Equilibrium State Stabilization of the Hamilton Systems with Quality Estimate of Control⋆

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Abstract. The problem of stabilizing the zero solution of a nonautonomous Hamiltonian system with guaranteed estimation of the quality of control is solved. This problem arises from the problem of optimal stabilization by minimizing the requirements for the functional: instead of minimizing it, it is necessary only that it does not exceed a predetermined estimate. The solution is obtained by synthesizing the active program control, applied to the system, and stabilizing the control based on the feedback principle. The problem is solved analytically on the basis of the Lyapunov direct method of stability theory using the Lyapunov function with sign-constant derivatives. As examples, problems on the synthesis and stabilization of program motions of a homogeneous variable-length rod and a mathematical pendulum of variable length in a rotating plane are solved.

Keywords: controlled mechanical system, equilibrium state, limiting functions, stabilization, Lyapunov's function, quality estimate, computer modeling, math modeling

1 Introduction

The problems of controlled motions of mechanical systems are relevant and attract the attention of many researchers. The problem of constructing and investigating the properties and stability conditions of such motions was considered in the works of many scientists, for example [14, 16, 18, 20, 21]. As a rule, ensuring the asymptotic stability of the solution of the problem reduces to investigating the zero solution of the non-autonomous system [6] and is carried out on the basis of the direct Lyapunov method [17]. The method of limit systems [5] and its modification [3] allow using the Lyapunov functions with constant-sign derivatives to significantly expand the class of functions used to construct the

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desired controls in closed analytic form in the class of continuous functions. This method has proved itself well in solving problems of constructing given motions of mechanical systems, and with its help only at the Samara State Aerospace University a lot of problems were solved by a group of authors on the construction of asymptotically stable program motions: a rigid body on a moving platform [8], a double pendulum variable length with a movable suspension point [13], the arms of the robot manipulator [9], a free gyrostat with variable moments of inertia, depending on the time [7], spherical motions of the satellite a circular orbit [10] fixed-rotor motions gyrostat with polostyu1 filled with a viscous liquid [12].

In many problems, in addition to stabilizing movements, it is important to assess the quality of the transient process, given by some quality functional that requires optimization [15]. A wide range of researchers have found the problems of optimal stabilization, for example [1, 19]. In [4], a formulation was given and sufficient conditions were obtained for the problem of stabilizing the zero solution of non-autonomous systems with a guaranteed estimate of the quality of control arising from the optimal problem.

This method can be used for mathematical and computer modeling of economic systems.

2 Problem Formulation

The governed mechanical system is considered. Its motion is described by Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H(t,p,q)}{\partial p}, \\ \dot{p} &= -\frac{\partial H(t,p,q)}{\partial q} + u, \end{aligned} \tag{1}$$

where $q = (q_1, \ldots, q_n)^T$, n-dimensional vector of generalized coordinates in a real linear space R^n, with norm $\|q\|$, $p = (p_1, \ldots, p_n)^T$, n-dimensional vector of generalized linear momentum, $p \in R^n$ with norm $\|p\|$ and $H(t, q, p)$ is the Hamiltonian of the system, $u(t, q, p) \in U$ are forces of control action. Sign T means transpose operation.

It is assumed that system (1) has trivial solution $q = p = 0$ for $u \equiv 0$.

Let control quality estimation of this system be equal to

$$I = \int_{t_0}^{\infty} W(t, x[t], u[t])\, dt. \tag{2}$$

This is done for transferring process with control $u[t]$ and trajectory $x[t]$ of system (1).

Integrated function $W[t, x, u]$, defining quality control estimation (2) of process $x[t]$ is continuous non-negative function defined on whole motion area in general.

We will give three diverse types of system stabilization problem.

Problem of stabilization. The solution of the stabilization problem is to find such control actions $u = u^0(t, q, p)$ of all $u(t, q, p) \in U$. These effects ensure the asymptotic stability of the unperturbed motion $q = p = 0$ of the system (1).

The problem of optimal stabilization is obtained from the previous problem under the condition of a minimum of the quality criterion (2) and consists in finding such control actions$u = u^0(t, q, p)$ of all $u(t, q, p) \in U$. This control ensures asymptotical stability of non-perturbed motion $q = p = 0$ of the system (1).

An equation satisfied for any other control action $u = u^*(t, q, p) \in U$, ensuring asymptotical stability of solution $q = p = 0$:

$$I^0 = \int_{t_0}^{\infty} W(t, q^0[t], p^0[t], u^0[t])\, dt \leq \int_{t_0}^{\infty} W(t, q^*[t], p^*[t], u^*[t])\, dt = I^*$$

for $t_0 \geq 0$, $q^0(t_0) = q^*(t_0) = q_0$, $p^0(t_0) = p^*(t_0) = p_0$.

The problems of stabilization and optimal stabilization were posed by N. N. Krasovsky in work [15].

The problem of optimal stabilization of motion of a controlled system on an infinite time interval reduces to finding the optimal Lyapunov function and optimal control actions satisfying the Bellman partial differential equation. This equation must be solved taking into account the additional inequality. The result is a rather difficult task. In this paper, we study a variable formulation of the problem of stabilization of motion-stabilization with a guaranteed estimate of the quality of control. It arises from the problem of optimal stabilization when the requirement for the functional is weakened: it is not necessary to minimize it, it is only necessary that it does not exceed some estimate [4].

Definition. Control action $u = u^0(t, q, p)$ is called stabilizing with guaranteed estimate of the control quality $P(t, q, p)$ if it ensures asymptotical stability of non-perturbed motion $q = p = 0$ of system (1). The following condition satisfied for any controlled motions $q^0[t]$, $p^0[t]$, $q^0(t_0) = q_0$, $p^0(t_0) = p_0$:

$$I = \int_{t_0}^{\infty} W(t, q^0[t], p^0[t], u^0[t]) dt \leq P(t_0, q_0, p_0). \qquad (3)$$

According to this definition, we place *the problem of stabilization with a guaranteed estimate of the quality of control* of Hamiltonian systems: to find control action $u = u^0(t, q, p)$ among all $u(t, q, p) \in U$. The control action should ensure asymptotical stability of non-perturbed motion $q = p = 0$ of system (1). The inequality (3) inequality is satisfied for any controlled motion.

The above statement of the problem due to the weakening of the requirement of minimization of the functional (2) makes it possible to simplify the problem [15] and substantially extend the class of solvable problems in comparison with the problem of optimal stabilization.

Lyapunov's function $V = V(t, x)$ and Bellman function

$$B[V, t, x, u] = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T X(t, x, u) + W(t, x, u), \qquad (4)$$

where $x = (q, p)^T$ is still important for getting solution of this problem.

Variables q, p can be considered as deviations from trivial solution $q = p = 0$ because we are investigating stabilization problem of zero solution of system (1). Equations system (1) is equations of perturbed motion. This will allow us to apply to methods of stabilization with a guaranteed estimation of the quality of control of Hamiltonian systems methods and results developed to study the stability and stabilization of the zero equilibrium position of non-autonomous systems [4].

We will consider classical systems with Hamiltonian:

$$H = \frac{1}{2} p^T A p + U(t, q), \tag{5}$$

where $A = A(t, q)$ is the symmetric coefficient matrix of non-negative quadratic form with variables p. Equations of perturbed motion (1) will be written, considering the Hamiltonian (5):

$$\begin{cases} \dot{q} = Ap, \\ \dot{p} = -\frac{1}{2} p^T \frac{\partial A}{\partial q} p - \frac{\partial U}{\partial q} + u. \end{cases} \tag{6}$$

3 Derivation of Stabilizing Control

We assume that C is the positive-definite, symmetric, not disappearing, limited matrix:

$$A_0 E \leq C \leq A_1 E, \quad 0 < A_0 < A_1 - const,$$

where E is the identity matrix.

Positive-definite on variables p and q Lyapunov's function is considered [11]. It admits of an infinitesimal upper bound.

$$V(t, q, p) = \frac{1}{2} p^T A p + \frac{1}{2} q^T C q. \tag{7}$$

The total derivative on time of Lyapunov's function by system (6) will take the form:

$$\begin{aligned} \dot{V} &= \left(\frac{\partial V}{\partial p} \right)^T \dot{p} + \left(\frac{\partial V}{\partial q} \right)^T \dot{q} + \frac{\partial V}{\partial t} = \\ &= \left(\frac{\partial V}{\partial p} \right)^T \cdot \left[-\frac{1}{2} p^T \frac{\partial A}{\partial q} p - \frac{\partial U}{\partial q} + u \right] + \left(\frac{\partial V}{\partial q} \right)^T \cdot (Ap) + \frac{\partial V}{\partial t}. \end{aligned} \tag{8}$$

We choose stabilizing control in form:

$$u^0 = \frac{\partial U}{\partial q} - Cq - A^{-1} D p, \tag{9}$$

where $D = D(t)$ is the symmetrical, positive-definite, $n \times n$ matrix. If we will add forces (9) to system (6), we have equations of controlled motion:

$$\begin{cases} \dot{q} = Ap, \\ \dot{p} = -\frac{1}{2}p^T \frac{\partial A}{\partial q} p - Cq - A^{-1}Dp. \end{cases} \tag{10}$$

Wherein the total derivative on time (8) of Lyapunov's function by system (10) will take the form:

$$\begin{aligned} \dot{V} = p^T A^T \left[-\frac{1}{2} p^T \frac{\partial A}{\partial q} p - \frac{\partial U}{\partial q} + \left\{ \frac{\partial U}{\partial q} - Cq - A^{-1}Dp \right\} \right] + \\ + \left(\left[\frac{1}{2} p^T \frac{\partial A}{\partial q} p \right]^T - q^T C^T \right) Ap + \frac{1}{2} p^T \frac{\partial A}{\partial t} p + \frac{1}{2} q^T \frac{\partial C}{\partial t} q = \\ = p^T A^T \left[-\frac{1}{2} p^T \frac{\partial A}{\partial q} p \right] + \left[\frac{1}{2} p^T \frac{\partial A}{\partial q} p \right]^T Ap + p^T A^T (-Cq) - \\ - p^T A^T A^{-1} Dp + q^T C^T Ap + \frac{1}{2} p^T \frac{\partial A}{\partial t} p + \frac{1}{2} q^T \frac{\partial C}{\partial t} q \end{aligned}$$

We discard the terms of the third order on variables p:

$$\dot{V} \approx -p^T \left(D - \frac{1}{2} \frac{\partial A}{\partial t} \right) p + \frac{1}{2} q^T \frac{\partial C}{\partial t} q \tag{11}$$

4 Basic Results

Basic results about stabilization of the Hamilton systems with quality estimate of zero solution $q = p = 0$ are the following statements.

Statement 1. *Let's suppose:*

1. *A and C do not depend on t. That is, the conditions $\frac{\partial A}{\partial t} = 0$, $\frac{\partial C}{\partial t} = 0$ are satisfied;*
2. *$B[V, t, p, q, u^0(t, p, q)] \leq 0$.*

Then control (9) *is stabilizing for motion $q = p = 0$ of system* (6) *with the guaranteed quality estimate:*

$$P(t_0, q_0, p_0) = V(t_0, q_0, p_0) = \frac{1}{2} p_0^T A(q_0) p_0 + \frac{1}{2} q_0^T C q_0. \tag{12}$$

Proof. We chose control in the form (9) for function $V(t, q, p)$ (7) according to conditions $\partial A/\partial t = 0$, $\partial C/\partial t = 0$ and (11), then:

$$\dot{V} \approx -p^T Dp \tag{13}$$

The function V is positive-definite on variables q, p and the derivative $\dot{V}$, according to (13), is negative-definite only on variable p. Then the solution $q =$

$p = 0$ is asymptotically stable by the theorem on asymptotic stability zero solution of non-autonomous system [3]. The matrixes A, C are limited. $p \to 0$, $q \to 0$ are satisfied under $t \to \infty$. Then we have estimate for Lyapunov's function V:

$$V(t, q(t), p(t)) = \frac{1}{2}p^T Ap + \frac{1}{2}q^T Cq \to 0.$$

According to (4), we have:

$$B = \dot{V} + W \tag{14}$$

We have $\dot{V} + W \leq 0$ or $W \leq -\dot{V}$, according to 2. We integrate this expression into the interval $[t_0, t]$:

$$\int_{t_0}^{T} W dt \leq V(t_0, q_0, p_0) - V(t, q, p) \tag{15}$$

Further, we take the limit by $t \to \infty$:

$$\int_{t_0}^{\infty} W dt \leq V(t_0, q_0, p_0) = P(t_0, q_0, p_0).$$

The statement is proved. □

Statement 2. *Let's suppose:*

1. $\frac{\partial A}{\partial t} = 0$, $\frac{\partial C}{\partial t} < 0$,
2. $B(V, t, p, q, u^0(t, p, q)) \leq 0$.

Then the control (9) *is stabilizing for motion* $q = p = 0$ *of system* (6) *with the guaranteed quality estimate* (12).

Remark 1. The proof of statement 2 is like the proof of statement 1. The difference is that derivative of the function (7), according to (11) and conditions $\frac{\partial C}{\partial t} < 0$ is negative-definite on variables q, p. Then the solution $q = p = 0$ is asymptotically stable by the Lyapunov's theorem on asymptotic stability.

Statement 3. *Let's suppose the conditions are satisfied:*

1. $\frac{\partial A}{\partial t} \neq 0$, $\frac{\partial C}{\partial t} = 0$,
2. $D > \frac{1}{2}\frac{\partial A}{\partial t}$,
3. $B(V, t, p, q, u^0(t, p, q)) \leq 0$.

Then the control (9) *is stabilizing for motion* $q = p = 0$ *of system* (6) *with the guaranteed quality estimate* (12).

Statement 4. *Let's suppose the conditions are satisfied:*

1. $\frac{\partial A}{\partial t} \neq 0$, $\frac{\partial C}{\partial t} < 0$,

2. $D > \frac{1}{2}\frac{\partial A}{\partial t}$,
3. $B(V, t, p, q, u^0(t, p, q)) \leq 0$.

Then the control (9) *is stabilizing for motion* $q = p = 0$ *of system* (6) *with the guaranteed quality estimate* (12).

Remark 2. The proofs of statements 3, 4 are like the proofs of statements 1, 2 respectively.

5 Stabilization of Homogeneous Rod's Motion

We consider a homogeneous heavy rod. The rod length is variable. A rod mass is $m = 1$. The rod moves without friction in the plane O_{xy}. The plane O_{xy} rotates with a constant angular velocity ω around a fixed vertical axis O_y (Fig. 1).

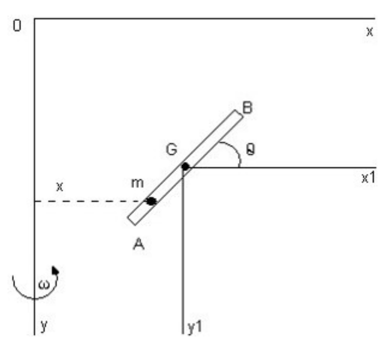

Fig. 1. The rod in a rotate plane

Let's suppose the ε, η are center mass coordinates of rod. The angle θ is the deviation angle of the rod from the horizontal. Suppose that the rod length changes according to law $k = k(t) = a + b\cos t$, $a = const > b = const > 0$. The system has three degrees of freedom.

The Hamiltonian of system takes form:

$$H(\varepsilon, \eta, \theta) = \frac{1}{2}(\dot{\varepsilon}^2 + \dot{\eta}^2 + k^2(t)\dot{\theta}^2 + \omega^2 k^2(t)\cos^2\theta + \omega^2\varepsilon^2) - g\eta$$

We chose program motion. Suppose that center mass of system moves on the circle with radius $R = 1$. The center of a circle lies in the point O (2, 2). The rod rotates in the plane O_{xy} around the center mass with constant angular velocity $\omega_0 = const$:

$$\begin{cases} \varepsilon^* = \cos t + 2, \\ \eta^* = \sin t + 2, \\ \theta^* = \omega_0 t. \end{cases}$$

We construct equations of controlled rod motion. We obtained the equations of perturbed motion by introducing deviations:

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2, \\ \dot{q}_3 = \frac{p_3}{k^2}, \\ \dot{p}_1 = \omega^2 q_1, \\ \dot{p}_2 = 0, \\ \dot{p}_3 = -\omega^2 k^2 \sin q_3 \cos(q_3 + 2\omega_0 t). \end{cases}$$

Suppose that quality estimate of transition process is given by the functional (2) with integrand

$$W(t, q, p) = d_1 p_1^2 + d_2 p_2^2.$$

It satisfies condition 3 of statement 3. We set the problem of stabilization zero solution $q = p = 0$ of disturbed motion set of equations with guaranteed quality estimate of the control for the proposed motion.

We chose the Lyapunov's function:

$$V = \frac{1}{2}\left(p_1^2 + p_2^2 + \frac{1}{k^2(t)} p_3^2 + A_1 q_1^2 + A_2 q_2^2 + A_3 q_3^2\right).$$

The control is chosen in form (9):

$$\begin{cases} c u_1 = -\omega^2 q_1 - c_1 q_1 - d_1 p_1, \\ u_2 = -c_2 q_2 - d_2 p_2, \\ u_3 = \omega^2 k^2 \sin q_3 \cos(q_3 + 2\omega_2 t) - c_3 q_3 - k^2 d_3 p_3. \end{cases}$$

Equations of motion of the stabilized system are obtained:

$$\begin{cases} l\dot{q}_1 = p_1, \\ \dot{q}_2 = p_2, \\ \dot{q}_3 = \frac{p_3}{k^2}, \\ \dot{p}_1 = -c_1 q_1 - d_1 p_1, \\ \dot{p}_2 = -c_2 q_2 - d_2 p_2, \\ \dot{p}_3 = -c_3 q_3 - k^2 d_3 p_3. \end{cases}$$

We have asymptotically stable solution $q = p = 0$ with guaranteed quality estimate by statement 3:

$$P(t_0, q_0, p_0) = V(t_0, q_0, p_0) = \frac{1}{2}\left(p_{10}^2 + p_{20}^2 + \frac{1}{k^2(t_0)} p_{30}^2 + A_1 q_{10}^2 + A_2 q_{20}^2 + A_3 q_{30}^2\right).$$

6 Conclusion

In this paper, the problem of determining the control stabilizing the motion of a mechanical system described by Hamilton's equations is formulated and solved with the additional condition of finding an assured estimate of the quality of control. The solution of the problem is reduced to the investigation of the zero solution of the nonautonomous system and was carried out on the basis of the direct Lyapunov method with the use of the method of limit systems, which made it possible to use the Lyapunov functions with sign-constant derivatives to construct the desired control in a closed analytic form in the class of continuous functions. Four statements that solve the problem are formulated and proved. Based on the results obtained, two illustrative examples are solved. The results of the work develop and generalize the corresponding results of [2, 4, 6, 11].

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