Stirling and Eulerian numbers of types B and D

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Abstract

In this paper we generalize a well-known identity relating Stirling numbers of the second kind and Eulerian numbers to Coxeter groups of types B and D.

1 Introduction

Stirling numbers of the second kind, denoted by $S(n,k)$, arise in a variety of problems in enumerative combinatorics. They first appeared as the coefficients of the expansion of the polynomial x^n in terms of the *falling polynomials* as presented in the following identity:

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot [x(x-1)\cdots(x-k+1)],$$

see the survey of Boyadzhiev [Boy12]. However, their most common combinatorial interpretation is as counting the number of partitions of the set $[n] := \{1,\ldots,n\}$ into k blocks (see [Sta12, page 81]). They count also the number of vertices of rank k of the intersection poset of the Coxeter hyperplane arrangement of type A_{n-1}, graded by co-dimension. In that context, they are also called *Whitney numbers* $W(n, n-k)$ (see Zaslavsky [Zas81] and Suter [Sut00] for more details).

The original definition of the *Eulerian numbers* was first given by Euler in an analytic context [Eul36, §13]. Later, they began to appear in combinatorial problems, as the Eulerian number $A(n,k)$ counts the number of permutations in the symmetric group S_n, having $k-1$ descents. We recall that a descent of $\sigma \in S_n$ is an element of

$$\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}, \qquad (1)$$

In: L. Ferrari, M. Vamvakari (eds.): Proceedings of the GASCom 2018 Workshop, Athens, Greece, 18–20 June 2018, published at http://ceur-ws.org

called the *descent set* of σ. We also denote $\mathrm{des}(\pi) := |\mathrm{Des}(\sigma)|$ the *descent number*.

Stirling numbers of the second kind and Eulerian numbers are closely related by the following classical identity, see e.g. [Bon04, Theorem 1.18].

Theorem 1.1. *For all positive integers n and r, we have*

$$S(n,r) = \frac{1}{r!}\sum_{k=0}^{r} A(n,k)\binom{n-k}{r-k}.$$

The aim of this note is to give two generalizations of this theorem to the Coxeter groups of types B and D.

2 Coxeter groups of types B and D and their Eulerian numbers

Let (W,S) be a Coxeter system. As usual, denote by $\ell(w)$ the *length* of $w \in W$, namely the minimum k for which $w = s_1 \cdots s_k$ with $s_i \in S$. The *right descent set* of $w \in W$ is defined to be

$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}.$$

A combinatorial characterization of $D_R(w)$ in type A, is given by Equation (1) above. Now we recall analogous descriptions in types B and D.

We denote by B_n the group of all bijections β of the set $[-n,n] \setminus \{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

for all $i \in [-n,n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank n. If $\beta \in B_n$ then we write $\beta = [\beta(1),\dots,\beta(n)]$ and we call this the *window* notation of β. Occasionally, we will use the *complete* notation of a permutation, e.g.

$$\pi = [3,-2,1,-4,5] = \left[\begin{array}{ccccccccccc} -5 & -4 & -3 & -2 & -1 & & 1 & 2 & 3 & 4 & 5 \\ -5 & 4 & -1 & 2 & -3 & & 3 & -2 & 1 & -4 & 5 \end{array}\right].$$

As set of generators for B_n we take $S_B := \{s_1^B,\dots,s_{n-1}^B,s_0^B\}$ where for $i \in [n-1]$

$$s_i^B := [1,\dots,i-1,i+1,i,i+2,\dots,n] \text{ and } s_0^B := [-1,2,\dots,n].$$

It is well known that (B_n,S_B) is a Coxeter system of type B (see e.g., [BB05, §8.1]). The following characterizations of the right descent set of $\beta \in B_n$ is well known [BB05].

Proposition 2.1. *Let $\beta \in B_n$. Then*

$$\mathrm{Des}_B(\beta) \quad = \quad \{i \in [0,n-1] \mid \beta(i) > \beta(i+1)\},$$

where $\beta(0) := 0$ (we use the usual order on the integers). In particular, $0 \in \mathrm{Des}_B(\beta)$ is a descent if and only if $\beta(1) < 0$. We set $\mathrm{des}_B(\beta) := |\mathrm{Des}_B(\beta)|$.

We set:

$$A_B(n,k) := |\{\beta \in B_n \mid \mathrm{des}_B(\beta) = k-1\}|,$$

and we call them the *Eulerian numbers of type B*.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation. It is usually called the *even-signed permutation group*. As a set of generators for D_n we take $S_D := \{s_0^D, s_1^D,\dots,s_{n-1}^D\}$ where for $i \in [n-1]$

$$s_i^D := s_i^B \text{ and } s_0^D := [-2,-1,3,\dots,n].$$

There is a well-known direct combinatorial way to compute the right descent set of $\gamma \in D_n$, (see, e.g., [BB05, §8.2]).

Proposition 2.2. *Let* $\gamma \in D_n$. *Then*

$$\mathrm{Des}_D(\gamma) \quad = \quad \{i \in [0, n-1] \mid \gamma(i) > \gamma(i+1)\},$$

where $\gamma(0) := -\gamma(2)$. *In particular,* $0 \in \mathrm{Des}_D(\gamma)$ *if and only if* $\gamma(1) + \gamma(2) < 0$. *We set* $\mathrm{des}_D(\gamma) := |\mathrm{Des}_D(\gamma)|$.

Then we set

$$A_D(n, k) := |\{\gamma \in D_n \mid \mathrm{des}_D(\gamma) = k-1\}|,$$

and we call it the *Eulerian number of type D*.

For example, if $\gamma = [1, -3, 4, -5, -2, -6]$, then $\mathrm{Des}_D(\gamma) = \{0, 1, 3, 5\}$, but $\mathrm{Des}_B(\gamma) = \{1, 3, 5\}$.

3 Set partitions of types B and D

It is well-known that the number of elements of rank k in the lattice of set partitions of $[n]$, $\pi_A(n)$, is counted by the Stirling numbers of the second kind $S(n, k)$.

For Coxeter groups of type B, Reiner [Rei97] defined a natural set partition lattice $\pi_B(n)$ which comes from the interpretation of the lattice $\pi_A(n)$ as the poset of intersection subspaces of subsets of hyperplanes in the root system of type A_{n-1}: $\{x_i = x_j \mid 1 \leq i < j \leq n\}$, ordered by reverse inclusion.

For example, the partition $\{\{1, 3, 6\}, \{2, 5, 4\}\}$ is interpreted as the subspace

$$\{\vec{x} = (x_1, x_2, \ldots, x_6) \in \mathbb{R}^6 \mid x_1 = x_3 = x_6,\ x_2 = x_5 = x_4\},$$

which can be written as the intersection of the hyperplanes $x_1 = x_3,\ x_1 = x_6,\ x_2 = x_5$ and $x_4 = x_5$.

The poset of intersection subspaces of the subspaces of the root system of type B:

$$\{x_i = \pm x_j \mid 1 \leq i < j \leq n\} \cup \{x_i = 0 \mid 1 \leq i \leq n\},$$

consists of subspaces which look typically like: $\{x \in \mathbb{R}^8 \mid x_1 = -x_3 = -x_4 = -x_8, x_2 = x_5 = 0, x_6 = -x_7\}$, and which can be represented in a simpler way like this:

$$\{\{1, -3, -4, -8\}, \{-1, 3, 4, 8\}, \{2, -2, 5, -5\}, \{6, -7\}, \{-6, 7\}\}.$$

This was Reiner's motivation for defining the partitions of type B as follows [Rei97]. Set $[\pm n] := \{\pm 1, \ldots, \pm n\}$.

Definition 3.1. A *set partition of type* B_n is a partition of the set $[\pm n]$ into blocks such that the following conditions are satisfied:

- If C appears as a block in the partition, then $-C$ also appears in that partition.
- There exists at most one block satisfying $-C = C$. This block is called the *zero-block* (if it exists, it is a set of the form $\{\pm i \mid i \in E\}$ for some $E \subseteq [n]$).

Definition 3.2. A *set partition of type* D_n is a set partition of type B_n with the additional restriction that the zero-block, if presents, contains at least two pairs.

For example, the set partition $\{\{1, 2\}, \{-1, -2\}, \{\pm 3\}\}$ is a set partition of type B_3 but not of type D_3, while $\{\{1\}, \{-1\}, \{\pm 2, \pm 3\}\}$ is a set partition of type D_3.

We denote by $S_B(n, k)$ (*resp.* $S_D(n, k)$) the number of set partitions of type B_n (*resp.* type D_n) having exactly k pairs of non-zero blocks. They are called *Stirling numbers of type B* (*resp. D*).

We define now the concept of an *ordered* set partition:

Definition 3.3. A set partition of type B_n (type D_n) is *ordered* if the set of blocks is totally ordered and the following conditions are satisfied:

- If the zero-block exists, then it appears as the first block.
- For each block C which is not a zero-block, the blocks C and $-C$ occupy adjacent places.

4 Main results

The main results of this paper are two generalizations of Theorem 1.1 to Coxeter groups of type B and type D.

Theorem 4.1. *For all positive integers n and r, we have*

$$S_B(n,r) = \frac{1}{2^r r!}\sum_{k=0}^{r} A_B(n,k)\binom{n-k}{r-k}.$$

Theorem 4.2. *For all positive integers n and r, we have*

$$S_D(n,r) = \frac{1}{2^r r!}\left(n2^{n-1}(r-1)!S(n-1,r-1) + \sum_{k=0}^{r} A_D(n,k)\binom{n-k}{r-k}\right),$$

where $S(n-1,r-1)$ is the usual Stirling number of the second kind.

Now, by inverting these formulas, similarly to [Bon04, Corollary 1.18], we get the following expression of the Eulerian numbers of type B (*resp.* type D) in terms of the Stirling numbers of type B (*resp.* type D).

Corollary 4.1. *For all positive integers n and r, we have*

$$A_B(n,k) = \sum_{r=1}^{k} (-1)^{k-r}\cdot 2^r r!\cdot S_B(n,r)\binom{n-r}{k-r}.$$

Corollary 4.2. *For all positive integers n and r, we have*

$$A_D(n,k) = \sum_{r=1}^{k}(-1)^{k-r}\left[2^r r!\cdot S_D(n,r) + n2^{n-1}(r-1)!S(n-1,r-1)\right]\binom{n-r}{k-r}.$$

5 Proof of Theorem 4.1 - Type B

This proof uses arguments similar to Bona's proof of Theorem 1.17 in [Bon04] for the corresponding identity in type A. Theorem 4.1 is equivalent to the following equation:

$$2^r r! S_B(n,r) = \sum_{k=0}^{r} A_B(n,k)\binom{n-k}{r-k}.$$

The number $2^r r! S_B(n,r)$ in the left-hand side counts the number of ordered set partitions of type B_n. Let us show that the right-hand side counts the same objects in a different way.

Given a signed permutation $\beta \in B_n$ with $\mathrm{des}_B(\beta) = k$, written in its window notation, we show how to construct ordered set partitions of type B_n having r blocks.

Split β into increasing runs by putting a separator right after every descent. If 0 is a descent we add a separator just before $\beta(1)$. This splits β into a set of blocks. Here by a block we mean the set of entries between two consecutive separators, where by convention the last separator is always right after $\beta(n)$. This set of blocks becomes an ordered set partitions of type B_n by performing the following two steps:

1. For each obtained block C, locate the block $-C$ right after it.

2. If $0 \notin \mathrm{Des}_B(\beta)$, define the *zero-block* to be equal to $\{\pm\beta(1),\ldots,\pm\beta(i)\}$, where i denotes the first descent of β. It will be located as the first block.

For illustrating the above construction, let $\beta = [3,-2,1,-4,5] \in B_5$. We add the separators after the descents, obtaining:

$$\beta = [\ 3 \mid -2 \quad 1 \mid -4 \quad 5\ ].$$

The associated ordered set partition of type B_5 is:

$$\{\{\pm 3\},\{-2,1\},\{-1,2\},\{-4,5\},\{-5,4\}\}.$$

When β is written in complete notation, the definition of the zero-block become more natural, since it appears as a usual block when the permutation is split by the separators (one after any descent), i.e.

$$\beta = [\ -5 \quad 4 \mid -1 \quad 2 \mid -3 \quad 3 \mid -2 \quad 1 \mid -4 \quad 5\ ].$$

Now, we distinguish between two cases:

1. If $r = k$ (where $\mathrm{des}_B(\beta) = k$), the above construction produces an ordered set partition of type $B_n$with exactly r pairs of blocks. In fact, if $0 \notin \mathrm{Des}_B(\beta)$, then the digits in the first increasing run in the window notation of β, together with all their signed copies, constitute the zero-block, C_0, and the k descents produce k pairs of blocks $\{C_i\}, \{-C_i\}$, where C_i denotes the increasing run starting after the ith descent.

   If $0 \in \mathrm{Des}_B(\beta)$, then there is no zero block, so that each increasing run contributes a pair of blocks which in total form a set partition of type B_n having $k+1$ pairs of blocks as described above.

2. If $r > k$, we add $r-k$ separators in places which are not descents. Note that if $0 \notin \mathrm{Des}_B(\beta)$ then one might also add a separator before the first place (which means that the first increasing run will contribute two regular blocks instead of one zero-block). Now we produce the desired ordered partition in the same way we did in the preceding case. The number of ordered partitions obtained from β in this way is $\binom{n-k}{r-k}$, and it is independent whether 0 is a descent of β or not. □

For example $\beta = [1, 4 \mid -5, -3, 2] \in B_5$ produces the ordered set partition of type B_5:

$$\{\{\pm 1, \pm 4\}, \{-5, -3, 2\}, \{5, 3, -2\}\}$$

with one pair of non-zero blocks. Moreover, β produces exactly $\binom{4}{1}$ partitions with two pairs of blocks, namely

$$\begin{gathered}\{\{1, 4\}, \{-1, -4\}, \{-5, -3, 2\}, \{5, 3, -2\}\},\\ \{\{\pm 1\}, \{4\}, \{-4\}, \{-5, -3, 2\}, \{5, 3, -2\}\},\\ \{\{\pm 1, \pm 4\}, \{5\}, \{-5\}, \{-3, 2\}, \{3, -2\}\},\\ \{\{\pm 1, \pm 4\}, \{-5, -3\}, \{5, 3\}, \{2\}, \{-2\}\},\end{gathered}$$

obtained by placing one extra separator in positions 0,1,3, and 4, respectively. For larger r, the idea is the same, by adding more separators.

6 Proof of Theorem 4.2 - Type D

The proof for type D is a bit more tricky. The basic idea is the same as before: obtaining the whole set of ordered set partitions of type D starting from permutations in D_n, by adding separators after every descent and in the non-descent spots, which we call *artificial separators.*

The problem which naturally arises now is that, if we follow the same procedure used in the previous section, we may obtain set partitions of type B which are not of type D, namely set partitions with zero-block containing exactly one pair of elements. This happens exactly if there is a separator (either induced by a descent or an artificial one) between $\gamma(1)$ and $\gamma(2)$, but not before $\gamma(1)$.

In order to solve this problem, for any such $\gamma \in D_n$, we toggle the sign of $\gamma(1)$, obtaining $\gamma' \in B_n \setminus D_n$, and we apply the type B procedure to γ', by obtaining a genuine set partition of type D without a zero-block. We call this the *switch operation.*

Note that changing the sign of the first entry of γ produces an element γ' having a descent in 0, i.e. $\gamma'(1) + \gamma'(2) < 0$. In other words, to obtain the block decomposition associated to γ', toggle the sign of the first entry of γ and move the separator from position 1 to position 0.

For example, let $\gamma = [3, -1, 4, -2, -6, -5] \in D_6$. After placing the separators induced by the descents, we have:

$$\gamma = [3 \mid -1, 4 \mid -2 \mid -6, -5].$$

Here, since $0 \notin \mathrm{Des}_D(\gamma)$, the zero-block is $\{\pm 3\}$ and applying the procedure in type B we obtain the set partition of type B_6

$$\{\{\pm 3\}, \{-1, 4\}, \{1, -4\}, \{2\}, \{-2\}, \{-5, -6\}, \{5, 6\}\},$$

which is not a legal set partition of type D_6, since the zero-block consists of only one pair. Toggling the sign of $\gamma(1)$, we have:

$$\gamma' = [\ |\ -3, -1, 4\ |\ -2\ |\ -6, -5] \in B_6 \setminus D_6,$$

where the first separator stands for the descent at 0. This will give us the following set partition of type D_6:

$$\{\{-3, -1, 4\}, \{3, 1, -4\}, \{2\}, \{-2\}, \{-6, -5\}, \{6, 5\}\}.$$

For the next step, we denote any ordered set partition of type D in an abbreviated form, by writing only the first block in each pair, e.g. $\{\{-3\}, \{3\}, \{-4, -2, 1\}, \{4, 2, -1\}\}$ will now be written as $\{\{-3\}, \{-4, -2, 1\}\}$. We call an ordered set partition of type D having an odd number of negative entries in this abbreviated notation an *odd partition*.

The following lemma characterizes the structure of the odd partitions, which can not be obtained from permutations of D_n by using the switch operation.

Lemma 6.1. *The ordered odd partitions having r blocks, which can not be obtained from permutations in D_n by a switch operation are exactly of the form*

$$P' = \{\{*\}, P\},$$

where $*$ stands for one element of $[\pm n]$, and P consists of the blocks of a usual ordered set partition of the set $[n] \setminus \{*\}$ with $r - 1$ blocks.

Proof. Note that the singleton $\{*\}$ cannot be a zero-block by the definition of an odd partition, and this is why we require the partition P to have $r - 1$ blocks.

First, it is easy to see that if P is an odd partition starting with a singleton, then P can not be obtained from any $\gamma \in D_n$ by the switch operation, since that operation removes the separator between $\gamma(1)$ and $\gamma(2)$, and hence it merges the two first blocks, and the first block has at least two elements.

On the other hand, if an odd partition P does not start with a singleton block, we now show that it can be obtained by a switch of a permutation in D_n. Assume that $B = \{a_1 < a_2 < \cdots < a_t\}$ is the first block of P, where $t > 1$. If P is obtained from a permutation γ' which itself is a switch of some $\gamma \in D_n$, then we must have $\gamma'(1) = a_1$ and $\gamma'(2) = a_2$. Hence $\gamma(1) = -\gamma'(1) = -a_1$ and $\gamma(2) = \gamma'(2) = a_2$. We deal with this situation case-by-case:

1. If $a_1 > 0$ and $a_2 > 0$, then γ' is obtained from γ by the switch operation, where $\gamma(1) = -a_1$ and $\gamma(2) = a_2$ and we add an artificial separator between $\gamma(1)$ and $\gamma(2)$. Since $0 \notin \mathrm{Des}_D(\gamma)$, the switch operation is required in order to get γ'.

2. If $a_1 < 0$ and $a_2 < 0$, then γ' is obtained from γ where $\gamma(1) = -a_1$ and $\gamma(a_2) = a_2$. Since $a_1 < a_2 < 0$, we have $-a_1 > a_2$, so there is a separator induced by a descent between $\gamma(1)$ and $\gamma(2)$, while $0 \notin \mathrm{Des}_D(\gamma)$ so that the switch is indeed required.

3. If $a_1 < 0$ and $a_2 > 0$, then there is a permutation $\gamma \in D_n$ such that $\gamma(1) = -a_1 > 0$ and $\gamma(2) = a_2 > 0$ so that $0 \notin \mathrm{Des}_D(\gamma)$ and the switch is required either due to a separator induced by a descent between $\gamma(1)$ and $\gamma(2)$, or due to an artificial separator that we added.

□

In the next lemma we count the number of odd partitions of the form $\{\{*\}, P\}$

Lemma 6.2. *The number of odd partitions which cannot be obtained from permutations in D_n by a switch operation is:*

$$n2^{n-1}(r-1)!S(n-1, r-1).$$

Proof. For constructing an odd partition, with structure given in Lemma 6.1, one can start by choosing the unique element in the singleton $\{*\}$, which can be done in n ways. Afterwards, one has to choose and order the $r - 1$ blocks in P, which can be done in $(r-1)!S(n-1, r-1)$ ways. Finally, one has to choose the sign of any entry in the partition $P' = \{\{*\}, P\}$, in such a way that an odd number of entries will be signed, and this can be done in 2^{n-1} ways. □

Finally, we can now finish the proof of Theorem 4.2.

Proof of Theorem 4.2. As before, the equation in the statement of Theorem 4.2 is equivalent to the following:

$$2^r r! S_D(n,r) = n 2^{n-1}(r-1)! S(n-1,r-1) + \sum_{k=0}^{r} A_D(n,k)\binom{n-k}{r-k}.$$

The left-hand side of the above equation counts the number of ordered set partitions of type D_n with r parts. The right-hand side counts the same set of partitions divided in two categories: those coming from the usual procedure or from the switch operation induced by permutations in $A_D(n,k)$, and those that are not, which are counted in Lemma 6.2. This completes the proof. □

References

[BB05] A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups.* Graduate Texts in Mathematics 231, Springer-Verlag, New York, 2005.

[Bon04] M. Bona. *Combinatorics of Permutations.* Chapman & Hall /CRC, 2004.

[Boy12] K.N. Boyadzhiev. Close Encounters with the Stirling Numbers of the Second Kind. *Mathematics Magazine*, 85(4):252–266, 2012.

[Car59] L. Carlitz. Eulerian numbers and polynomials. *Mathematics Magazine*, 32(5):247–260, 1959.

[Eul36] L. Eulero. Methodus universalis series summandi ulterious promota. *Commentarii academiæscientiarum imperialis Petropolitanæ*, 8:147–158, 1736. Reprinted in his Opera Omnia, series 1, volume 14, 124–137.

[Rei97] V. Reiner. Non-crossing partitions for classical reflection groups. *Discrete Mathematics*, 177(1–3):195–222, 1997.

[Sta12] R. P. Stanley. *Enumerative combinatorics, Vol. 1, Second edition.* Cambridge Studies in Advanced Mathematics 49, Cambridge Univ. Press, Cambridge, 2012.

[Sut00] R. Suter. Two Analogues of a Classical Sequence. *Journal of Integer Sequences*, 3, Article 00.1.8, 18 pp., 2000.

[Zas81] T. Zaslavsky. The geometry of root systems and signed graphs. *American Mathematical Monthly*, 88:88–105, 1981.