi_i(\sigma_i, x)$, which, in the case of saturated controls, depend generally on all components of the state vector x:

$$\Phi_i(\sigma_i, x) = -f_i(x) + g_i(x)\bar{u}_i \operatorname{sign}\left(\frac{\sigma_i(x^i) + f_i(x)}{g_i(x)}\right) \equiv -f_i(x) + |g_i(x)|\bar{u}_i \operatorname{sign}(\sigma_i(x^i) + f_i(x)). \tag{8}$$

In a neighborhood of the origin where controls do not reach saturation, $\Phi_i(\sigma, x) = \sigma_i(x^i)$, and the subsystems are linear and independent. Inclusion of σ_i into the arguments of functions Φ_i emphasizes the fact that the right-hand side of the system depends not only on the current values of the state variables but also on the value of the linear function $\sigma_i(x^i)$.

The problem we consider here is to find an estimate of the attraction domain of the zero solution of system (7). Note that, in order that $x = 0$ could be a stable equilibrium state of system (7), the following condition must be satisfied:

$$\left|\frac{f_i(0)}{g_i(0)}\right| < \bar{u}_i,\ \ i = 1, \ldots, m. \tag{9}$$

Note also that, if $x = 0$ is an equilibrium state of the closed-loop system, i.e., $f_i(0) = 0$, $i = 1, \ldots, m$, then conditions (9) are automatically satisfied.

In what follows, we assume that the domain of system (1) does not generally coincide with the entire space, $D_x \subseteq R^n$, and that the right-hand side of the system satisfies conditions (9).

3 Estimate of the Attraction Domain

3.1 Comparison System

Along with system (7), we consider the linear nonstationary systems

$$\dot{x}_1^i = x_2^i, \ldots, \dot{x}_{r_i-1}^i = x_{r_i}^i, \dot{x}_{r_i}^i = -\beta_i(t)\sigma_i(x^i),\ i = 1, \ldots, m, \tag{10}$$

that are absolutely stable in sectors $(\beta_{i0}, 1]$, $i = 1, \ldots, m$, respectively. Recall that a linear nonstationary system is said to be absolutely stable in the angle (sector) $(\beta_{i0}, 1]$ if its zero solution is asymptotically stable for any measurable functions $\beta_i(t)$ satisfying the inequalities [Aizerman, 1964, Pyatnitskii, 1970]

$$0 < \beta_{i0} < \beta_i(t) \leq 1. \tag{11}$$

Systems (10) will be referred to as the *comparison systems* for the nonlinear system (7).

If the right-hand side of the ith subsystem in (7) satisfied the “sector” condition

$$0 < \beta_{i0}\sigma_i^2 < \Phi_i(\sigma_i, x)\sigma_i \leq \sigma_i^2, \tag{12}$$

for any $x \in R^n$, then $x^i = 0$ would be an asymptotically stable solution of this system on the whole (absolute stability of a nonlinear subsystem in (7) in the angle $(\beta_{i0}, 1]$ follows from absolute stability of the corresponding linear nonstationary system in the angle $(\beta_{i0}, 1]$ [Pyatnitskii, 1970]). Accordingly, the fulfillment of conditions (12) for all $i = 1, \ldots, m$ would imply absolute stability of the n-dimensional nonlinear system (7) consisting of m subsystems.

For one-dimensional subsystems ($r_i = 1$) in (7), the zero solution of the corresponding (scalar) comparison system is asymptotically stable for any $\beta_i(t) > 0$; i.e., $\beta_{i0} = 0$, and the sector condition (12) takes the form $\Phi_i(x^i, x)x^i > 0$.

Conditions (12), however, are generally not satisfied in the entire coordinate space R^n (to say nothing of the fact that system (1) is not generally defined in the entire coordinate space). Nevertheless, study of stability of the nonlinear system (7) still can be reduced to study of absolute stability of linear nonstationary systems (10) if we require that the sector conditions (12) hold in a *positive invariant set* (further, simply *invariant set*) of the n-dimensional system (7) rather than in the entire coordinate space [Pesterev, 2017], i.e., in a set in R^n that, together with any point belonging to the set, contains the entire half-trajectory of system (7) that begins at this point. This brings us at the question of how to find an invariant set of system (7)?

3.2 Invariant Set of the System

In the case of the scalar control ($m = 1$), for an invariant set, one can take an invariant set of the comparison system provided that the sector condition (12) holds in it [Pesterev, 2017]. In this paper, we extend this result to the multi-input case. Namely, we will seek an invariant set of the nonlinear system in the form of a Cartesian product of invariant sets of the comparison systems $\Upsilon = \Upsilon_1 \times \cdots \times \Upsilon_m$, where Υ_i is a (positive) invariant set of the ith comparison system (10), i.e., a set in R^{r_i} that, together with any point belonging to the set, contains the entire half-trajectory of the ith system in (10) that begins at this point for any measurable functions $\beta_i(t)$ satisfying condition (11).

Theorem 1 *Let linear nonstationary systems in* (10) *be absolutely stable for any measurable functions* $\beta_i(t)$ *satisfying inequalities (11). Let* $\Upsilon_i \subseteq R^{r_i}$ *be invariant sets of these systems and* $\Upsilon = \Upsilon_1 \times \cdots \times \Upsilon_m \subseteq D_x$*. If the right-hand sides of the nonlinear system* (7) *satisfy the sector conditions* (12) *in* Υ*, then* Υ *is an invariant set of system* (7) *and, for any* $x_0 \in \Upsilon$*, solution* $x(t)$ *of system* (7) *with the zero condition* $x(0) = x_0$ *tends to the equilibrium state* $x = 0$ *as* $t \to \infty$*.*

The proof of the theorem is similar to that for one-input affine systems given in [Pesterev, 2017].

Thus, if the comparison systems (10) are absolutely stable in sectors $(\beta_{i0}, 1]$, $i = 1, \ldots, m$, construction of an estimate of the attraction domain for the nonlinear system (7) reduces, basically, to (a) finding families of invariant sets of the comparison systems and (b) selecting from these families (possibly, with the use of some optimality criterion) those sets belonging to the domain of system (1) in which sector conditions (12) hold.

The first task is easily solved if Lyapunov functions of the comparison systems (10), (11) are known. Constructive solution of both tasks seems to be possible if we confine ourselves to ellipsoidal invariant sets. In this case, solution of the first task reduces to finding quadratic Lyapunov functions $\mathcal{L}_i(x^i) = (x^i)^{\mathrm{T}} P_i x^i$, where P_i is a positive definite matrix of order r_i, and invariant sets of the comparison systems are ellipsoids

$$\Omega_i(P_i) = \{x^i : (x^i)^{\mathrm{T}} P_i x^i \le 1\}. \tag{13}$$

The desired invariant set of the nonlinear system (7) is sought as the Cartesian product of invariant ellipsoids:

$$\Omega = \Omega_1(P_1) \times \cdots \times \Omega_m(P_m). \tag{14}$$

Further in the paper, we consider only ellipsoidal invariant sets of the comparison systems.

3.3 Invariant Ellipsoids of the Comparison Systems

Let us rewrite the ith equation in (10) when $r_i > 1$ in the equivalent matrix form

$$\dot{x}^i = A_{\beta_i}(t) x^i, \tag{15}$$

where

$$A_{\beta_i}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_i(t)c_{i1} & -\beta_i(t)c_{i2} & -\beta_i(t)c_{i3} & \cdots & -\beta_i(t)c_{ir_i} \end{bmatrix}. \tag{16}$$

Let $A_{\beta_{i0}}$ denote the constant matrix obtained from (16) by substituting $\beta_i(t) \equiv \beta_{i0}$. For $\beta_i(t) \equiv 1$, we obtain matrix A_i given by (4).

It is known [Boyd, 1994] that the sufficient condition of absolute stability of system (15) in a sector $[\beta_{i0}, 1]$ is existence of a common Lyapunov function for the linear systems $\dot{x}^i = A_i x^i$ and $\dot{x}^i = A_{\beta_{i0}} x^i$. In turn, a common quadratic Lyapunov function $\mathcal{L}_i = (x^i)^{\mathrm{T}} P_i x^i$ for these two systems exists if and only if the LMI system

$$P_i A_i + A_i^{\mathrm{T}} P_i < 0, \; P_i A_{\beta_{i0}} + A_{\beta_{i0}}^{\mathrm{T}} P_i < 0 \tag{17}$$

has a nontrivial solution [Boyd, 1994]. Hence, the sufficient condition of absolute stability of the linear nonstationary system (15) in a sector $[\beta_{i0}, 1]$ is existence of a positive definite matrix P_i satisfying inequalities (17). Ellipsoid (14) the matrix of which satisfies LMIs (17) will be referred to as an *invariant ellipsoid of the comparison system*. The desired matrices P_i are obtained independently for each comparison system.

For $r_i = 1$ (one-dimensional subsystem), the comparison system is absolutely stable for any $\beta_i(t)$ satisfying the condition $\beta_i(t) > 0$, and its invariant set is an arbitrary interval of axis x^i containing the zero point, with the invariant ellipsoid being degenerated into an interval symmetric with respect to the origin ($P_i > 0$ is a scalar).

In accordance with Theorem 1, in order that the Cartesian product Ω of ellipsoids (14) whose matrices are solutions of the LMIs (17) be an invariant set of the closed-loop system (7), it is sufficient that $\Omega \subseteq D_x$ and inequalities (12) hold in Ω. First, let us show that, if the right-hand sides of system (1) satisfy certain additional conditions, then the fulfillment of the sector inequalities (12) for $r_i > 1$ is ensured by adding one matrix inequality to each system (17).

3.4 Ensuring Fulfillment of Sector Conditions

Consider the functions $U_i(x) \equiv |g_i(x)|\bar{u}_i - |f_i(x)|$, $i = 1, \dots, m$, defined on D_x. The following assertion is valid.

Lemma 1 *Let Ω be the Cartesian product of the ellipsoids $\Omega_i \subseteq R^{r_i}$ such that $\Omega \subseteq D_x$ and the following conditions hold:*

$$U_i(x) > 0 \;\forall x \in \Omega. \tag{18}$$

Let also, for all the subsystems with $r_i > 1$, matrices P_i of the ellipsoids $\Omega_i(P_i)$ satisfy the LMIs

$$P_i \geq c_i c_i^{\mathrm{T}} \frac{\beta_{i0}^2}{U_{i0}^2}, \tag{19}$$

where $U_{i0} = \min_{x \in \Omega} U_i(x)$. Then, the sector conditions (12) hold in Ω.

The proof of the lemma is similar to that for one-input affine systems given in [Pesterev, 2017] and is omitted to save room.

The assumptions of the lemma are trivially fulfilled if $D_x = R^n$ and the lower bounds of the functions $U_i(x)$ in R^n are strictly positive. In this case, the desired ellipsoids $\Omega_i(P_i)$ are found by solving the LMIs (17), (19) for $i = 1, \dots, m$, with U_{i0} being the lower bounds of the functions $U_i(x)$ in R^n.

If conditions (18) do not hold in the entire R^n and/or $D_x \neq R^n$, then the fulfillment of the sector conditions in Ω can be ensured if ellipsoids Ω_i are constructed in regions $\Pi_i \subseteq R^{r_i}$ such that $\Pi \equiv \Pi_1 \times \cdots \times \Pi_m \subseteq D_x$ and conditions $U_i(x) > 0$, $i = 1, \dots, m$ hold for all $x \in \Pi$. Such regions Π_i exist by virtue of conditions (9) and continuity of functions $f_i(x)$, $g_i(x)$. In this case, one arrives at the problem of inscribing an ellipsoid into a given region Π_i.

3.5 Inscribing Ellipsoid into a Region

The problem of inscribing an ellipsoid into a given region Π_i is most easily solved when Π_i is convex and its boundary is formed by first- and/or second-order surfaces. In this case, the condition $\Omega_i(P_i) \subseteq \Pi_i$ can be written as an LMI system of the form [Pesterev, 2017]

$$l_j^i(P_i) \leq 0,\; j = 1, \dots, s_i, \tag{20}$$

where $l_j^i(P_i)$ is a linear form of matrix P_i. A region Π_i the belonging of an ellipsoid $\Omega_i(P_i)$ to which can be written as an LMI system (20) will be further referred to as a *region with simple boundary* (in what follows, the notation Π_i is used only for regions with simple boundary).

The selection of regions Π_i is generally not unique. A particular shape of the boundary of region Π_i is selected with regard to specifics of the problem under consideration and depends on the form of functions on the right-hand side of system (1). Without loss of generality, we assume that, for a fixed shape of region Π_i, its size is determined by one parameter α_i (all subsequent discussions are easily extended to the case where the size of the region is a function of several parameters). The minimum of function $U_i(x)$ in the general case depends on the sizes of all regions Π_i and is a monotonically nonincreasing function of α_i's: $U_{i0} = U_{i0}(\alpha_1, \dots, \alpha_m)$, $i = 1, \dots, m$.

3.6 An Algorithm to Construct Attraction Domain Estimates

From the above-said, it follows that an estimate of the attraction domain is constructed essentially in two stages. On the first stage, a subdomain $\Pi \subseteq D_x \subseteq R^n$ of the form $\Pi = \Pi_1 \times \cdots \times \Pi_m$ is constructed, where $\Pi_i \subseteq R^{r_i}$ are regions with simple boundaries in which conditions (18) hold. If $D_x = R^n$ and $\forall i\; U_{i0} = \inf_{x \in D_x} U_i(x) > 0$, we may set $\Pi_i = R^{r_i}$ and $\Pi = R^n$. As soon as such regions Π_i are constructed, the problem is decomposed into

m independent problems of constructing ellipsoidal estimates inscribed into the regions Π_i for the subsystems with scalar controls composing system (7), i.e., reduces to the problem considered in [Pesterev, 2017].

Recall how an estimate of the attraction domain for a system with scalar control is found when an approximation of the system domain by a region with a simple boundary is available (in the given case, these are regions Π_i, $i = 1, \ldots, m$). A value $\beta_{i0} \in (\beta_{i0}^*, 1]$ is taken, where $(\beta_{i0}^*, 1]$ is the greatest sector in which system (10) has a quadratic Lyapunov function the derivative of which by virtue of the system is negative for any $\beta_i(t)$ belonging to the sector. For small dimensions r_i and/or special choice of the feedback coefficients, β_{i0}^* can be found analytically. For instance, for $r_i = 1$, the comparison system is obviously absolutely stable for any positive functions $\beta_i(t)$; i.e., $\beta_{i0}^* = 0$. For $r_i > 1$, if matrix A_i has only one repeated eigenvalue $\lambda < 0$, β_{i0}^* does not depend on λ [Pesterev, 2016] and, in particular, for $r_i = 2$, $\beta_{i0}^* = 1/9$ [Pesterev, 2011]. Numerical estimation of the boundary of the stability sector in the general case of arbitrary distribution of eigenvalues of a Hurwitz matrix A_i presents no problem. To this end, it will suffice to solve the LMI system (17) for several values of β_{i0} and take the least of them for which the system had a solution to be β_{i0}^* [Pesterev, 2017]. Then, the system of LMIs (17), (19), (20) is solved, and the ellipsoid $\Omega_i(P_i)$ is constructed.

It should be noted that the second stage is not needed for one-dimensional ($r_i = 1$) subsystems. Indeed, in this case, the corresponding comparison system is absolutely stable for any positive $\beta_i(t)$, and any interval of axis x^i containing the zero point is its (positive) invariant set. On the other hand, a region with simple boundary Π_i in the one-dimensional case is an interval where condition (18) holds. Since condition (18) holds at any point of Π_i, any interval of axis x^i containing the zero point that belongs to Π_i (in particular, the entire interval Π_i) can be taken to be Ω_i. From the above discussion, it also follows that the estimate obtained does not depend on particular (positive) values of the feedback coefficients in the one-dimensional subsystems.

Let us see what the shape of the desired estimate Ω is for small n and $m > 1$ (for $m = 1$, Ω is an n-dimensional ellipsoid [Pesterev, 2017]). For $n = m = 2$ (a system with vector relative degree $\{1, 1\}$), "invariant ellipsoids" are intervals of two coordinate axes, so that the estimate of the attraction domain is a rectangle. For $n = 3$, there exist two normal forms with relative degrees $\{1, 1, 1\}$ ($m = 3$) and $\{2, 1\}$ ($m = 2$). In the former case, we have three one-dimensional subsystems, and the estimate is a parallelepiped. In the latter case (one- and two-dimensional subsystems), the estimate is an elliptic cylinder (Cartesian product of an ellipse and an interval). Note also that, for an arbitrary n and $m = n$, the estimate is an n-dimensional parallelepiped.

3.7 Finding an Optimal Estimate of the Attraction Domain

With each domain $\Omega = \Omega_1(P_1) \times \cdots \times \Omega_m(P_m)$, we associate a functional $F(\Omega)$ characterizing its "size" (e.g., volume of the domain) and pose the problem of finding ellipsoids $\Omega_1, \ldots, \Omega_m$ the Cartesian product of which is an invariant set of the nonlinear system under study such that the functional $F(\Omega) \equiv F(\Omega_1, \ldots, \Omega_m)$ takes its maximum value on this domain. Since matrices of the ellipsoid depend on the parameters U_{i0} andβ_{i0}, where $U_{i0} = U_{i0}(\alpha_1, \ldots, \alpha_m)$, $F(\Omega)$ is a function of $2m$ variables: $F(\Omega) \equiv F(\alpha_1, \ldots, \alpha_m, \beta_{10}, \ldots, \beta_{m0})$. Clearly, variables α_i in the general case cannot be selected independently from one another. Indeed, when any α_i increases, the domain Π_i (and, hence, the domain Π) also increases. As a result, the minimums of (generally, all) functions $U_j(x)$ in the general case decrease (even if the domain Π_j remains the same) and may become negative. Hence, α_i, $i = 1, \ldots, m$ may not vary independently; i.e., vector $\alpha \equiv [\alpha_1, \ldots, \alpha_m]^{\mathrm{T}}$ belongs to some domain $\mathcal{A} \subset R^m$ in the m-dimensional space of parameters α_i. Thus, the problem of finding the best estimate reduces to solving a constrained optimization problem for a function of $2m$ variables (the number of variables can be less than $2m$ if some subsystems composing the systems are one-dimensional, since $\beta_{i0} = 0$ for such subsystems) $F(\alpha_1, \ldots, \alpha_m, \beta_{10}, \ldots, \beta_{m0})$ under the constraints $\alpha \in \mathcal{A}$ and $\beta_{i0} \in [\beta_{i0}^*, 1]$, $i = 1, \ldots, m$. The values of the function $F(\alpha_1, \ldots, \alpha_m, \beta_{10}, \ldots, \beta_{m0})$ are determined as a result of solving m LMI systems (17), (19), (20).

Now, note that, except for the case of $m = n$, the estimate found by solving the above-described optimization problem is not, strictly speaking, the best estimate in the sense of the given performance index F. Indeed, since the LMI system (17), (19), (20) has infinitely many solutions, function $F(\alpha_1, \ldots, \alpha_m, \beta_{10}, \ldots, \beta_{m0})$ is defined ambiguously and its value depends on the criterion used in the LMI solver to select a single matrix P_i. Given the same values of the arguments of function F, the value of the function will be different if we use different solvers. Thus, when stating the problem of finding the "best" estimate of the attraction domain, it is required to specify not only the performance index but also the criterion used in the LMI solver for selecting the single solution. To emphasize this fact, we will write the functional F in the form $F_\varphi(\Omega) \equiv F_\varphi(\alpha_1, \ldots, \alpha_m, \beta_{10}, \ldots, \beta_{m0})$, where the subscript φ means that the solution of the LMI system outputted by the LMI solver used minimizes functional $\varphi(P_i)$ (in Matlab, functionals φ are allowed to be linear functions of entries of the desired matrix; e.g., matrix

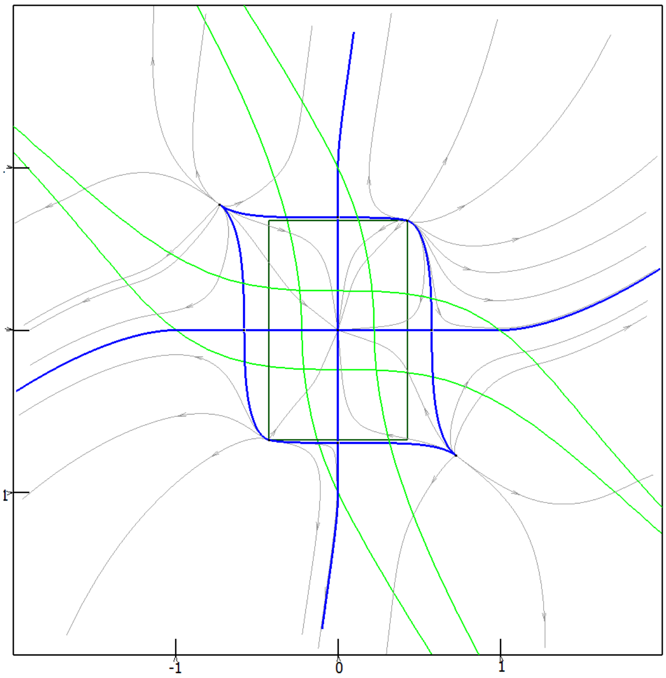

Figure 1: Phase portrait of the closed-loop system (21)–(23) and the best rectangular attraction domain estimate.

trace), and call the estimate obtained by maximizing the functional $F_\varphi(\alpha_1, \dots, \alpha_m, \beta_{10}, \dots, \beta_{m0})$ *quasi-optimal estimate* of the attraction domain in the sense of the criterion F. In other words, the estimate obtained by maximizing functional F_φ is the best estimate of the sought form in the sense of the criterion F under the condition that the solution of the LMIs (17), (19), (20) minimizes the functional φ. Determination of the global maximum seems to be unrealistic, since this requires searching for an infinite number of matrices P_i for any fixed set of arguments of function F.

In the case of $m = n$, we have n one-dimensional subsystems, and the desired estimate is an n-dimensional parallelepiped. As discussed above, construction of the estimate does not require solving LMIs, and, given a performance index F, finding the "best" estimate reduces to solving a conventional constrained optimization problem: find $\max F(\alpha)$ under the constraint $\alpha \in \mathcal{A}$.

4 Numerical Example

As an illustration, we consider a two-dimensional control systems with two constrained inputs represented in the normal form:

$$\dot{x}_1 = f_1(x) + u_1,\ \dot{x}_2 = f_2(x) + u_2,\ D_x = R^2, \tag{21}$$

where

$$f_1(x) = 3x_1^2 \operatorname{sign}(x_1) + x_2^2 \operatorname{sign}(x_2),\ f_2(x) = x_1^3 + 3x_2^3. \tag{22}$$

The open-loop system has one unstable equilibrium at the origin. Our goal is to stabilize the system at the origin by applying constrained controls $|u_1| \le 1$ and $|u_2| \le 1$ in the form of the feedback

$$u_i(x) = -\mathrm{sat}_1(f_i(x) + \mu_i x_i)),\ \mu_i > 0,\ i = 1, 2. \tag{23}$$

In accordance with the above-described algorithm, we need to find regions with simple boundaries Π_1 and Π_2 such that $U_i(x) > 0$, $i = 1, 2$, $\forall x \in \Pi = \Pi_1 \times \Pi_2$, where $U_1(x) = 1 - |3x_1^2 \operatorname{sign}(x_1) + x_2^2 \operatorname{sign}(x_2)|$ and $U_2(x) = 1 - |x_1^3 + 3x_2^3|$. Since the subsystems are one-dimensional, Π_1 and Π_2 are intervals of axes x_1 and x_2 containing the zero point (see Section 3.6), and Π is a rectangle. From the symmetry considerations, we set $\Pi_1 = \{x_1 : -a < x_1 < a\}$ and $\Pi_2 = \{x_2 : -b < x_2 < b\}$, $a, b > 0$. The minimums of functions $U_1(x)$ and $U_2(x)$ on such a set are achieved at the corners of the rectangle with the coordinates (a, b) and $(-a, -b)$. Substituting these values into the functions and equating the latter to zero, we obtain a system of two nonlinear equations, the solution of which is easily found numerically: $a = 0.426$ and $b = 0.675$. As shown in Section 3.6, the set of points belonging to the rectangle $(-a, a) \times (-b, b)$ is an invariant set of the system and can be taken to be an estimate of the attraction domain. It is not difficult to show that the estimate obtained is the best estimate in the considered class of (rectangular) estimates: the rectangle obtained has the largest square (as well as the greatest perimeter) among all invariant rectangles.

The phase portrait of the closed-loop system (with the feedback coefficients $\mu_1 = \mu_2 = 5.0$) is shown in Fig. 1. The bold blue lines show separatrices and the boundary of the attraction domain. The lines bounding the linearity regions of the susbsystems are depicted by the green bold curves. It can be seen from the figure that, in addition to the stable equilibrium at the origin, the closed-loop system has eight unstable equilibria: four saddle points at the intersections of the boundary of the attraction domain with the coordinate axes and four focuses at the "corners" of the attraction domain. The optimal invariant rectangle is depicted by the thin line. The figure demonstrates that the estimate obtained is a pretty good approximation of the attraction domain.

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