Basic Algorithms of the Rule of Inference for a Logical-Type Systems with Many Fuzzy Inputs

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Abstract. In this paper considered an algorithm of fuzzy inference for logical-type systems. Noticed, that well-known methods of inference often not applicable in systems with a large number of inputs due to the increasing computational complexity. Proposed new algorithm for the rule of inference in systems with many fuzzy inputs based on the fuzzy truth value. Considered some basic operations (conjunction and disjunction) and proposed numerical algorithm for its calculating. Presented fuzzy implications that can be used in the tuning of fuzzy systems based on the fuzzy truth value.

1. Introduction

Fuzzy systems are widely used in many fields of technology for several decades [1]. Their popularity is based on a simple and intuitive approach to the uncertainty that is unavoidable in the review and evaluation processes in the surrounding world. Over the years, researchers have developed various approaches to the problem of fuzzy inference. Theoretical basics were introduced by Zade [2] and since other solutions have been proposed. The most popular applications in fuzzy inference are approaches of Mamdani [3], Larsen [4], Takagi and Sugeno [5], and also Tsukamoto.

These solutions are usually used because of the simplicity and efficiency of their implementation. At the same time, the methods don't fully correspond to Zade's theory because of the considerable simplification. An incompatibility can be seen in systems with many fuzzy inputs, i.e. when the membership functions of facts differ from singletons.

As is known, a special case of the compositional rule of inference is the generalized rule of modus ponens [2], which is described by the relation:

$$\mu_{B'}(y) = \sup_{x\in X}\left\{\mu_{A'}(x) *^T I\big(\mu_A(x), \mu_B(y)\big)\right\},$$

where $\mu_{A'}(x)$, $\mu_A(x)$, $\mu_{B'}(y)$, $\mu_B(y)$ – membership functions, $*^T$ – t-norm, which is the intersection of fuzzy fact A' and fuzzy implication I, the argument of which are input A and output B. Fuzzy sets are described in the space of reasoning X for premise and fact, and in Y for value B and result of inference B'.

Unfortunately, for compound premises, computation by the above method becomes very complicated due to multidimensional analysis. This is the main reason why the method in this form is not used for very common rules with a compound premise containing many links. For example, intelligent data mining solutions for the classification of genes that produce rules with thousands of premises [6].

An important advantage of the approach proposed in the article is the inference within a single space of truth for all premises. This is achieved by transforming the relationship between fact and premise into a so-called fuzzy truth value. Bringing all the relationships between different facts and premises into one fuzzy truth space simplifies computation of the composite truth function. Therefore, this approach devoid the problems of multivariate analysis and it is better suited for solving problems of intellectual analysis.

2. The Statement of a Problem

The task that is solved using fuzzy production system is formulated as follows. Consider the system with n inputs $x = [x_1,\ x_2, \ldots, x_n]$ and one output y. The relationship between inputs and outputs is described using N fuzzy rules, represented in the following form:

$$R_k: If\ x_1\ is\ A_{1k}\ and \ldots\ x_n\ is\ A_{nk}\ then\ y\ is\ B_k, k = \overline{1, N} \tag{2.1}$$

where $x \in \boldsymbol{X} = X_1 \times X_2 \times \ldots \times X_n,\ y \in \boldsymbol{Y}$ and $\boldsymbol{A_k} = A_{1k} \times A_{2k} \times \ldots \times A_{nk} \subseteq \boldsymbol{X},\ B_k \subseteq \boldsymbol{Y}$ are fuzzy sets.

The feature of logical-type systems according to the classification represented in [1], is that the rule (2.1) is formalized using fuzzy implication as (n+1)-ary fuzzy relation $R_k \subseteq X_1 \times \ldots \times X_n \times Y$ in the following way:

$$R_k = A_{1k} \times A_{2k} \times \ldots \times A_{nk} \times Y \to X_1 \times X_2 \times \ldots \times X_n \times B_k \tag{2.2}$$

where «→» – fuzzy implication, expressing the causal relationships between antecedent «$x_1\ is\ A_{1k}\ and \ldots\ x_n\ is\ A_{nk}$» and consequent «$y\ is\ B_k$». The task is to determine the fuzzy inference $B'_k \subseteq Y$ for the system represented in the (2.1) form, If the inputs are fuzzy sets $A' = A'_1 \times \ldots \times A'_n \subseteq X$ or $x_1\ is\ A'_1\ and \ldots and\ x_n\ is\ A'_n$.

3. Inference method based on fuzzy truth value

Fuzzy output B'_k for systems with n inputs using fuzzy truth value (FTV) based on the generalized modus ponens rule [2], following [7], is described as follows:

$$\mu_{B'_k}(y) = \sup_{t \in [0,1]} \{\tau_{A_k/A'}(t) *^T \dot{I}(\tau, \mu_{B_k}(y))\}, k = \overline{1, N}, \tag{3.1}$$

where $\dot{I}(*)$ – fuzzy implication operation; $*^T$ – t-norm; $\mu_{B'_k}(y)$ and $\mu_{B_k}(y)$ – membership functions of fuzzy sets $B'_k,\ B_k$; $\tau_{A_k/A'}(t)$ – membership function of FTV A_k provided that A'_k is true, which for independent inputs in accordance with [7] is equal to:

$$\tau_{A_k/A'}(t) = \mu_{CP(A_k,A')}(t) = \sup_{\substack{(t_1,\ldots,t_n) \in [0,1]^n \\ T(t_i)=t \\ i=\overline{1,n}}} \left\{ \mathrm{T}_{i=\overline{1,n}} \left(\mu_{CP(A_{ik},A'_i)}(t_i) \right) \right\}, \tag{3.2}$$

where $t \in [0, 1], k = \overline{1, N};\ CP(A_{ik}, A'_i)$ – fuzzy compatibility degree of A_{ik} with an input fuzzy value A'_i. Using the notation of the degree of compatibility, (2.2) can be represented as follows:

$$CP(A_k, A') = \tilde{T}_{i=\overline{1,n}}(CP(A_{ik}, A'_i)), k = \overline{1, N}, \tag{3.3}$$

where $\tilde{T}$ – extended by the generalization principle n-local t-norm [7].

4. Basic operations with FTV

To obtain the resulting FTV $\tau_{A_k/A'}(t)$ of R_k, it is necessary to convolve the fuzzy truth values calculated for each input, in accordance with the structure of the rule. Determine based on the generalization principle [2] operations of conjunction and disjunction in the truth space.

Consider a crisp mapping f space X to space Y:

$$f: X \to Y.$$

A – fuzzy set in space X, i.e. $A \subseteq X$,

$$A = \sum_{i=1}^{n} \frac{\mu_i(x_i)}{x_i}.$$

If the map f is one-to-one, then the generalization principle consists in the fact that the generated by this mapping and defined in space Y fuzzy set B defined by:

$$B = f(A) = \sum_{i=1}^{n} \frac{\mu_i(x_i)}{f(x_i)}.$$

Consider the case in which more than one element of the set X is mapped to the same element $y \in Y$ (f is not a one-to-one mapping). In this situation, membership degree of an element y to the fuzzy set $B = f(A)$ is equal to the maximum membership degree among those elements of the set X that are mapped into the same element of the set Y.

Denote by $f^{-1}(y)$ set of elements $x \in X$, which are mapped into the element $y \in Y$ by the transformation f. If $f^{-1}(y)$ is an empty set, i.e. $f^{-1}(y) = \varnothing$, then the membership degree of the y element to the fuzzy set B is zero. Hence:

Definition

If there is some clear mapping $f: X \to Y$ and given a fuzzy set $A \subseteq X$, then the principle of generalization is that the fuzzy set generated by this mapping$B$ has the form:

$$B = f(A) = \left\{\frac{\mu_B(y)}{y} | y = f(x), x \in X\right\},$$

where

$$\mu_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & \text{если } f^{-1}(y) \neq \varnothing, \\ 0, & \text{если } f^{-1}(y) = \varnothing. \end{cases}$$

This definition covers the space X with both finite and infinite number of elements. In the second case, B is represented by:

$$B = f(A) = \int_y \frac{\mu_B(x)}{f(x)}.$$

Generalize this assertion to the multidimensional case.

Let X be the Cartesian product of crisp sets $X_1 \times \ldots \times X_n$.

If there is a crisp mapping $f: X_1 \times X_2 \times \ldots \times X_n \to Y$, and also some fuzzy sets $A_1 \subseteq X_1, A_2 \subseteq X_2, \ldots, A_n \subseteq X_n$, then the principle of generalization says that the image formed by the mapping f the fuzzy set B has the form:

$$B = f(A_1, A_2, \ldots, A_n) = \left\{\frac{\mu_B(y)}{y} | y = f(x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_n) \in X\right\},$$

in this case:

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1, x_2, \ldots, x_n) \in f^{-1}(y)} \min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\right\}, & \text{если } f^{-1}(y) \neq \varnothing, \\ 0, & \text{если } f^{-1}(y) = \varnothing. \end{cases}$$

In this expression, the operation *min* can be replaced by an algebraic product, or by a more general t-norm.

Let there be given two fuzzy sets, which are two fuzzy truth values. The membership functions are denoted like $\mu_1(\tau_1)$, $\mu_2(\tau_2)$, $\tau_1, \tau_2 \in [0, 1]$.

Define the t-norm $\tilde{T}$ for FTV based on the principle of generalization:

$$\mu_{con}(\tau) = \mu_1(\tau_1) \tilde{T}\, \mu_2(\tau_2),$$

and also define s-norm (t-conorm) $\tilde{S}$:

$$\mu_{dis}(\tau) = \mu_1(\tau_1)\tilde{S}\,\mu_2(\tau_2),$$

where $\mu_{con}(\tau),\ \mu_{dis}(\tau)$ – functions of the result of the conjunction and disjunction, $\tau \in [0, 1]$.
In this case, the map for t $\tilde{T}$:

$$y = f(\tau_1, \tau_2) = \tau_1 *^T \tau_2,$$

where $*^T$ – some t-norm.

Hence, following the principle of generalization:

$$\mu_{con}(\tau) = \mu_1(\tau_1)\tilde{T}\,\mu_2(\tau_2) = \sup_{\substack{\tau_1,\tau_2 \in [0,1] \\ \tau_1 *^T \tau_2 = \tau}} [\mu_1(\tau_1)\ *^T\ \mu_2(\tau_2)]. \tag{4.1}$$

Mapping for $\tilde{S}$:

$$y = f(\tau_1, \tau_2) = \tau_1\ *^S \tau_2,$$

where $*^S$ – some s-norm.

Hence, following the principle of generalization:

$$\mu_{dis}(\tau) = \mu_1(\tau_1)\tilde{S}\,\mu_2(\tau_2) = \sup_{\substack{\tau_1,\tau_2 \in [0,1] \\ \tau_1 *^S \tau_2 = \tau}} [\mu_1(\tau_1) *^T\ \mu_2(\tau_2)]. \tag{4.2}$$

5. Numerical algorithm for calculating the conjunction (disjunction) for FTV
Consider an algorithm for obtaining a discrete variant of the conjunction and disjunction for FTV.
Since the software implementation of conjunction and disjunction means a discrete representation of fuzzy sets, so fuzzy sets given analytically must be discretize.

Suppose that two discrete fuzzy sets are given A_1 и A_2, defined in space X, which are fuzzy truth values:

$$A_1 = \sum_{i=1}^{n} \frac{\mu_1(\tau_1^i)}{\tau_1^i},$$

$$A_2 = \sum_{j=1}^{m} \frac{\mu_2(\tau_2^j)}{\tau_2^i},$$

where $\mu_1(\tau_1)$ и $\mu_2(\tau_2)$ the membership functions of sets A_1 и A_2, $\tau_1^i, \tau_2^j \in [0, 1], i = \overline{1, n}, j = \overline{1, m}$.

To compute the discrete version of conjunction between the fuzzy sets in accordance with (4.4), perform the following stages:

1. Initialize the resulting fuzzy set $A_{con} = A_1 *^T A_2 = \emptyset$. The membership function is denoted by $\mu_{con}(\tau_{con})$, $\tau_{con} \in [0, 1]$, $*^T$ – t-norm. Initialize counters $i = 1,\ j = 1$.
2. Calculate the value of the argument of the resulting membership function
$$\tau = \tau_1^i *^T \tau_2^j.$$
3. Perform a search in the set A_{con} of the membership function value $\mu_{con}(\tau_{con})$ with argument $\tau_{con} = \tau$.
   3.1. When the item $\frac{\mu_{con}(\tau_{con})}{\tau_{con}} = \frac{\mu_{con}(\tau)}{\tau}$ is found then replace its value with $\frac{\max\left(\mu_{con}(\tau),\ \mu_1(\tau_1^i) *^T \mu_2(\tau_2^j)\right)}{\tau}$.
   3.2. In the absence of coincidence, add into a set A_{con} new value of the membership function $\frac{\mu_{con}(\tau_{con})}{\tau_{con}}$, where $\tau_{con} = \tau,\ \mu_{con}(\tau_{con}) = \mu_1(\tau_1^i) *^T \mu_2(\tau_2^j)$.
4. Increment i и j.
5. Repeat stages 2-4 for $i = \overline{1, n},\ j = \overline{1, m}$.

To calculate the disjunction by (4.5), the t-norm should be changed by the s-norm (t-conorm) in stage 2: $\tau = \tau_1^i *^S \tau_2^j$.

Figure 1 shows the result of the algorithm for computing the conjunction of FTV, obtained in accordance with [8] for fuzzy sets with membership functions, given in the form of Gaussian curves. As the t-norm, the operation min was used.

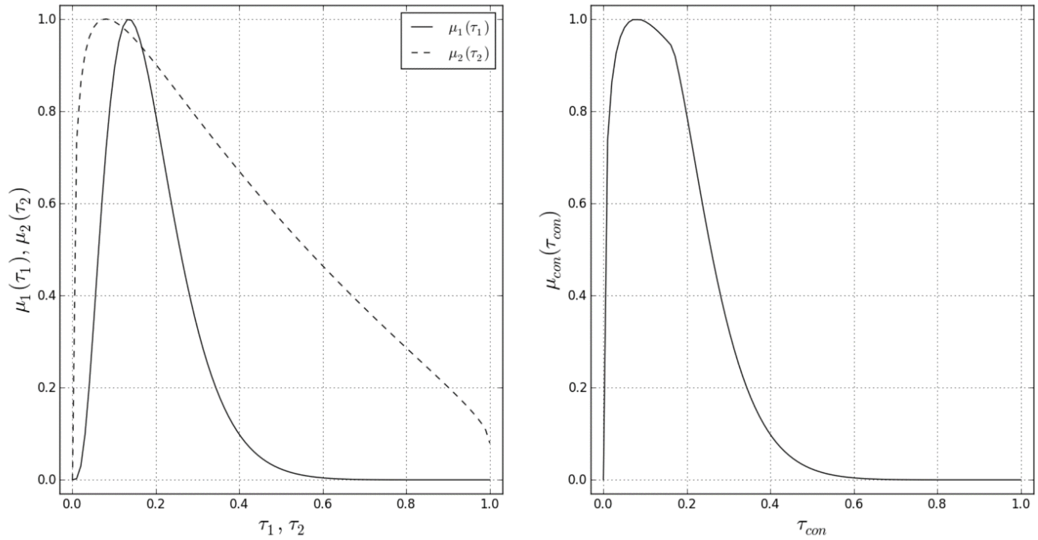

Figure 1. The result of FTV conjunction.

6. Fuzzy implications

Need to use a variety of implications take place when tuning fuzzy systems (see, for example, [9]). Using the most frequently used fuzzy implications [10], define $I(\tau, \mu_{B_k}(y))$.

Figures 2-11 show families of fuzzy implication, considered below, for values $\mu_{B_k}(y)$ from 0 to 1 in increments of 0,1.

- Kleene-Dienes implication

$$I(\tau, \mu_{B_k}(y)) = \max[1 - \tau, \mu_{B_k}(y)]; \tag{6.1}$$

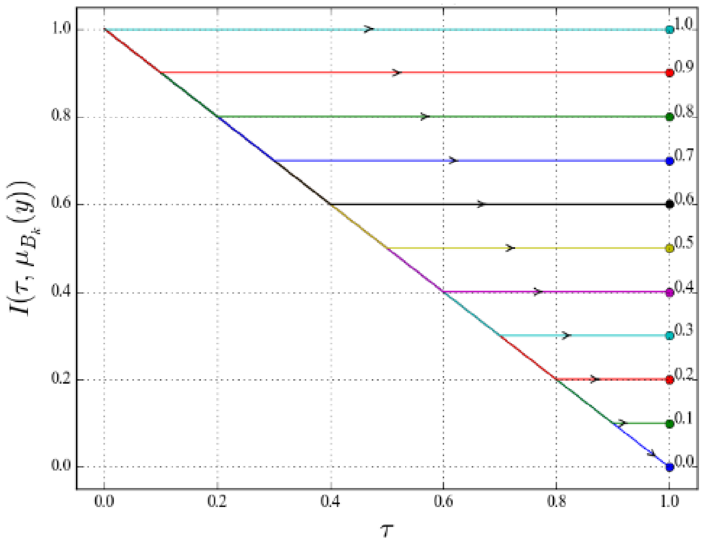

Figure 2. Kleene-Dienes implication.

- Lukasiewicz implication

$$\mathrm{I}\big(\tau, \mu_{B_k}(y)\big) = \min\big[1, 1 - \tau + \mu_{B_k}(y)\big]; \tag{6.2}$$

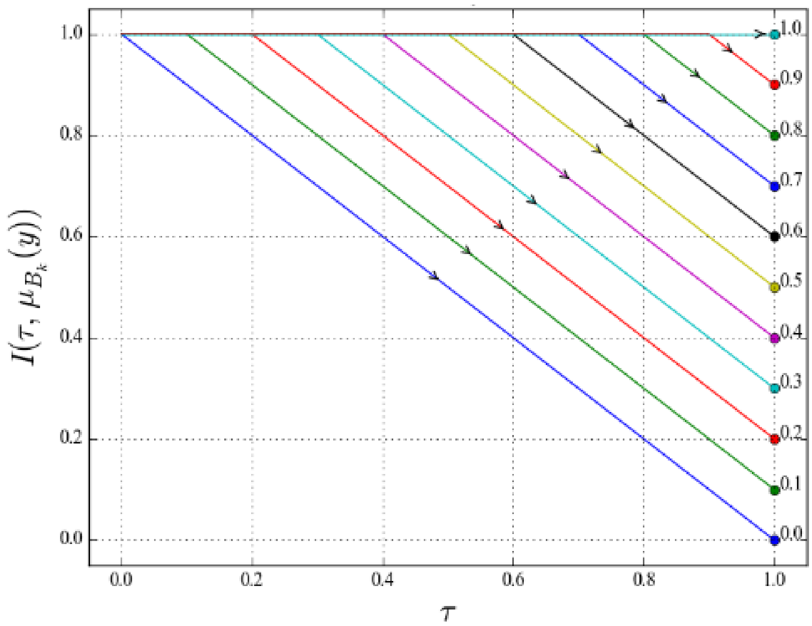

Figure 3. Lukasiewicz implication.

- Reichenbach implication

$$\mathrm{I}\big(\tau, \mu_{B_k}(y)\big) = 1 - \tau + \tau \cdot \mu_{B_k}(y); \tag{6.3}$$

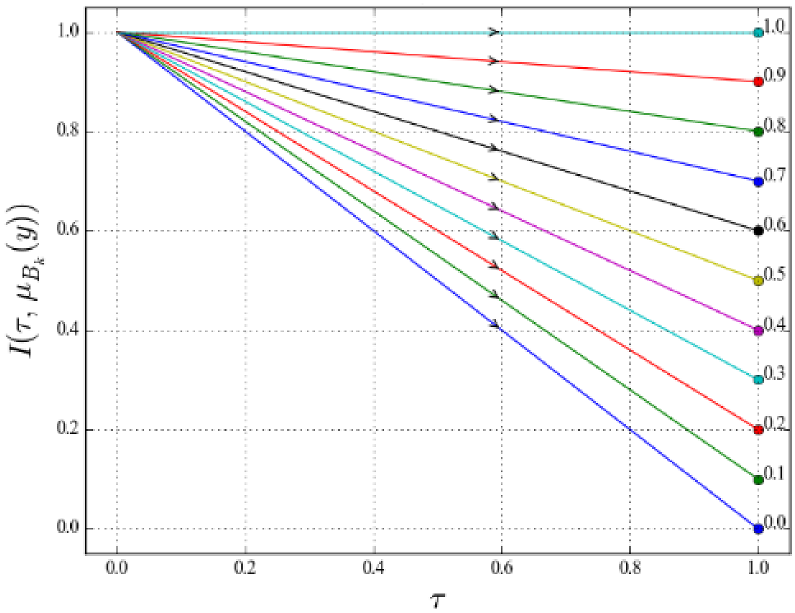

Figure 4. Reichenbach implication.

- Fodor implication

$$\mathrm{I}\big(\tau, \mu_{B_k}(y)\big) = \begin{cases} 1, & \tau \le \mu_{B_k}(y), \\ \max\,[1 - \tau, \mu_{B_k}(y)], & \tau > \mu_{B_k}(y); \end{cases} \tag{6.4}$$

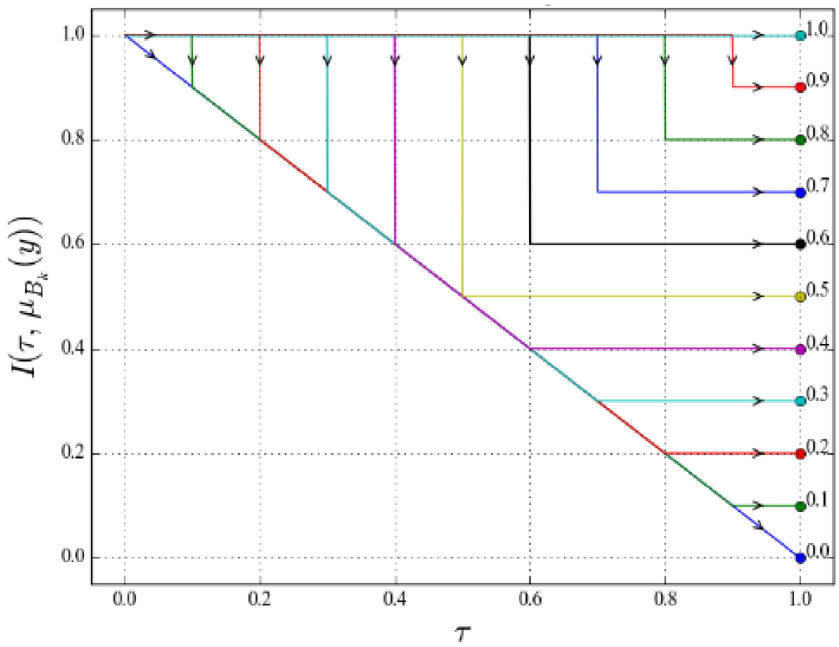

Figure 5. Fodor implication.

- Rescher implication

$$I(\tau, \mu_{B_k}(y)) = \begin{cases} 1, & \tau \le \mu_{B_k}(y), \\ 0, & \tau > \mu_{B_k}(y); \end{cases} \tag{6.5}$$

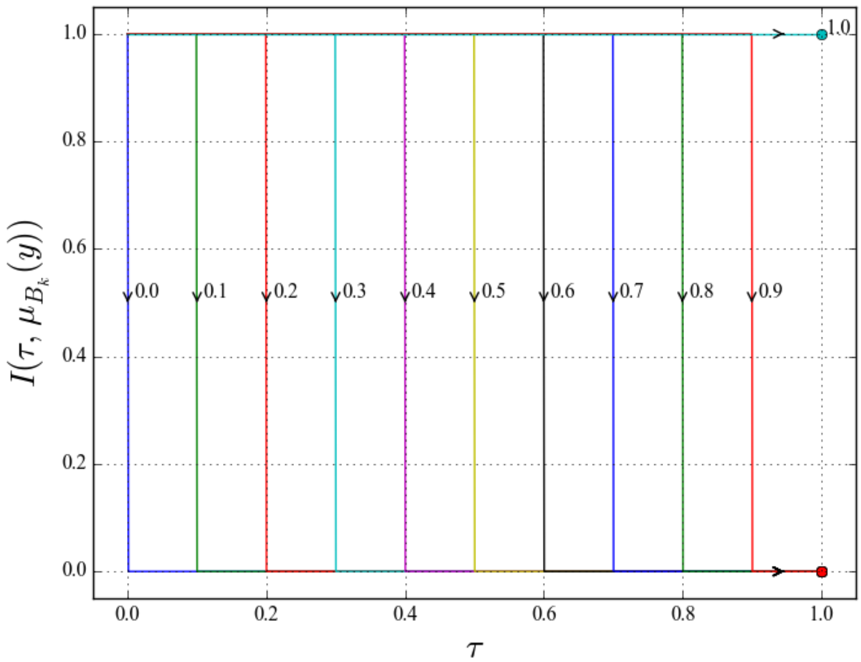

Figure 6. Rescher implication.

- Goguen implication

$$I(\tau, \mu_{B_k}(y)) = \begin{cases} 1, & \tau = 0, \\ \min\left[1, \frac{\mu_{B_k}(y)}{\tau}\right], & \tau > 0; \end{cases} \tag{6.6}$$

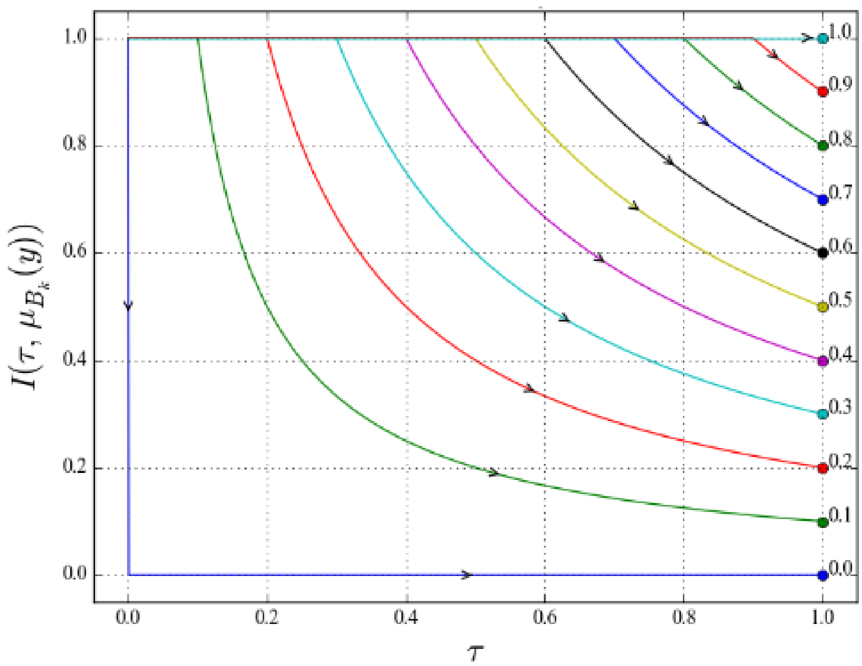

Figure 7. Goguen implication.

- Godel implication

$$\mathrm{I}(\tau, \mu_{B_k}(y)) = \begin{cases} 1, & \tau \le \mu_{B_k}(y), \\ \mu_{B_k}(y), & \tau > \mu_{B_k}(y); \end{cases} \quad (6.7)$$

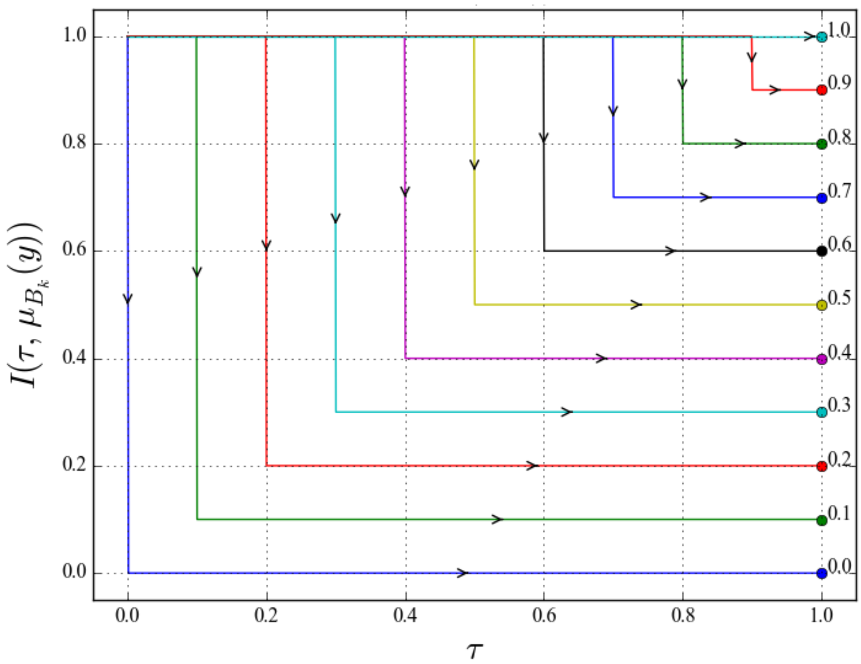

Figure 8. Godel implication.

- Yager implication

$$\mathrm{I}(\tau, \mu_{B_k}(y)) = \begin{cases} 1, & \tau = 0, \\ \mu_{B_k}(y)^{\tau}, & \tau > 0; \end{cases} \quad (6.7)$$

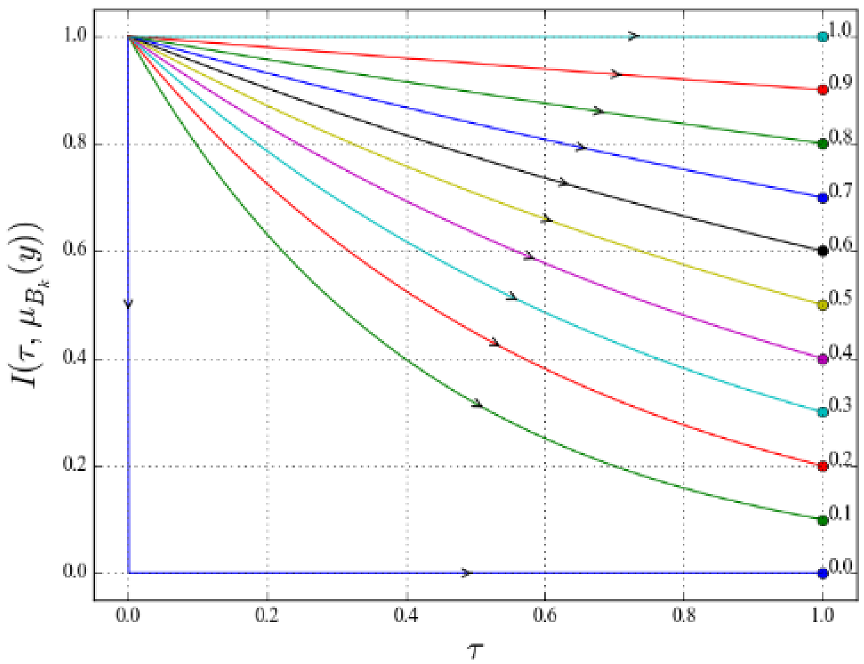

Figure 9. Yager implication.

- Zadeh implication

$$\mathrm{I}\left(\tau, \mu_{B_k}(y)\right) = \max\left\{\min\left[\tau, \mu_{B_k}(y)\right], 1 - \tau\right\}; \qquad (6.8)$$

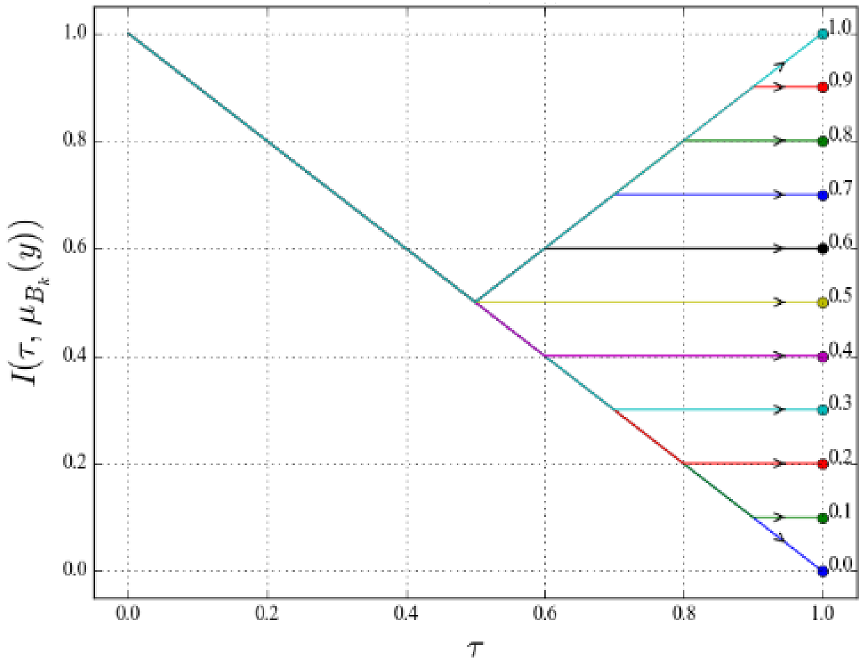

Figure 10. Zadeh implication.

- Wilmott implication

$$\mathrm{I}\left(\tau, \mu_{B_k}(y)\right) = \min\left\{\max\left[1 - \tau, \mu_{B_k}(y)\right], \max\left[\tau, 1 - \mu_{B_k}(y), \min\left(\mu_{B_k}(y), 1 - \tau\right)\right]\right\}; \qquad (6.9)$$

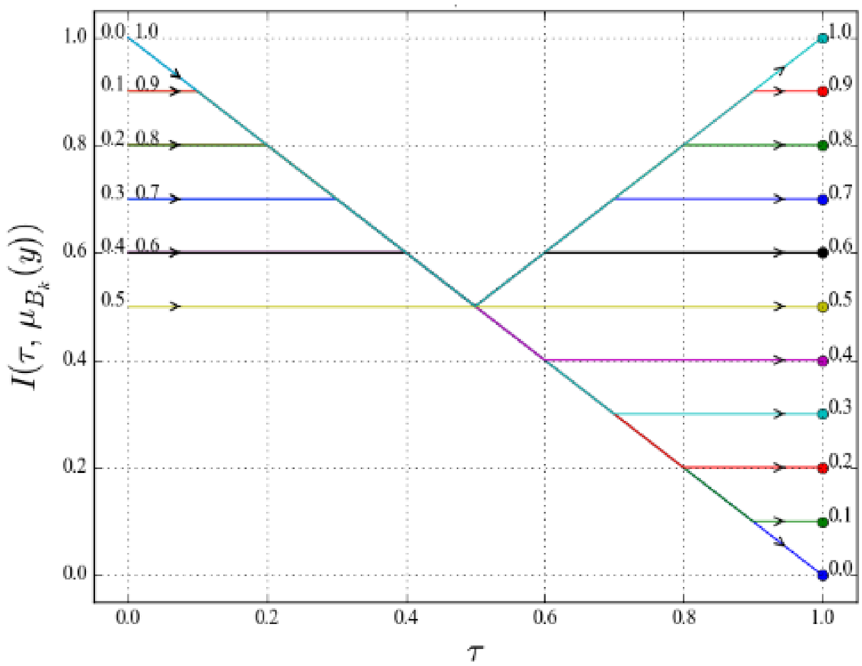

Figure 11. Wilmott implication.

7. Calculation of fuzzy output value

When using the logical model of a fuzzy system, aggregating the values of the outputs $B'_1, \dots, B'_N$ for each of the N rules is performed by using of intersection operation of fuzzy sets:

$$B' = \bigcap_{k=1}^{N} B'_k.$$

Membership function B' is calculated using the t-norm, that is,

$$\mu_{B'}(y) = \mathop{\mathrm{T}}_{k=\overline{1,N}} \mu_{B'_k}(y). \tag{7.1}$$

Next, a fuzzy set B′ mapped into crisp scalar value $\bar{y}$. Using method of the center of gravity in its discrete version:

$$\bar{y} = \frac{\sum_{l=1}^{N} \bar{y}_l \cdot \mu_{B'}(\bar{y}_l)}{\sum_{l=1}^{N} \mu_{B'}(\bar{y}_l)}, \tag{7.2}$$

where $\bar{y}_l,\ l = \overline{1,N}$ – values of the centers of membership functions μ_{B_k} [11].

When considering logical-type systems $I(*)$ works like a fuzzy implication. Next using the S-implication [12]:

$$I\left(\tau, \mu_{B_k}(y)\right) = S\left(1 - \tau, \mu_{B_k}(y)\right). \tag{7.3}$$

Following (3.1), (7.1), and (7.3), the fuzzy set B' will be determined like:

$$\mu_{B'}(y) = \mathop{\mathrm{T}}_{k=\overline{1,N}} \left\{ \sup_{\tau \in [0,1]} \left\{ \mu_{CP_k}(\tau) \overset{\mathrm{T}}{*} S\left(1 - \tau, \mu_{B_k}(y)\right) \right\} \right\},$$

where $\mu_{CP_k}(\tau)$ – FTV, obtained for the k-th rule as a result of convolution by the algorithm considered above.

Hence the expression (7.2) takes the form:

$$\bar{y} = \frac{\sum_{l=1}^{N} \bar{y}_l \cdot \mathop{\mathrm{T}}_{k=\overline{1,N}} \left\{ \sup_{\tau\in[0,1]} \left\{ \mu_{CP_k}(\tau) \overset{\mathrm{T}}{*} S\left(1-\tau, \mu_{B_k}(\bar{y}_l)\right) \right\} \right\}}{\sum_{l=1}^{N} \mathop{\mathrm{T}}_{k=\overline{1,N}} \left\{ \sup_{\tau\in[0,1]} \left\{ \mu_{CP_k}(\tau) \overset{\mathrm{T}}{*} S\left(1-\tau, \mu_{B_k}(\bar{y}_l)\right) \right\} \right\}} \tag{7.4}$$

Figure 12 illustrates the definition of the supremum in (7.4) with the using of Lukasiewicz's implication, that is:

$$S\left(1-\tau, \mu_{B_k}(y)\right) = \min\left\{1, 1-\tau+\mu_{B_k}(\bar{y}_l)\right\}.$$

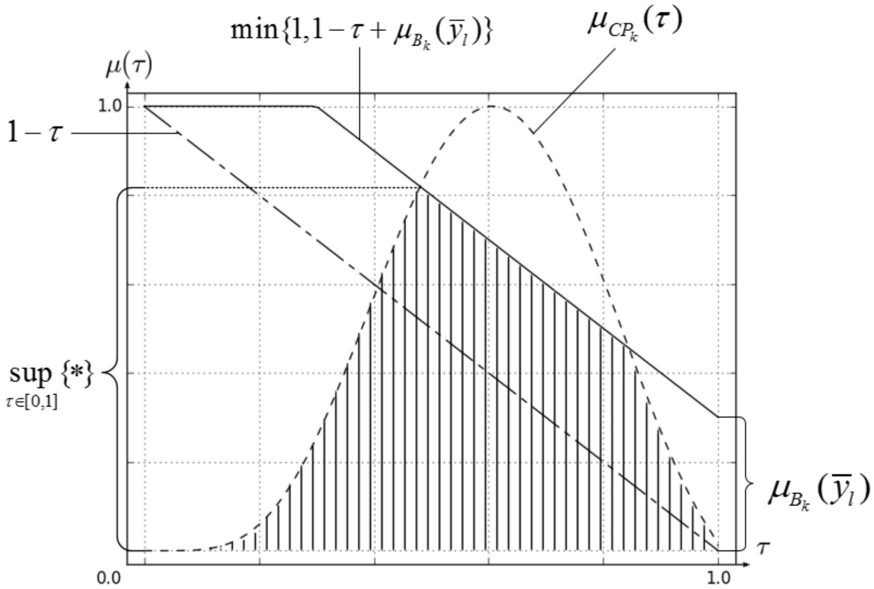

Figure 12. Graphical interpretation of the supremum definition for the expression (7.4).

In figure 13, the relationship (7.4) is represented as the network structure of the system. Herewith:

$$F_{kl}\left(\mu_{CP_k}(\tau), \bar{y}_l\right) = \sup_{\tau\in[0,1]} \left\{ \mu_{CP_k}(\tau) \overset{\mathrm{T}}{*} S\left(1-\tau, \mu_{B_k}(\bar{y}_l)\right) \right\}$$

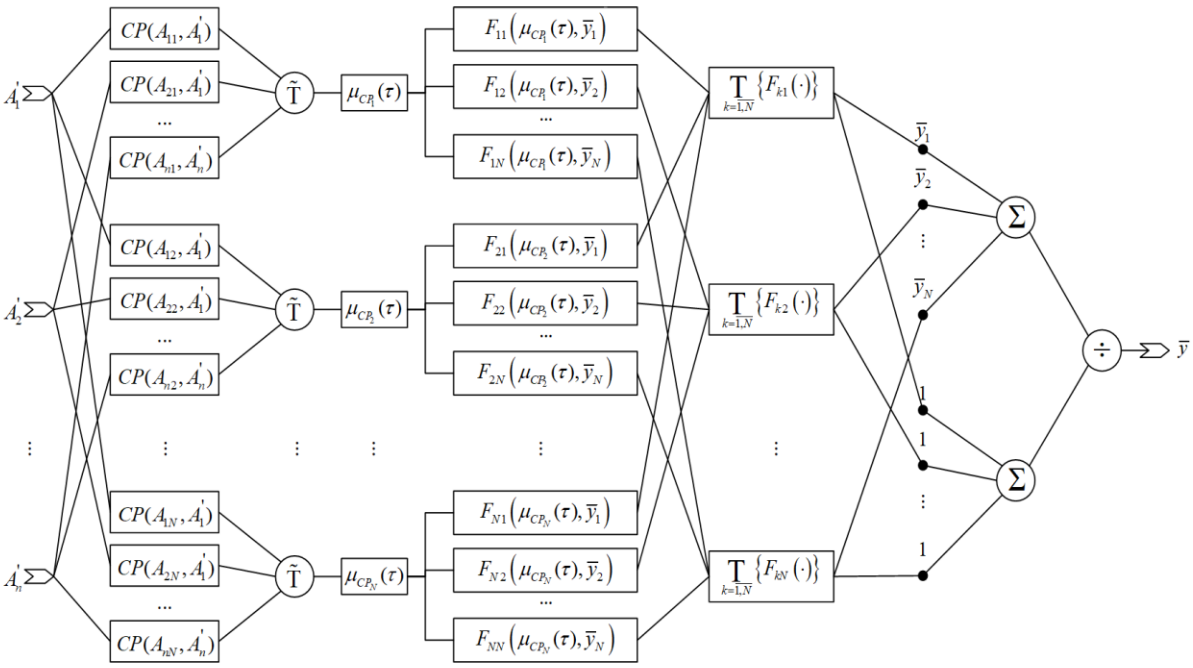

Figure 13. The network structure corresponding to the relation (7.4).

8. Conclusion

In this paper, proposed a method of fuzzy inference for logical-type systems whose inputs are fuzzy values. Presented an algorithm of conjunctions and disjunctions operations for FTV. Graphically illustrates functioning of the algorithm, as well as using of some fuzzy implication functions.

The presented method has a polynomial computational complexity that allows using it for solving problems of modeling systems with a large number of fuzzy inputs, such as diagnostics, prediction and control (see, for example, [6]). The further research task is the development of learning algorithms necessary for the transformation of network structures obtained based on the received expressions.

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