Creative Insights: Dual Cognitive Processes in Perspicuous Diagrams

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Abstract. This paper explores the role of the subject in the production or interpretation of a diagram drawn up as a first conjectural step towards a creative solution of a mathematical problem. When discussing diagrammatic reasoning mechanisms, the role the agent of discovery has received considerably less attention than that of the object achieved, however, it is by no means less important. The purpose is to analyze the strategies behind the subject's perspicuity when finding these solutions, providing an account of such 'perspicuous' diagrams through insightful techniques labelled here as 'dual cognitive strategies', mechanisms for exploring new visualizations through diagrams.

Keywords: Perspicuity, Diagrams, Creative Insights, Dual Cognitive Strategies.

1 Introduction

Diagrams serve many purposes; nonetheless, the focus of this work will be the case of the construction of diagrams for outlining original ideas that may solve creative and innovative mathematical problems. Such ideas, implicitly expressed through diagrams, usually -not always though-, appear in the form of creative insights. An insight is defined herein as "the sudden appearance of a solution to a problem that one has been working on without any conscious sense of progress" (Vartanian, 2011: 166). Although this phenomenon has been discussed exhaustively by the theorists of creativity and is the subject of a heated debate, there exist a systematic series of studies on the neural correlates of insight that account for the phenomenon in question (Kounios & Jung-Beeman, 2009; Vartanian et al., 2013). In fact, a mathematical problem can be solved either analytically or with insight coupled with the fact that subjects can reliably judge which of the two processes led to the solution. Therefore, brain activity, at the time immediately prior to a solution, can distinguish between analytic and insight solutions.

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A characteristic of such diagrams of primal ideas consists in their visualization. The ability to think in visual images generally is considered of great help in solving mathematical problems. Actually, in some cases they can be the whole answer[1]. Visual imagery is one of the various heuristics or cognitive strategies often used in, but not limited to, mathematical creativity. In this paper, we will discuss visualizations in mathematics from a systematic and historical perspective. We will examine a historical case study, which stems from Ancient Greek Mathematics, more precisely from Plato's mathematical discussions. In addition, from a systematic standpoint, we will consider the matter of what a visualization in mathematical research can accomplish. We will especially focus on the intervention of the creative subject in the construction of diagrams that express primitive creative ideas, even when these ideas only express plausible conjectures that are not necessarily conclusive, and said diagrams do not constitute irrefutable proof of such conjectures.

Thus, our proposal lies at the intersection of visualization research and diagrammatic reasoning. In contemporary literature, visualization research and diagrammatic reasoning are usually considered two different work areas, with some points in common. Creativity related research in mathematical domains have hinted at a complex interaction between visualization and reasoning from diagrams during problem solving. While 'visualization', in this paper, refers to the process of creating a mental understanding of an object or phenomenon, from which information is conveyed to the mind through our sensory perception channels, the set of these techniques rely on our senses, especially on vision, to analyze the content of images. Essentially, however, it has to do with the cognitive interpretation ability of the visualization bearer. During creative discoveries, mathematicians move from data to insight, often building diagrams, which help in the understanding of the construction of a plausible conjecture in the way to a solution of a posed problem.

In this paper, we will focus on such primordial diagrams as manifestations of creative insights. Specifically, we will take into consideration who the bearer of such an insightful visualization is, i.e. who is supposed to make the discovery, arguing that diagrams do not speak for themselves but through the minds of their creators or interpreters. When discussing diagrammatic reasoning mechanisms, the role of the agent of discovery has received considerably less attention than that of the object achieved, however, it is by no means less important.

By means of this approach, we would be able to describe certain diagrammatic strategies, common to various creative processes, which tend to produce insights, that we have called 'dual cognitive strategies'[2]. The chief goal is to direct attention not to the diagram in question but to the subject producer or interpreter of the diagram, characterizing what we call 'perspicuous diagrams'. Said approach focusses the work towards the

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1 Cf. Peirce and its logic diagrams, especially the Existential graphs, (Peirce, 1992a, 1992b, 1998).

2 "Dual cognitive strategy or mechanism" should not be confused with similar terms often used as "dual coding theory" by Pavio and "dual process theory" by Evans, Over, Kahneman and many others. The meaning used here corresponds to some extent but not totally with what Shimojima (2015) describes as "cognitive potentials". We like to thank the anonymous reviewers for extensive and helpful written comments on an earlier version of this paper.

description of certain techniques of mathematical visualization. Visual imagery plays an important role in creativity and it is crucial in the description of insights as disruptive instances "by which a problem solver suddenly moves from a state of not knowing how to solve a problem to a state of knowing how to solve it" (Mayer, 1992)

The notion of perspicuity regarding subjects, in this article, is applied metaphorically to diagrams, understanding that a perspicuous diagram is a representation, which makes perspicuous what is presented to someone to whom that diagram is given. What has been proposed aims at characterizing the notion of perspicuity in general terms, and, in particular, how it applies to mathematical diagrams. This will lead to conclude in a systematic description of perspicuous diagrams, products of creative insights, originated through dual cognitive processes.

The paper is organized as follows. In Section 2, we briefly review the notion of insight in relation to mathematical problems. Next, in Section 3, we describe concisely some of the frequent meanings and uses of the term 'perspicuity' in history, but only those applicable to diagrams. In Section 4, we present a historical case study that illustrates how a mathematician, typically, carries out a process of constructing visualizations and implicitly provides mechanisms for exploring the emergence of diagrams in mathematics. Section 5 intends to provide a systematic characterization of the notion of perspicuous diagrams. Particularly, we expose the dual-strategy mechanism proposed herein, applied to the case study, a case of a simple diagram which solves a mathematical problem applying two modes of perspicuity between the participants of the mathematical outcome. Finally, we conclude in Section 6, where we outline directions for future work.

2 Creative insights: a dual underlying mechanism

Among the many different definitions, we choose the following characterization of 'insight':

> The concept of insight is closely related to those of understanding and comprehension. To gain insight is to understand (something) more fully, to move from a state of relative confusion to one of comprehension (…) as in 'getting' a joke or reading some material that seems murky but then becomes clear. The difference between 'I don't understand' and 'I see' is what is intended. The occurrence of insight is associated with the 'Aha!' experience, with the proverbial bulb going on over one's head (…) the product of a process of restructuring. (Dominowski & Dallob, 1995: 37-38)

Graham Wallas (1926) modelled the creative process in four steps: preparation, incubation, illumination and verification. In this context, an insight is usually understood as part or the whole stage of illumination, occurring after having experienced an incubation. According to Wallas, during the incubation, the problem "rests", eventually resulting in a dissolution of the previous structures that made up the problem and a weakening of the previously acquired assumptions. This is where imagination is put into practice, altering the supposedly rigid foundations of the problem, resulting, at best, with the emergence of a creative insight. This notion had an important impulse from the Gestalt psychologists (Köhler, 1925, Koffka, 1935, Duncker, 1945, Wertheimer, 1945, 1959), and later, from authors such as Nęcka (2011), Sternberg & Davidson (1995), Kaplan & Simon (1990), Ohlsson (1984, 1992), Bowden & Jung-Beeman (2003), Aziz.Zadeh, Kaplan & Iacoboni (2009), Schooler & Melcher (1995), among many others[3]. The following description is enlighting: "Insight is a sudden realization of the essence of a complex, paradoxical, or not well-understood situation, particularly the essence of a problem at hand (…) It produces realization of what the difficulty comes from, what the obstacles are to a good solution, and why the previous attempts to solve the problem were futile". (Nęcka, 2011: 667) In this regard, George Polya says:

> After the coming of the idea, we see more -more meaning, more purpose, and more relations. The coming of the idea is similar to switching on the light in a dark room (…) A suddenly arising idea, a spectacular new element amid dramatic rearrangement, has an impressive air of importance and carries strong conviction. This conviction is expressed by such exclamations as 'Now I have it!' 'I have got it!' 'That's the trick' (Polya, 1962: 60)

In relation to whether the phenomenon of insight exists in creative processes, there are two opposite tendencies[4] that, although, have somewhat varied in their proposals, remain the same since the 80s of the 20th century-. On the one hand, a position led by Perkins and Weisberg, the "Nothing-Special View", or more recently, the "Business as Usual Perspective". They consider that insight is merely extension of ordinary processes of perceiving, recognizing, learning and conceiving (Perkins, 1981, 2000; Weisberg, 1986). On the other hand, the "Special-Process View", most often associated with the above-mentioned Gestalt Psychologists, and the great majority of recent theorists, who generally "sustained semiconscious and perhaps unconscious processes were applied to problem in parallel with, and eventually in concert with, a conscious process. And the process was not at all mundane or typical" (Feldman, 1988: 287). Although an insight emerges to the consciousness of the creative subject instantaneously, fleetingly and suddenly, it does not alter the fact that an insight is the result of an imaginative and memory related process: it is not carried out as an instance disconnected from key stages of development, in which great expert knowledge and arduous systematic work play an essential role. Usually insight is described in terms of metaphors, analogies, graphic images or diagrams, which tend to admit an instantaneous understanding even when their characterization often escapes the usual logical frames.

In this regard, Wallas pays attention to a phenomenon that precedes, accompanies or constitutes a new thought: "A new thought (…) may be accompanied by, or may consist of, a visual or audible 'image' (…) The image may be a picture that has only incompletely and with difficulty been made visible to the mind by a severe effort of concentration, but which is accompanied by an unusually intense and vivid emotion (…) largely due to the amazing clearness of his power of sensory imagination." (Wallas, 1926: 110-111)

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[3] A more complete list and its references can be found in (Kaufman & Sternberg, 2010).

[4] These opposite tendencies appeared in the field of philosophy but not in psychology. We are grateful to an anonymous reviewer for this commentary.

With these brief notes regarding the notion of insight in mind, we ask ourselves what are the reasons that lead us to succinctly deal with the notion of creative insight in this paper. As we have clarified *ut supra*, such insights incite a change of perspective from an unfinished and unresolved situation that provokes ignorance, to another completely opposite, that allows access to previously missing knowledge.

We consider that the original unresolved situation is expressed through the diagrammatic formulation of a problem. It is analyzed under a certain perspective P, a graphical way of visualizing the problem. However, such a perspective P does not result in a solution, even after incessant work where many roads have been tried to no avail. With this, the analysis of the problem stagnates and the researcher is blocked. This leads to the initiation of what Graham Wallas (1926) called the period of 'incubation', a stage where an impasse is produced, to "put the problem out of mind, while paradoxically remaining sensitive to stimuli and ideas that might be related to the unsolved problem" (Smith, 2011: 657).

If after the incubation, the mathematician succeeds in finding a different path of resolution that opens the blockade, there is a change in perspective of the problem, from P to P', which eventually allows us to go from not knowing to knowing. What occurred here is an association of two different ways to visualize the same problem -P and P'- and a Gestalt switch of the passage from P to P'.

If the first perspective P emphasized a property or properties $a_1, a_2, ..., a_j$ (in the case study problem developed in section 4, it is a single property *a*), the second diagrammatic perspective P' highlights another property or properties $b_1, b_2, ..., b_k$ (in our problem, $k = 1$ and $b_1 = b$). If subsequently, we restrict our analysis to j=1 and k=1, we have that property *b* of problem X, originally, according to perspective P, was a secondary and peripheral property. Therefore, property *b*, from the perspective P', becomes central and relevant, to offer a perspicuous solution of the problem X.

The dual cognitive mechanism that we intend to define in this paper focuses on this property *b* and its vertiginous change of status, from the periphery to the core of the problem: *b* visualized from P perspective is not relevant, and even goes unnoticed. Instead *b*, analyzed through perspective P', solves the problem in an ingenious way. That is why, in this context, an insight is defined as a sudden change of perspective due to the application of a dual cognitive strategy.

It is interesting to analyze the subject that produces the change of perspective when developing some perspicuous or ingenious skill that allows the focus on *b*, through perspective P', to make the solution of X evident[5]. The ability to visualize the same object -in this case problem X even when it has undergone several modifications throughout its resolution- from two different points of view, and the cognitive leap produced from P to P', implies a task of ingenuity. Indeed, as we will see in the next section, the different authors herein cited introduce the notion of perspicuity as the ability to be able to change perspective in such a manner that the second way of visualization

[5] It should be noted that perspicuity may not necessarily be applied to the P' perspective, arising serendipitously. Sometimes in these random situations, surprising perspicuous techniques of immediate application are used after their discovery, in the understanding of it, a situation that also involves creative leaps.

clarifies everything that previously remained in the shadows. Let us move on to section 3.

3 The notion of perspicuity throughout history

The notion of perspicuity, as a mean to capture creative insightful diagrams, is reduced to “grasping” an implicit cognitive mechanism during the creative process. This grasping produces graphic solutions to problems through a Gestalt switch, which dramatically changes the entire former configuration of a problem, and suddenly making it clear and transparent with a mere glimpse of the solution diagram.

We have labeled such a mechanism as a ‘dual cognitive strategy’, since the same problem is perceived and understood through two different diagrammatic configurations that, surprisingly, connect to each other in order to solve the problem. This idea of connection or association, in general, is implicit in any characterization of the concept of perspicuity. Indeed, the notion of perspicuity has been outlined at least since the works of Aristotle: “Acumen (is the) talent for hitting upon the middle term in an imperceptible time (…) For seeing the extremes he becomes familiar with all the explanatory middle terms” (Aristotle, 1984: 147)[6]. Thus, *acumen or perspicuity is the demonstrative ability to find the suitable middle term* that, being the real cause of the problem at task, allows us to derive a conclusion from the premises in our possession. The key to the ingenious act lies in connecting two extreme terms to each other through a medium term found, thus forming a mathematical proportion. The curious thing about this finding is the unexpectedness of such a connection since, in principle, the extreme terms are radically different from each other and it would prove difficult to link them, unless a perspicuously obtained idea comes forth.

Moreover, instead of a single middle term C, there could be a finite quantity of them, establishing a continuous proportion: $A:C_1 :: C_1:C_2 :: ... :: C_n:B$, with *n* natural. Aristotle, in *On Memory and Reminiscence*, 452a17, offers the following example: milk is to white, as white is to fog, as fog is to wet, as wet is to autumn. Thus, surprisingly "milk" is put in conjunction with "Autumn", to refer to the fog season, and are the terms "white", "fog" and "humid" that explain the surprising conjunction "Autumnal milk", meaning that Autumn is characterized by a vaporous and humid mist that dyes the air white, like milk.

The above-mentioned argument indicates that, now, it is clear how a creative insight is obtained from a set of initial data (*milk*) to a result (*autumn*) from the surprising fact: through the combination via a few middle terms, which is precisely what being perspicuous is described as. Such surprising fact has to do with the strangeness and sudden nature of the connection between the ends of a chain of middle terms included thereof. In this regard, perspicuity refers to the swiftness of invention for matters that arise unexpectedly, and consists in the ability to find the medium easily.

[6] Cf. Aristotle, *Posterior Analytics*, Book I, Chapter 34, 89b10.

In this sense, Giambattista Vico (2005 [1709]) raises his notion of *ingenium* in *De Ratione*, even though the context of application is not limited to the domain of mathematics or logic: “Ingenium is the power of conjoining and unifying things that are disparate and far apart. The Latins called it acute or obtuse, both terms being derived from geometry. An acute wit penetrates more quickly and unites diverse things”.

What has been exposed up to this point connects these ideas with Ludwig Wittgenstein, who introduces the concept of a ‘perspicuous presentation’ (*übersichtliche Darstellung*) as a way of achieving a transparent understanding of the use of grammar, a clear philosophical view of how we use our words:

> A main source of our failure to understand is that we do not have a *clear view* of the use of words. –Our grammar is lacking in this sort of perspicuity. A perspicuous presentation produces just that understanding which consists in ‘seeing the connections’. Hence the importance of finding and inventing *intermediate links*. The concept of a perspicuous presentation is of fundamental significance for us. It earmarks the form of account we give, the way we look at things. (Wittgenstein, 1958, §122)

Although various interpreters have arrived at divergent views[7], all extracted from paragraph §122 of the *Philosophical Investigations* and several other passages of Wittgenstein’s works, for the purpose of this paper, we will only take into account the function that it fulfills in such applications and not the context to which the perspicuous adjective is applied. This is precisely what Wittgenstein borrowed from Goethe through his Spengler[8] readings around 1930. In addition, this is indeed what allows us to posit the dual cognitive strategy offered by perspicuous diagrams.

The influence of Goethe and Spengler is reflected in Wittgenstein's observations on Frazer's *Golden Bough*, which pose some of the first references about the concept of perspicuous presentation, which consists of 'seeing connections' through associating bonds, i.e. ‘intermediate links’ (*Zwischenglieder*) which in turn form chains:

> (...) A hypothetical connecting link, the only thing it does, in this case, is to draw attention to the similarity, to the connection between the facts. In the same way that a relation between the circle and the ellipse was illustrated, insofar as an ellipse gradually became a circle; but not to assert that a certain ellipse had originated in fact, historically from a circle (evolutionary hypothesis), but only in order to sharpen our gaze to see a formal connection. (Wittgenstein, 1993)

In this work, Wittgenstein agrees with Goethe in rejecting causal explanations applied to science as well as philosophical Positivism. This presumably leads Wittgenstein to apply the notion of perspicuous presentation to a type of approach to language and philosophy based on the procedure used by Goethe in the constitution of 'morphology', a discipline that had deal with the transformation of living beings.

[7] Cf. three standard concepts of perspicuous presentation advanced by Peter Hacker (2001, 2005, 2007), Gordon Baker (2004, 2005), and Hans Sluga (2011) respectively, among other commentators, for more details.

[8] It is worth noting that in 1918 Oswald Spengler conceived his book *The Decline of the West*, assimilating in it the morphological method of Goethe to the analysis of history.

Subsequently, we will briefly describe Goethe's morphological method and Wittgenstenian adaptation of it, leaving aside the biological content of Goethe's work and the philosophical content of Wittgenstein's. This effort of methodological abstraction has the purpose of capturing the characteristics of the perspicuous presentations in both authors, allowing, from there, to sketch the bases of the cognitive dual mechanism involved in the notion of perspicuity that we developed in section 5.

In contrast to the predominant scientific methodology at the time of Goethe, based on causal explanations, his proposal of scientificity consists of approaching any phenomenon of study through observation and not starting from the formulation of theories to be tested. This method is based on seeking of a multiplicity of samples, trying to construct a sequence of cases among which there are relations of similarity detected -notwithstanding their qualitative differences-, oriented to capture the unity in diversity. Moreover, it does not see only resemblances among these cases, but also passages and transitions.

> We intend to present morphology as a new science, no longer according to the object, which is known, but according to the point of view and the method that should give this doctrine its own form and also assign its place to the other beliefs. (...) It intends only to expose and not explain (...) Morphology orders bodies in groups as well as in series, according to their visible forms and the properties that are determined and studied. Thus, it allows having a vision of totality, from the enormous mass of the data. (Goethe, 1988)

All these processes allow seeing the mentioned phenomenon in a different light, recognizing a pattern through diversity. Phenomena do not always manifest plainly to our sight the network of relations that perspicuously connect them, and it is precisely such association of phenomena by which morphology looks to Goethe, and philosophical investigations looks to Wittgenstein, respectively.

This network of relations provides a new perspective on the phenomenon. The pattern achieved allows us to understand the complete ordered series that governs the transformation suffered by the initial phenomenon to become a perspicuous reformulation of it, a true insight, which finally provides a full understanding of the problematic original situation.

Thus, the scientist manages to apprehend a set of internal connections, a series of correlations between the various elements, in the process of transforming the original phenomenon into its new synthetic formulation -which may well be presented in a diagrammatic format-, revealing perspicuously the progressive development of such initially problematic situation.

Therefore, the process that first involves a vast recollection of specimens, and then orders them in an associative network, is oriented to describe rather than explain causally; seeks access to an understanding rather than a justification of the phenomenon in question: "Every explanation has to disappear and only the description has to take its place" (Wittgenstein, 1958, §109).

Such understanding is embodied in an insightful representation, whose result constitutes a perspicuous description of the phenomenon, allowing one to have a completely clear perspective before one’s eyes. This perspicuous presentation exhibits in full view some of its features -and properly ordered- which had until now been unrecognized, overlooked or misconceived.

4 Case study: the slave boy passage in Plato's *Meno*

There are two mathematical problems discussed in *Meno*. Let´s consider the first one: the problem of doubling the area of a given square and finding out what is the length of the side of the new square. Traditionally Proclus (1992) has stated that classical Greek mathematicians worked with two kinds of mathematical propositions: theorems, which asserted properties of given objects, and problems, which asserted that an object with given properties can be constructed. The proposition *partially* proven in the slave boy passage -as we will see in our argument- is of the latter kind: a new square can be constructed. Contextualizing this mathematical problem, the aforementioned dialogue suggests that during a discussion between Socrates and Meno concerning the nature of virtue emerges the so-called *Meno*'s paradox. This paradox questions the possibility of knowledge unless it pre-existed as reminiscence -i.e. all knowledge is past experience recollected-, a proposal that Socrates presents to Meno. In order to illustrate this theory, Socrates engages in a conversation with one of Meno's slave boys, who never had any formal training in geometry (85e) but understands and speaks Greek language.

In a first stage, Socrates asks the boy to explain the following question: given this figure (a square drawn by him in the sand) whose length of the sides are two feet, what is the length side of a similar figure of double the area? (82d-e2) In successive steps, the problem will be solved by breaking the given square into a finite number of sections and then constructing a larger square, double the size, in which some parts of the original are involved in this process. After applying a strategy that consists in dividing or doubling sides of the original square, that came to a dead end -an *aporia*-, a critical examination drove the resolution to divide areas, finally coming to the division by the diagonal of the original square. Then, the slave boy is led to visually recognize in this construction, the side of the double square as the diagonal line of the original one, learning during this process what a diagonal is.

The diagram shows a square *on* the diagonal of the given square (figure 1). Both squares are drawn along successive stages within a bigger square four times the original (figure 2), so that each of its four sides has a midpoint: by connecting these four midpoints, the four diagonals are drawn, which form a square consisting of exactly four triangular areas, each of which is half as large as the original square. Therefore, the square enclosed by the diagonal is twice the area of the original square, and its sides are equal in length to the diagonal of the original square (85b). When the correct construction was completed, its geometrical truth became evident and clear from the diagram, for which Socrates made sure that even someone with no special mathematical training, as in the case of the boy, could see the solution to the problem. However, is this a true solution to the problem mentioned above, as it was raised in 82d2-e2? The answer is no. In fact, Socrates demands a numerical value, i.e. the length of the side of the proposed new square with a surface twice that of the original square -or a ratio in proportion between the sides of both squares.

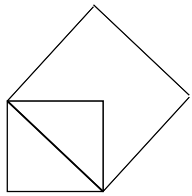

Fig. 1

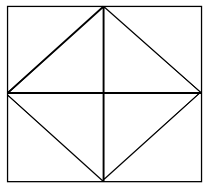

Fig. 2

The obtained result brings with it a bigger problem that makes it difficult to find a complete solution: since the diagonal of the original square is an irrational number, the boy could never give a number that represented the length of the side of the double square. Socrates in the *Meno* says, "The figure of eight is not made out of a line of three? -No.-But from what line? -tell me exactly; and *if you would rather not reckon, try and show me the line*.-Indeed, Socrates, I do not know". Ancient Greek Geometry requires such calculations, and if these calculations were not accessible, they could be replaced by a geometric visual expression.

Thus, there is no way to give a complete explanation of a numeric value of the diagonal of the square, but only a geometrical relation that represents it. Although this construction is possible, we affirm that the proposition is *partially* but not totally proven due to something missing in the process. A geometric response is provided but not an arithmetic solution, as required. Consequently, we see that the slave boy achieved true opinion but he could not acquire full knowledge. The result achieved in the case of the slave only allows him to *discover* what the solution is, provided by the existence of the diagonal of the original square taken as the solution for the side of the new square. Nevertheless, the slave does not *justify* such a procedure. This is the case of a visual argument, which is sound although its soundness is not entirely clear from the diagram. This is an instance where perspicuity functions better in Socrates' case than in the slave boy situation. Marcus Giaquinto provides the following comment, appropriate for the *Meno* case:

> (V)isual arguments (other than symbol manipulations) often lack the transparency required for proof. Even when a visual argument is in fact sound, its soundness may not be clear, in which case the argument is not a way of *proving* the truth of the conclusion, though it may

> be a way of discovering it. But this is consistent with the claim that visual non-symbolic thinking can be (and often is) part of a way of proving something. (Giaquinto, 2016)

If the problem originally proposed by Socrates was to indicate what number *b* measures, the side of the new square B with twice the area of the original square A on the side *a*, in reality, they end up solving a derived problem: to provide a geometric measure of the side *b* of square B, which doubles in area the previous square A. This new problem can be considered a "reduction" or *apagogé* of the first problem posed, which does receive a complete demonstration, an original and creative response with cognitive gain, a truly perspicuous kind of knowledge, a true insight corresponding to a question derived from the first. Regarding this last statement, we concur with Giaquinto (1993: 81) that the problem has "epistemological value, good methodology and true assumptions". Nevertheless, we differ with that author with respect to the fact that the obtained resolution is correct in the required data. He could never have arrived to the solution given the prevailing notion of number at that time. Instead, seen anachronistically, today it would be a complete answer, because every geometric measure currently has a numerical correlate, in this case through an irrational number, $b = a\sqrt{2}$, with *a* a positive integer. In this regard, Giaquinto states: "That passage has been taken to illustrate something important about mathematical discovery (...) although just *what* truths are illustrated is unclear" (1993: 81).

Although our proposal accepts to a great extent the admirable perspective of Giaquinto regarding the visualization in mathematics throughout his diverse works, -especially in his affirmation that there are reasons to "take seriously the possibility of mathematical discovery without demonstrable justification" (1992: 384)-, in this paper, we are not talking only about discovering *truths* but also discovering plausible ideas which eventually will lead to truths. In this case, at the moment of discovery "no violation of epistemic rationality is involved in the believer's having the belief" (1992: 383), even if the result is still not guaranteed, and it may never be, because it may turn out to be a false belief. We believe it to be a more realistic condition for a mathematician to work with plausible though not necessarily true ideas. *Meno*'s case would fit the case that Giaquinto qualifies as one in which "the thinking involved in making such a discovery would not constitute a proof" (1992: 384).

What makes the answer offered to the second reduced problem perspicuous is that: (a) it is unusual, unexpected and hits like a knockout, causing surprise by the sudden idea that emerges and that has impact; (b) the impact caused is due to the fact that the idea is correct; (c) such success is surprising, it is strange, it comes out of the norm, it is something out of the box; and (d) above all, the response obtained has no other way of expressing itself than in the geometric visual way. The reason being that analytically, at that time, it was impossible to offer a numerical value associated with said geometric diagonal, which is clear and perspicuously visible. It could not have been solved except in a diagrammatic way. The only analytical characterization would have been the mere mention of *b* as a diagonal of a square, without a numerical correlation. This fourth reason is what makes this example a model of diagrammatic reasoning, and for that reason it is so cited, despite its mathematical simplicity. In this regard, Giaquinto states: "The diagrams are used as a source of evidence by the slave for the conclusions he reaches", "The use of diagrams is a non-incidental element of the process illustrated by the dialogue" (1993: 83).

We believe that the response that Giaquinto offers to the *Meno* problem in the 1993 paper does not reach its greatest splendor. However, a better argument could be one that Giaquinto brilliantly uses himself in a later work (2011), in this case, for another problem but which seems to adapt well to this case, contrary to what this author has always maintained when referring to empirical evidence, that here, under these conditions, seems justifiable. Therefore, we have paraphrase it:[9]

> The domain of generalization of any [square, its side and its diagonal] is quite homogeneous [in the sense that any square] is geometrically just the same as any other (…) That eliminates the major kind of unreliability affecting generalizations from diagrams (…) These geometrical concepts are perceptual concepts of a kind. [Their] spatial properties are those that visible [squares] must appear to have in order to look perfect. Even if the [squares] in a diagram do not look perfect, we can often tell whether a feature of a diagram would be preserved in any corresponding diagram with perfect-looking [squares], and this opens the possibility of discovering from the diagram a feature of the represented geometrical situation (…) Only in restricted circumstances [like this one] (…) the concepts of the relevant mathematical entities must be suitably related to perceptual concepts. (Giaquinto, 2011: 304-305)

A vital hidden element in the way in which Socrates guides the slave is the manner in which Socrates draws the original square and the rest, except the last one, that is the solution to the problem: they all rest on a base side that is drawn horizontally to the observers. Such horizontal positioning of these squares makes it difficult for the slave to imagine, a natural consequence, the appearance of a square inclined at 45 degrees as a solution to the problem.

What underlies this question is a typical psychological fixation around the drawing of squares on the plane: the general interpretation that a square must be drawn with its base resting on a horizontal line. Any other orientation apparently, at first sight, as if it were out of thin air, is partially unexpected. If it is abstracted from the intuitive and current notion that a square must have a parallel orientation in relation to the edge of the sheet where it is traced -i.e. with its horizontal base and not rotated at any angle between 0 and 90 degrees-, then a certain type of rhombuses -whose internal angles all have 90 degrees-, can also be assimilated as squares. This procedure allows us to expand the space of ideas where to search for solutions to the problem.

One sees that all the options to the solution posed by the slave boy adhere to this fixation. Socrates must then produce a change of perspective: take the diagonal of the original square, rotate the view and base a square on such diagonal, taking it as the side of a new inclined square. To the naked eye, what is visualized is a rhombus that, clearly, but not immediately, also turns out to be a square. Moreover, precisely, the square sought. In this regard, the point of view of the observer affects the possibilities of accessing the solution. It takes a lot of ingenuity to transgress psychological fixations like this one. In the next section, we will revisit this matter, which shows how beliefs and

[9] The square brackets were added at our discretion.

past assumptions can hinder the possibility of solving a problem, unless we are willing to give up the persistence of such fixations, appealing to a certain wit or perspicuity of the observer in the search of solutions.

5 Dual cognitive strategies in perspicuous diagrams: a characterization

A perspicuous diagram A of a mathematical problem X is a representation that brings about a Gestalt switch by highlighting a new and unexpected aspect of the meaning of X. As follows, X now acquires two meanings; consequently, the perspicuous meaning is the one that clarifies the problematic situation in which X was involved, thus enabling the resolutions of latent doubts. It is the way this knowledge is framed, rather than the knowledge itself, what leads to creative insight in the case of creative problem solving. Perspicuous representations exploit unarticulated knowledge to imagine peculiar circumstances making said knowledge stand out. In fact, secondary features that occupy the periphery of the meaning of an object X, subsequently turn out to be considered as primary characteristics of the object, by thinking outside the box.

The strategy we present consists of capturing a dual meaning of the object in question. In diagram A, initial configurator of the problem X, there is an element or aspect that is emphasized in its description. In the case of Meno's boy slave, a characterization of square A, which rests on a horizontal line, described from its side *a*, is highlighted, regardless of it being taken horizontally or vertically. In the case of square B, the result of the problem, of an area which doubles that of square A, -which now rests on one of its vertices-, focuses its attention on its side *b*, which ends up being the diagonal of the initial square A. Thus, *b* fulfils a dual function: *b* is the diagonal of A and, furthermore, *b* is the side of square B. The gist of perspicuity in this simple problem lies in identifying the diagonal of A by fulfilling a new function as the side of another square strategically located, inclined 45 degrees with respect to the first. In mathematics, many problems are solved following a strategy of refunctionalization of a property that becomes a different instrument in another context. The mathematical proportion explaining this idea indicates $a:b::b:2a$, where $2a$ is the diagonal of square B.

The purpose of the dual strategy is to capture certain versatility of a mathematical design that previously seemed rigidly constructed. Having two different views of a mathematical problem, one as a starting yet unsolved approach, and the other one as the final solution to it, makes the transition from one to the other a matter of discovery. Said transition is the product of the manipulation of the original and its successive transformations. At the beginning, the slave boy thought that the solution to the problem would be a 4 foot per side square, double the size of the original one. In that case, the resulting area of 16 feet overpass the expected value of 8. Then he proposes a side of 3 feet, yet another failed attempt: the area now is of 9 feet, larger than 8. Some cases, which are source of information of the problem at hand, support the selection of a single interpretation from the initial conditions of the problem, due to previously fixed ideas of the researcher. The person is blocked at such point, as for her/him it is not possible to switch from this interpretation to any other one.

What the dual cognitive strategy can do is to alternate, or switch, between the first blocked interpretation and the second good one. Twofoldness, however paradoxical, may be sound. Finally, the expected diagrammatic manipulation consists in choosing, as the side of the solution square, the diagonal of the original. Such manipulations could be realized in a perspicuous way, so as to represent the information in a transparent manner. Any transparent representation allows recognizing the solution to the problem, that is, it makes explicit the implicit connection between the premises and the expected conclusion, perceiving their logical interrelations. Every perspicuous representation, when they provoke the leap from the unknown to the known, generates a 'surprise factor' each time it is carried out, describing an apparent psychological instantaneous effect. This surprise is eliminated when we find a perspicuous middle term that unites the extreme terms provoking an unexpected remote association. Thus, perspicuous representations are obtained by uniting distant and remote events. This task allows to describe them as generators of creative acts via imagination, producing an Eureka moment or an "Aha!" effect, a unique insight, clearing the path from the unknown towards the known. In these cases, conclusions could be easily read and understood with the given information, exhibited by means of a perspicuous selected diagram. As in the example of the slave boy, as he is being informed of Socrates' solution, he notices the diagonal line drawn in the original square from its opposite corners. Note that this strategy is based on the observatory style, provided by Goethe-Wittgenstein, when it comes to the search of connections: reviewing each element of the original square, its sides, angles, diagonals, and manipulating such elements while applying different points of view. All the conditions are present for the detection of the new square as a product derived from the first, forming a more general unified diagram, like the one in figure 1.

The foregoing ideas allows us to conclude that a perspicuous diagram associated with a problem is a representation that, in a clear and obvious manner, exhibits the solution to said problem, as a consequence of having gone through a Gestalt switch or creative insight, -as a result of the application of a dual cognitive strategy-, product of modifications and/or constructive manipulations of the diagram that expresses the original problem.

6 Conclusions

We have presented a cognitive mechanism that is involved in the development of perspicuous diagrams, the dual cognitive strategy. Although the case study was simple, it is possible to apply this strategy to more complex mathematical situations. Our mechanism works for all diagrams that are the result of creative insight. The focus has been placed on detecting certain process strategies that are useful when looking for original mathematical ideas. As a result, the dual cognitive strategy for enhancing creativity allows overcoming stumbling blocks in the path to creativity by finding other uses to certain elements in diagrams. These findings support the statement that some creative insights operate under dual cognitive mechanisms. However, this type of dual strategy still requires a more complex characterization. A greater number of study cases are needed in order to gain in-depth understanding of other mathematical insights.

Some of the future work suggestions include:

- How can we explore this kind of problems better and in more complete ways, in order to characterize other aspects concerning this mechanism?
- What kind of diagram properties we can identify through the application of the dual cognitive strategy?
- Is it possible to search for dual uses of different elements in any mathematical diagram?
- Can we predict plausible solutions to a mathematical problem by searching methodologically the properties of an initial diagram conforming said problem?

Further study is needed to ascertain which other features of a perspicuous diagram could produce an evident and transparent representation of a mathematical problem that exhibits directly the possible solutions.

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