Cyclic descent extensions and distributions

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Abstract

The notion of descent set is classical both for permutations and for standard Young tableaux (SYT). Cellini introduced a natural notion of cyclic descent set for permutations, and Rhoades introduced such a notion for SYT, but only of rectangular shapes. In this paper, we describe cyclic descents for SYT of various other shapes. Motivated by these findings, we define cyclic extensions of descent sets in a general context, and we show that they exist for SYT of almost all shapes. Finally, we introduce the ring of cyclic quasisymmetric functions and apply it to analyze the distributions of cyclic descents over permutations and SYT.

1 Introduction

The notion of descent set, for permutations as well as for standard Young tableaux (SYT), is classical. Cellini introduced a natural notion of *cyclic descent set* for permutations, and Rhoades introduced such a notion for SYT — but only for rectangular shapes.

In [ER17a] and [AER18] we defined cyclic descent set maps for SYT of various shapes; see Theorem 3.2 below and the remark following it. Motivated by these results, *cyclic extensions* of descent sets have been defined in a general context and shown to exist for SYT of almost all shapes [ARR17]; see Theorem 4.1 below. The proof applies nonnegativity properties of Postnikov's toric Schur polynomials, providing a new interpretation of certain Gromov-Witten invariants. Finally, the ring of cyclic quasisymmetric functions has been introduced and studied in [AGRR+18], and further applied to analyze the resulting cyclic Eulerian distributions.

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2 Background

The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group $\mathfrak{S}_n$ on n letters is defined as

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$. For example, $\mathrm{Des}([2,1,4,5,3]) = \{1,4\}$. Its *cyclic descent set* was defined by Cellini [Cel98] as

$$\mathrm{cDes}(\pi) := \{1 \le i \le n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n], \tag{1}$$

with the convention $\pi_{n+1} := \pi_1$. For example, $\mathrm{cDes}([2,1,4,5,3]) = \{1,4,5\}$. This cyclic descent set was further studied by Dilks, Petersen, Stembridge [DPS09] and others. It has the following important properties. Consider the two actions of the cyclic group $\mathbb{Z}$, on $\mathfrak{S}_n$ and on the power set of $[n]$, in which the generator p of $\mathbb{Z}$ acts by

$$\begin{aligned} [\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n] &\overset{p}{\longmapsto} [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}], \\ \{i_1, \ldots, i_k\} &\overset{p}{\longmapsto} \{i_1 + 1, \ldots, i_k + 1\} \bmod n. \end{aligned}$$

Then for every permutation π, one has the following three properties:

$$\mathrm{cDes}(\pi) \cap [n-1] = \mathrm{Des}(\pi) \qquad \text{(extension)} \tag{2}$$

$$\mathrm{cDes}(p(\pi)) = p(\mathrm{cDes}(\pi)) \qquad \text{(equivariance)} \tag{3}$$

$$\varnothing \subsetneq \mathrm{cDes}(\pi) \subsetneq [n] \qquad \text{(non-Escher)} \tag{4}$$

The term *non-Escher* refers to M. C. Escher's drawing "Ascending and Descending", which paradoxically depicts the impossible cases $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$.

There is also an established notion of descent set for a *standard (Young) tableau* (SYT) T of a skew shape λ/μ:

$$\mathrm{Des}(T) := \{1 \le i \le n-1 \,:\, i+1 \text{ appears in a lower row of } T \text{ than } i\} \quad \subseteq [n-1].$$

For example, the following SYT T of shape $\lambda/\mu = (4,3,2)/(1,1)$ has $\mathrm{Des}(T) = \{2,3,5\}$:

	1	2	7
	3	5	
4	6		

For the special case of an SYT T of *rectangular* shape, Rhoades [Rho10, Lemma 3.3] introduced a notion of *cyclic descent set* $\mathrm{cDes}(T)$, possessing the above properties (2), (3) and (4) with respect to the $\mathbb{Z}$-action in which the generator p acts on tableaux via Schützenberger's *jeu-de-taquin promotion* operator. A similar concept of $\mathrm{cDes}(T)$ and accompanying action p was introduced for two-row shapes and certain other skew shapes (see Subsection 3.2 for the list) in [AER18, ER17a], and used there to answer Schur positivity questions.

3 Cyclic descents: definition and examples

3.1 Definition

Let us begin by formalizing the concept of a cyclic extension. Recall the bijection $p : 2^{[n]} \longrightarrow 2^{[n]}$ induced by the cyclic shift $i \mapsto i+1 \pmod n$, for all $i \in [n]$.

Definition 3.1. Let $\mathcal{T}$ be a finite set. A *descent map* is any map $\mathrm{Des} : \mathcal{T} \longrightarrow 2^{[n-1]}$. A *cyclic extension* of Des is a pair (cDes, p), where $\mathrm{cDes} : \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,

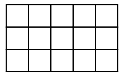
$$\begin{aligned} \text{(extension)} \quad & \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\ \text{(equivariance)} \quad & \mathrm{cDes}(p(T)) = p(\mathrm{cDes}(T)), \\ \text{(non-Escher)} \quad & \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n]. \end{aligned}$$

The non-Escher axiom is used to prove the (essential) uniqueness of the cyclic extension.

3.2 Examples

Cyclic extensions of descent maps have been given previously in several cases:

- For $\mathcal{T} = \mathfrak{S}_n$, the descent set $\mathrm{Des}(\pi)$ of a permutation π was described in Section 2, as was Cellini's original cyclic extension (cDes, p). Note that $n \geq 2$ is required for the non-Escher property.
- Let $\mathcal{T} = \mathrm{SYT}(\lambda)$ where $\lambda = (a^b)$ has *rectangular* shape, e.g.

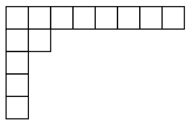
  $\lambda = (5^3) =$ 

  Consider the usual notion of descent set $\mathrm{Des}(T)$ on standard tableaux, as in Section 2. As mentioned earlier, Rhoades [Rho10, Lemma 3.3] showed that Schützenberger's jeu-de-taquin promotion operation p provides a cyclic extension (cDes, p). Again, we require $a, b \geq 2$ for the non-Escher property.

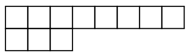
New examples are given in [AER18].

Theorem 3.2.

1. *Let* $\mathcal{T} = \mathrm{SYT}(\lambda)$ *where* λ *is a* hook plus internal corner, *namely* $\lambda = (n-2-k, 2, 1^k)$ *for* $0 \leq k \leq n-4$*; e.g.,*

   $\lambda = (8, 2, 1, 1, 1) =$ 

   There is a unique cyclic descent map cDes *on* $\mathcal{T}$*, defined as follows: by the extension property, it suffices to specify when* $n \in \mathrm{cDes}(T)$*, and one decrees this to hold if and only if the entry* $T_{2,2} - 1$ *lies strictly west of* $T_{2,2}$ *(namely, in the first column of* T*); see [AER18] for more details. Note that for most shapes in this family there are several possible shifting bijections* p*, so that the cyclic extension* (cDes, p) *is not unique.*
2. *Let* $\mathcal{T} = \mathrm{SYT}(\lambda)$ *with* λ *of* two-row *shape, namely* $\lambda = (n-k, k)$ *for* $2 \leq k \leq n/2$*; e.g.,*

   $\lambda = (8, 3) =$ 

   There exists a cyclic extension of Des *defined as follows; see [AER18]. Decree that* $n \in \mathrm{cDes}(T)$ *if and only if both*

   - *the last two entries in the second row of* T *are consecutive, and*
   - *for every* $1 < i < k$*,* $T_{2,i-1} > T_{1,i}$*.*

Other examples involve direct sums of shapes. Given t (skew) shapes $\nu^1, \ldots, \nu^t$, the *direct sum* $\nu^1 \oplus \nu^2 \oplus \cdots \oplus \nu^t$ is the skew shape consisting of t diagrams of shapes $\nu^1, \ldots, \nu^t$, for each i the diagram of ν^{i+1} is strictly northeast of the diagram of ν^i, with no rows or columns in common.

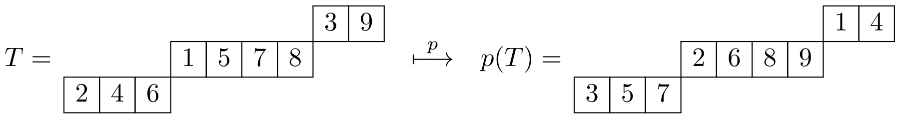
- Given any (strict) *composition* α of n, that is, an ordered sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ with $\sum_i \alpha_i = n$, consider the associated *horizontal strip* skew shape

  $$\alpha^{\oplus} := (\alpha_1) \oplus (\alpha_2) \oplus \cdots \oplus (\alpha_t)$$

  whose rows, from southwest to northeast, have sizes $\alpha_1, \ldots, \alpha_t$. For T in $\mathcal{T} = \mathrm{SYT}(\alpha^{\oplus})$, we define

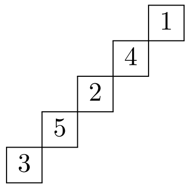
  $$\mathrm{cDes}(T) := \{1 \leq i \leq n : i+1 \text{ is in a lower row than } i\},$$

  where $n+1$ is interpreted as 1, as well as a bijection $p : \mathrm{SYT}(\alpha^{\oplus}) \to \mathrm{SYT}(\alpha^{\oplus})$ which first replaces each entry j of T by $j+1 \pmod{n}$ and then re-orders each row to make it left-to-right increasing. One can check that this (cDes, p) is a cyclic extension of Des, with $t \geq 2$ required for the non-Escher property. For example, when $\alpha = (3,4,2)$ (and $n = 9$), one has the following standard tableaux T of shape $\alpha^{\oplus}$:



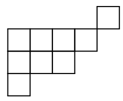
$$\text{cDes}(T) = \{1,3,5,9\} \overset{p}{\longmapsto} \text{cDes}(p(T)) = \{1,2,4,6\}$$

This generalizes the case of (cDes, p) on $\mathfrak{S}_n$, since for $\alpha = (1^n) = (1,1,\ldots,1)$ one has a bijection $\mathfrak{S}_n \to \text{SYT}(\alpha^{\oplus})$ which sends a permutation w to the tableau whose entries are $w^{-1}(1), \ldots, w^{-1}(n)$ read from southwest to northeast; e.g., for $n = 5$,

$w = [5,3,1,4,2] \longmapsto$ 

This bijection maps Cellini's cyclic extension (cDes, p) on $\mathfrak{S}_n$ to the one on SYT($\alpha^{\oplus}$) for $\alpha = (1,1,\ldots,1)$ defined above[1].

- Furthermore, for *any* nonempty partition $\lambda \vdash n-1$, the partition $\lambda \oplus (1)$, e.g.

$\lambda = (4,3,1) \oplus (1) =$ 

has an explicit cyclic extension of Des described in [ER17a]. This case was, in fact, our original motivation, and the question of existence of a cyclic extension of Des on SYT(λ/μ) appears there as [ER17a, Problem 5.5].

4 Cyclic descents for standard Young tableaux

Our first main result is a necessary and sufficient condition for the existence of a cyclic extension cDes of the descent map Des on the set SYT(λ/μ) of standard Young tableaux of shape λ/μ, with an accompanying $\mathbb{Z}$-action on SYT(λ/μ) via an operator p, satisfying properties (2), (3) and (4). In this story, a special role is played by the skew shapes known as *ribbons* (connected skew shapes containing no 2×2 rectangle), and in particular *hooks* (straight ribbon shapes, namely $\lambda = (n-k, 1^k)$ for $k = 0,1,\ldots,n-1$). Early versions of [AER18] and [ER17a] conjectured the following result.

Theorem 4.1. *[ARR17, Theorem 1.1] Let λ/μ be a skew shape. The descent map* Des *on* SYT(λ/μ) *has a cyclic extension* (cDes, p) *if and only if λ/μ is not a connected ribbon. Furthermore, for all $J \subseteq [n]$, all such cyclic extensions share the same cardinalities* $\#\text{cDes}^{-1}(J)$.

Our strategy for proving Theorem 4.1 is inspired by a result of Gessel [Ges84, Theorem 7] that we recall here. For a subset $J = \{j_1 < \ldots < j_t\} \subseteq [n-1]$, the composition (of n)

$$\alpha(J,n) := (j_1, j_2 - j_1, j_3 - j_2, \ldots, j_t - j_{t-1}, n - j_t) \tag{5}$$

defines a *connected ribbon* having the entries of $\alpha(J,n)$ as row lengths, and thus an associated *(skew) ribbon Schur function*

$$s_{\alpha(J,n)} := \sum_{\varnothing \subseteq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha(I,n)} \tag{6}$$

[1]This cyclic descent map can be further generalized to *strips*, which are the disconnected shapes each of whose connected components consists of either one row or one column; see [AER18].

with the following property: for any skew shape λ/μ, the descent map $\mathrm{Des}: \mathrm{SYT}(\lambda/\mu) \longrightarrow 2^{[n-1]}$ has fiber sizes given by

$$\#\mathrm{Des}^{-1}(J) = \langle s_{\lambda/\mu}, s_{\alpha(J,n)}\rangle \qquad (\forall J \subseteq [n-1]), \tag{7}$$

where $\langle -, - \rangle$ is the usual inner product on symmetric functions.

By analogy, for a subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ we define the corresponding *cyclic composition* of n as

$$\alpha^{\mathrm{cyc}}(J,n) := (j_2 - j_1, \ldots, j_t - j_{t-1}, j_1 + n - j_t), \tag{8}$$

with $\alpha^{\mathrm{cyc}}(J,n) := (n)$ when $J = \{j_1\}$; note that $\alpha^{\mathrm{cyc}}(\varnothing, n)$ is not defined. The corresponding *affine (or cyclic) ribbon Schur function* is then defined as

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J\setminus I)} h_{\alpha^{\mathrm{cyc}}(I,n)}. \tag{9}$$

We then collect enough properties of this function to show that there must exist a map $\mathrm{cDes}: \mathrm{SYT}(\lambda/\mu) \to 2^{[n]}$ and a $\mathbb{Z}$-action p on $\mathrm{SYT}(\lambda/\mu)$, as in Theorem 4.1, such that fiber sizes are given by

$$\#\mathrm{cDes}^{-1}(J) = \langle s_{\lambda/\mu}, \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}\rangle \qquad (\forall\ \varnothing \subsetneq J \subsetneq [n]). \tag{10}$$

The nonnegativity of this inner product when λ/μ is not a connected ribbon ultimately relies on relating $\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}$ to a special case of Postnikov's *toric Schur polynomials*, with their interpretation in terms of *Gromov-Witten invariants* for Grassmannians [Pos05].

5 Cyclic quasisymmetric functions

5.1 The ring of cyclic quasisymmetric functions

Recall from [Ges84] the following basic definitions: A *quasi-symmetric function* is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $i'_1 < \cdots < i'_t$ of positive integers, and any sequence $(m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal. Denote by QSym the set of all quasi-symmetric functions, and by QSym_n the set of all quasi-symmetric functions which are homogeneous of degree n.

The *fundamental quasi-symmetric function* corresponding to a subset $J \subseteq [n-1]$ is defined by

$$F_{n,J} := \sum x_{i_1} \cdots x_{i_n},$$

where the sum extends over all sequences $(i_1, \ldots, i_n)$ of positive integers such that $j \in J \Rightarrow i_j < i_{j+1}$ and $j \notin J \Rightarrow i_j \leq i_{j+1}$. The set $\{F_{n,J} : J \subseteq [n-1]\}$ forms a basis for the additive abelian group QSym_n.

The cyclic analogues of these concepts are introduced in [AGRR+18].

Definition 5.1. A *cyclic quasi-symmetric function* is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $i'_1 < \cdots < i'_t$ of positive integers, any sequence $m = (m_1, \ldots, m_t)$ of positive integers, and any cyclic shift $m' = (m'_1, \ldots, m'_t)$ of m, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m'_1} \cdots x_{i'_t}^{m'_t}$ in f are equal.

Denote by cQSym the set of all cyclic quasi-symmetric functions, and by cQSym_n the set of all cyclic quasi-symmetric functions which are homogeneous of degree n.

Observation 5.2. QSym, cQSym and the set Sym of symmetric functions (sometimes denoted Λ) are graded abelian groups satisfying

$$\mathrm{Sym} \subsetneq \mathrm{cQSym} \subsetneq \mathrm{QSym} \tag{11}$$

It is not too difficult to check that they are also rings, that is, closed under the multiplication operation on power series. For Sym, QSym this is well-known; the proof for cQSym is given in [AGRR+18].

5.2 Fundamental cyclic quasisymmetric functions

Definition 5.3. For each subset $J \subseteq [n]$ denote by $P^{\text{cyc}}_{n,J}$ the set of all pairs (w, k) consisting of a word $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ and an index $k \in [n]$ which satisfy

(i) The word w is "cyclically weakly increasing" from index k, namely $w_k \le w_{k+1} \le \ldots \le w_n \le w_1 \le \ldots \le w_{k-1}$.

(ii) If $j \in J$ then $w_j \neq w_{j+1}$, where indices are computed modulo n. (This condition is vacuous for $J = \varnothing$.)

Remark 5.4. The index k is uniquely determined by the word w, unless all the letters of w are equal; in which case, any index $k \in [n]$ will do. These "constant words" are relevant only for $J = \varnothing$, and the definition implies that each of them is counted n times (and not just once) in $P^{\text{cyc}}_{n,\varnothing}$.

Example 5.5. Let $n = 5$ and $J = \{1, 3\}$. The pairs $(12345, 1)$, $(23312, 4)$ and $(23122, 3)$ are in $P^{\text{cyc}}_{5,\{1,3\}}$ (see Figure 1), but the pairs $(12354, 1)$, $(22312, 4)$ and $(23112, 3)$ are not.

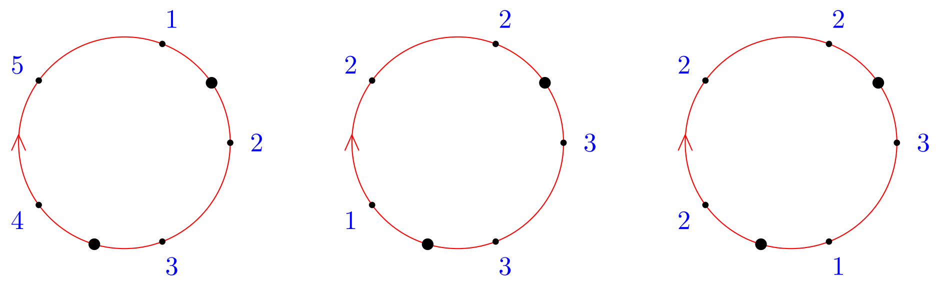


Figure 1: The pairs $(12345, 1)$, $(23312, 4)$ and $(23122, 3)$ in $P^{\text{cyc}}_{5,\{1,3\}}$.

Let $D_{n,J} := \{i \in \mathbb{Z}/n\mathbb{Z} \,:\, J + i \equiv J \pmod n\}$ be the stabilizer of J under the action of $\mathbb{Z}/n\mathbb{Z}$ by cyclic shifts, and let $d_{n,J} := \# D_{n,J}$.

Definition 5.6. The *fundamental cyclic quasisymmetric function* corresponding to a subset $J \subseteq [n]$ is defined by

$$F^{\text{cyc}}_{n,J} := d^{-1}_{n,J} \sum_{(w,k) \in P^{\text{cyc}}_{n,J}} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

Let $2^{[n]}_0$ be the set of all *nonempty* subsets of $[n]$, and let $c2^{[n]}_0$ be the set of orbits of elements of $2^{[n]}_0$ under cyclic shifts. If J and J' are in the same orbit then $F^{\text{cyc}}_{n,J} = F^{\text{cyc}}_{n,J'}$.

Theorem 5.7. *The set* $\{F^{\text{cyc}}_{n,J} \,:\, J \in c2^{[n]}_0\}$ *is a* $\mathbb{Z}$*-basis for* cQSym_n.

Furthermore, letting $s_{\lambda/\mu}$ be the skew Schur function indexed by a non-ribbon shape λ/μ, we have

$$\sum_{T \in \text{SYT}(\lambda/\mu)} F^{\text{cyc}}_{n,\text{cDes}(T)} = s_{\lambda/\mu}. \tag{12}$$

6 Cyclic Eulerian distributions

The distribution of descents on sets of permutations and other combinatorial objects, known as the *Eulerian distribution*, has been studied extensively; see, e.g., [BBS09], [Pet15, p. 91] and references therein. In this section we study the distribution of cyclic descents over SYT and compare it to its distribution over permutations.

6.1 Univariate generating functions

The *descent number* is the size of the descent set. For any skew shape λ/μ of size n there is a known expression [Sta99, equation (7.96)] for the generating function of the descent number, des, on standard Young tableaux of shape λ/μ:

$$\sum_{T\in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{des}(T)} = (1-t)^{n+1}\sum_{m\geq 0} s_{\lambda/\mu}(1^{m+1})t^m. \tag{13}$$

Here $s_{\lambda/\mu}(1^m)$ is the specialization of the skew Schur function $s_{\lambda/\mu}(x_1,x_2,\ldots)$ given by $x_1=\ldots=x_m=1$ and $x_{m+1}=\ldots=0$. Note that when $\mu=\varnothing$ this becomes even more explicit, through the *hook-content formula* [Sta99, Cor. 7.21.4] for the specialization $s_\lambda(1^m)$. In particular, for the skew shape $(1)^{\oplus n}$ this gives the well-known *Carlitz formula* for the *Eulerian distribution* on $\mathfrak{S}_n$:

$$\mathfrak{S}_n^{\mathrm{des}}(t) := \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)} = (1-t)^{n+1}\sum_{m\geq 0}(m+1)^n t^m \tag{14}$$

An analogous expression for the *cyclic descent number* cdes is proved in [ARR17].

Corollary 6.1. *For any skew shape λ/μ of size n which is not a connected ribbon,*

$$\sum_{T\in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{cdes}(T)} = n(1-t)^{n}\sum_{m\geq 1} s_{\lambda/\mu}(1^{m})\frac{t^m}{m}. \tag{15}$$

In particular, for the skew shape $(1)^{\oplus n}$ this gives

$$\mathfrak{S}_n^{\mathrm{cdes}}(t) := \sum_{w\in\mathfrak{S}_n} t^{\mathrm{cdes}(w)} = n(1-t)^{n}\sum_{m\geq 1} m^{n-1} t^m = nt\,\mathfrak{S}_{n-1}^{\mathrm{des}}(t) \qquad (n\geq 2). \tag{16}$$

For two-row shapes, [ARR17, Lemma 2.4] is applied in [AER18] to deduce the following.

Theorem 6.2. *For any $2\leq k\leq n/2$,*

$$\sum_{T\in \mathrm{SYT}((n-k,k))} t^{\mathrm{cdes}(T)} = \sum_{d=1}^{k}\frac{n}{d}\left[\binom{k-1}{d-1}\binom{n-k-1}{d-1} - \binom{k-2}{d-1}\binom{n-k}{d-1}\right]t^d.$$

6.2 Multivariate generating functions

Next we compare the distribution of cDes on $\mathrm{SYT}(\lambda)$ to the distribution of cDes on $\mathfrak{S}_n$. Recall [Sag01, Theorem 3.1.1 and §5.6 Ex. 22(a)] that the *Robinson-Schensted correspondence* is a bijection between $\mathfrak{S}_n$ and the set of pairs of standard Young tableaux of the same shape λ (and size n), having the property that if $w\mapsto(P,Q)$ then $\mathrm{Des}(w)=\mathrm{Des}(Q)$. Consequently

$$\sum_{w\in\mathfrak{S}_n} \mathbf{t}^{\mathrm{Des}(w)} = \sum_{\lambda\vdash n} f^\lambda \sum_{T\in\mathrm{SYT}(\lambda)} \mathbf{t}^{\mathrm{Des}(T)}.$$

Here $\mathbf{t}^S := \prod_{i\in S} t_i$ for $S\subseteq\{1,2,\ldots\}$, while $\lambda\vdash n$ means λ is a partition of n, and $f^\lambda := \#\mathrm{SYT}(\lambda)$. Note that Theorem 4.1 implies that any *non-hook* shape λ, as well as any disconnected skew shape λ/μ, will have $\sum_{T\in\mathrm{SYT}(\lambda/\mu)}\mathbf{t}^{\mathrm{cDes}(T)}$ well-defined and independent of the choice of cyclic extension (cDes,p) for Des on $\mathrm{SYT}(\lambda/\mu)$. Recall from Section 3 the notation $\oplus$ for direct sum of shapes.

By Equation (12), or alternatively by Equation (10), we deduce the following second main result.

Theorem 6.3. *For any $n\geq 2$*

$$\sum_{w\in\mathfrak{S}_n}\mathbf{t}^{\mathrm{cDes}(w)} = \sum_{\substack{\text{non-hook}\\ \lambda\vdash n}} f^\lambda \sum_{T\in\mathrm{SYT}(\lambda)}\mathbf{t}^{\mathrm{cDes}(T)} \quad + \quad \sum_{k=1}^{n-1}\binom{n-2}{k-1}\sum_{T\in\mathrm{SYT}((1^k)\oplus(n-k))}\mathbf{t}^{\mathrm{cDes}(T)},$$

where cDes *is defined on $\mathfrak{S}_n$ by Cellini's formula* (1) *and on standard Young tableaux (of the relevant shapes) as in Theorem 4.1.*

The explicit description of the unique cyclic descent map on near-hook SYT given in [AER18], is applied there to deduce the following.

Proposition 6.4. *1. For any $n \geq 2$*

$$\sum_{k=1}^{n-1} \sum_{T \in \mathrm{SYT}((1^k)\oplus(n-k))} \mathbf{t}^{\mathrm{cDes}(T)} = \prod_{i=1}^{n}(1+t_i) - 1 - t_1 \cdots t_n.$$

2. For any $n \geq 4$

$$\sum_{k=2}^{n-2} \sum_{T \in \mathrm{SYT}((n-k,2,1^{k-2}))} \mathbf{t}^{\mathrm{cDes}(T)} = 1 + t_1 \cdots t_n + \prod_{i=1}^{n}(1+t_i) \cdot \left[-1 + \sum_{j=1}^{n} \frac{t_j}{(1+t_{j-1})(1+t_j)} \right].$$

6.3 Cyclic Eulerian distribution on $\mathfrak{S}_n$

We now focus on $\lambda/\mu = (1)^{\oplus n}$, where we can take $\mathcal{T} = \mathfrak{S}_n$ and use the extra symmetry to get more refined results. Consider the *multivariate* generating functions

$$\mathfrak{S}_n^{\mathrm{Des}}(\mathbf{t}) = \mathfrak{S}_n^{\mathrm{Des}}(t_1, \ldots, t_{n-1}) := \sum_{w \in \mathfrak{S}_n} \mathbf{t}^{\mathrm{Des}(w)}$$

and

$$\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t}) = \mathfrak{S}_n^{\mathrm{cDes}}(t_1, \ldots, t_{n-1}, t_n) := \sum_{w \in \mathfrak{S}_n} \mathbf{t}^{\mathrm{cDes}(w)}.$$

Note that $\mathfrak{S}_n^{\mathrm{Des}}(\mathbf{t})$ and $\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t})$ are, respectively, the *flag h-polynomials* for the type A_{n-1} Coxeter complex and for the reduced Steinberg torus considered by Dilks, Petersen, and Stembridge [DPS09]. The two are related by an obvious specialization

$$\left[\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t})\right]_{t_n=1} = \mathfrak{S}_n^{\mathrm{Des}}(\mathbf{t}). \tag{17}$$

On the other hand, $\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t})$ and $\mathfrak{S}_{n-1}^{\mathrm{Des}}(\mathbf{t})$ are also related in a slightly less obvious way. Define an action of the cyclic group $\mathbb{Z}/n\mathbb{Z} = \langle c \rangle = \{e, c, c^2, \cdots, c^{n-1}\}$ on $\mathbb{Z}[t_1, \ldots, t_n]$ by shifting subscripts modulo n, i.e. $c(t_i) = t_{i+1 \pmod n}$.

Proposition 6.5. *For $n \geq 2$, one has*

$$\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t}) = \sum_{i=1}^{n} c^i \left(t_n\, \mathfrak{S}_{n-1}^{\mathrm{Des}}(\mathbf{t})\right) \tag{18}$$

and also

$$\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t}) = g(\mathbf{t}) \; + \; \mathbf{t}^{[n]} g(\mathbf{t}^{-1}), \tag{19}$$

where

$$g(\mathbf{t}) = g(t_1, \ldots, t_{n-1}) := \left[\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t})\right]_{t_n=0} = \sum_{i=1}^{n-1} t_i \cdot \left[c^i\, \mathfrak{S}_{n-1}^{\mathrm{Des}}(\mathbf{t})\right]_{t_n=0}. \tag{20}$$

Remark 6.6. Formulas (17) and (18) imply the following interesting (and seemingly new) recurrence for the ordinary multivariate Eulerian distribution $\mathfrak{S}_n^{\mathrm{Des}}(\mathbf{t})$:

$$\mathfrak{S}_n^{\mathrm{Des}}(\mathbf{t}) = \left[\sum_{i=1}^{n} t_i \cdot c^i \mathfrak{S}_{n-1}^{\mathrm{Des}}(\mathbf{t}) \right]_{t_n=1}.$$

One can specialize $\mathfrak{S}_n^{\mathrm{cDes}}(\mathbf{t})$ to a *bivariate* generating function

$$\mathfrak{S}_n^{\mathrm{cdes}}(t, u) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)} u^{\mathrm{cdes}(w) - \mathrm{des}(w)}$$

by setting $t_1 = t_2 = \cdots = t_{n-1} := t$ and $t_n := u$. The following result generalizes an observation of Fulman [Ful00] and Petersen [Pet05].

Proposition 6.7. *For $n \geq 2$ one has*

$$\mathfrak{S}_n^{\mathrm{cdes}}(t,u) = \left(nt + (u-t)\frac{d}{dt}t\right)\mathfrak{S}_{n-1}^{\mathrm{des}}(t). \tag{21}$$

Remark 6.8. The coefficients of $f(t) = \frac{d}{dt}t\mathfrak{S}_{n-1}^{\mathrm{des}}(t)$ appear as OEIS entry `A065826`.

The preceding calculations lead to an exponential generating function for $\mathfrak{S}_n^{\mathrm{cdes}}(u,t)$, generalizing work of Petersen [Pet15, Proposition 14.4]. For more details see [ARR17, §6].

7 Final remarks and open problems

Detailed proofs of Theorems 4.1 and 6.3 may be found in the full version paper [ARR17]. Our proof of the existence of (cDes, p) in Theorem 4.1, whose strategy was sketched above, is indirect and involves arbitrary choices.

Problem 7.1. *Find a natural, explicit map* cDes *and cyclic action p on* $\mathrm{SYT}(\lambda/\mu)$ *as in Theorem 4.1.*

Problem 7.2. *Find a Robinson-Schensted-style bijective proof of Theorem 6.3.*

For $J = \{j_1, \cdots, j_t\} \subseteq [n]$ let $-J := \{-j_1, \ldots, -j_t\}$ (interpreted modulo n).

Corollary 7.3. *Let λ/μ be a skew shape of size n which is not a connected ribbon. For any $J \subseteq [n]$ and any cyclic extension* cDes *of the usual descent map on* $\mathrm{SYT}(\lambda/\mu)$*, the fiber size*

$$\#\mathrm{cDes}^{-1}(J) = \#\mathrm{cDes}^{-1}(-J).$$

Problem 7.4. *For a solution of Problem 7.1, find an involution on* $\mathrm{SYT}(\lambda/\mu)$ *which sends the cyclic descent set to its negative.*

Problem 7.5. *For non-ribbon shapes λ/μ, can one choose the operator p in Theorem 4.1 and a polynomial $X(q)$ to obtain a cyclic sieving phenomenon (CSP) ?*

This problem was solved by Rhoades [Rho10] for rectangular shapes and by Pechenik [Pec14] for shapes $(k, k, 1^{n-2k})$. Recalling from [ER17a] the cyclic descent extension for $\mathrm{SYT}(\lambda \oplus (1))$, Equation (10) in the current paper has been applied in [ARS+18] to obtain a refined CSP on SYT of these skew shapes.

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