A novel heuristic for the coloring of planar graphs

Guillermo De Ita, Cristina López-Ramírez, and Adriana Luna

Facultad de Ciencias de la Computación, BUAP
deita@cs.buap.mx, cristyna2001@hotmail.com, adrylu88@gmail.com

Abstract. A novel algorithm is proposed for the coloring of planar graphs based on the construction of a maximal independent set S. The maximal independent set S must fullfil certain characteristics. It must contain the vertex that appears in the maximum number of odd cycles of G. The construction of S considers the internal-face graph of the input graph G in order to select the vertices that decomposes a maximal number of initial faces of G.

The traversing in pre-order of the internal-face graph G_f, of the planar graph G, provides us of a strategy in the construction of the maximal independent set S that reduces $(G - S)$ in a polygonal tree that will be 3-colorable.

1 Introduction

The graph vertex coloring problem consists of coloring the vertices of the graph with the smallest possible number of colors, so that two adjacent vertices can not receive the same color. If such a coloring with k colors exists, the graph is k-colorable. The chromatic number of a graph G, denoted as $X(G)$, represents the minimum number of colors for proper coloring G. The k-colorability problem consists of determining whether an input graph is k-colorable.

The inherent computational complexity, associated with solving NP-hard problems, has motivated the search for alternative methods, which allow in polinomial time the solution of special instances of NP-hard problems. For example, in the case of the vertex coloring problem, 2-coloring is solvable in polinomial time. Also, in polynomial time has been solved the 3-colorability for some graphs topologies, such as: AT-free graphs and perfect graphs, as well as to determine $X(G)$ for some classes of graphs such as: interval graphs, chordal graphs, and comparability graphs [1] . In all those cases, special structures (patterns) have been found to characterize the classes of graphs that are colorable in polinomial time complexity.

Graph vertex coloring is an active field of research with many interesting subproblems. The graph coloring problem has many applications in areas such as: scheduling problems, frequency allocation, planning, etc. [2–4].

Coloring planar graphs represents a relevant area of interest in graph and complexity theory, since it involves the frontier between efficient and intratable

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computational procedures. Into the set of planar graphs, the polygonal chain graphs have been a relevant issue of researching in mathematical chemistry, maybe because they express molecular graphs used to represent the structural formula of chemical compounds [5, 7, 8].

The 3-coloring problem has been shown to be a hard problem (NP-complete problem). We propose a novel greedy algorithm for the coloring on planar graphs. Our proposal is based on the logical specification of the constraints given by a 3-coloring of an polygonal array as the core of the coloring on planar graphs.

2 Preliminaries

Let $G = (V, E)$ be an undirected simple graph (i.e. finite, loop-less and without multiple edges) with vertex set V (or $V(G)$) and set of edges $E(orE(G))$. Two vertices v and w are called adjacent if there is an edge $v, w \in E$, joining them. The Neighborhood of $x \in V$ is $N(x) = \{y \in V : \{x, y\} \in E\}$ and its closed neighborhood is $N(x) \cup \{x\}$ which is denoted by $N[x]$. Note that v is not in $N(v)$, but it is in $N[v]$. We denote the cardinality of a set A, by $|A|$. The degree of a vertex $x \in V$, denoted by $\delta(x)$, is $|N(x)|$. The maximum degree of G, or just the degree of G, is $\Delta(G) = max\{\delta(x) : x \in V\}$.

A path from a vertex v to w is a sequence of edges: $v_0v_1, v_1v_2, ..., v_{n-1}v_n$ such that $v = v_0, v_n = w$, v_k is adjacent to v_{k+1} and the length of the path is n. A simple path is a path such that $v_0, v_1, ..., v_{n-1}, v_n$ are all different. A cycle is just a nonempty path in which the first and last vertices are identical; and a simple cycle is a cycle in which no vertex is repeated, except the first and last vertices. A k-cycle is a cycle of length k (it has k edges). A cycle of odd length is called an odd cycle, while a cycle of even length is called an even cycle. A graph without cycles is called acyclic.

Given a subset of vertices $S \subseteq V$, the subgraph of G where S is the set of vertices and the set of edges is $\{\{u, v\} \in E : u, v \in S\}$, is called the subgraph of G induced by S, and it is denoted by $G|S$. $G - S$ denotes the graph $G|(V - S)$. The subgraph induced by $N(v)$ is denoted as $H(v) = G|N(v)$, which contains all the nodes of $N(v)$ and all the edges that connect them.

An independent or stable set is a set of vertices in a graph such that none of its vertices is adjacent to another. That is, it is a set $S \subseteq V(G)$ of vertices such that for any pair of them there is not an edge that connects them.The size of an independent set is the number of vertices it contains. An independent set is maximal if it is not a proper subset of another another independent set, and it is maximum in G if there is not another independent set in G with a cardinality higher than $|S|$.

A coloring of a graph $G = (V, E)$ is an assignment of colors to its vertices. A coloring is proper if adjacent vertices always have different colors. A k-coloring of G is a mapping from V into the set {1,2,...,k} of k "colors". The k- colorability problem consists of deciding wheter an input graph is k-colorable. The chromatic number of G denoted by $X(G)$ is the minimum value k such that G has a proper k-coloring. If $X(G) = k$, then G is said to be k-chromatic or k- colorable.

Let $G = (V, E)$ be a graph. G is a bipartite graph if V can be partitioned into two subsets U_1 and U_2, called partite sets, such that every edge of G joins a vertex of U_1 to a vertex of U_2. If $G = (V, E)$ is a k-chromatic graph, then it is possible to partition V into k independent sets $V_1, V_2, ..., V_k$ called color classes, but it is not possible to partition V into $k - 1$ independent sets.

3 Planar Graphs

Planar graphs play play an important role both in the graph theory and in the graph drawing area. In fact, planar graphs have several interesting properties: they are sparse, four-colorables, and their inner structure is described succintly and elegantly [9].

A drawing Γ of a graph G maps each vertex v to a distinct point $\Gamma(v)$ of the plane and each edge (u, v) to a simple open Jordan curve $\Gamma(u, v)$ with endpoints $\Gamma(u)$ and $\Gamma(v)$. A drawing is planar if no two distinct edges intersect except, possibly, at common endpoints. A graph is planar if it admits a planar drawing. A planar drawing partitions the plane into connected regions called faces. The unbounded face is usually called external face or outer face. If all the vertices are incident to the outer face the planar drawing is called outerplanar, and the graph admitting it is an outerplanar graph.

Given a planar drawing, the (clockwise) circular order of the edges incident to each vertex is fixed. Two planar drawings are equivalent if they determine the same circular orderings of the edges incident to each vertex (sometimes called rotation scheme). A (planar) embedding is an equivalent class of planar drawings and is described by the clockwise circular order of the edges incident to each vertex. A graph put together with one of its planar embedding is sometimes referred to as a plane graph. A non-connected graph is planar if and only if all its connected components are planar.

Perhaps the most renown property is the one stated by Euler's Theorem, which shows that planar graphs are sparse. Namely, given a plane graph with n vertices, m edges and f faces, we have $n - m + f = 2$. A simple corollary that can be deduced from the Euler's rule, is that for a maximal planar graph with at least three vertices, where each face is a triangle, then ($2m = 3f$), and as $m = 3n - 6$, and, therefore, for any planar graph we have $m \leq 3n - 6$. This number reduces to $m = 2n - 3$ for maximal outerplanar graphs with at least three vertices (and $m \leq 2n - 3$ for general outerplanar graphs). Also, if $n \geq 3$ and the graph has no cycle of length 3, then $m \leq 2n - 4$. Finally, if the graph is a tree, then $m = n - 1$.

These considerations allow us to replace m with n in any asymptotic calculation involving planar graphs, while for general graphs only $m \in O(n^2)$ can be assumed. From a more practical perspective, they allow us to decide the non-planarity of denser graphs without reading all the edges (which would yield a quadratic algorithm).

The first complete characterization of planar graphs is due to Kuratowsky [14], and states that a graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$, where K_5 is the complete graph of order 5 and $K_{3,3}$ is the complete bipartite graph with 3 vertices in each of the sets of the partition. A similar result, recasted in terms of graph minors, is Wagners theorem that states that a graph G is planar if and only if it has no K_5 or $K_{3,3}$ as minor, that is, K_5 or $K_{3,3}$ cannot be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices [15, 16]. Observe that the two characterizations are different since a graph may admit K_5 as minor without having a subgraph that is a subdivision of K_5.

4 Coloring a planar graph

From now on, we consider only planar graphs as working graphs. The purpose of this research is to determine when a planar graph is 3 or 4-colorable. The famous four coloring theorem (4CT) [10–12] guarantees us that all planar graph is 4-colorable. Furthermore, 4CT provides an $O(n^2)$ time algorithm to 4-color any planar graph [11, 12]. However, the current known proof for the 4CT is computer assisted. In addition, the correctness of the proof is still lengthy and complicated [17].

On the other hand, any planar graph triangle-free is 3-colorable, according to the relevant theorem by Grotszch [13]. In a similar sense, it is easy (in lineartime on the size of the graph) to recognize if an input graph is 2-colorable, since it involves to recognize if the graph has (or not) even cycles.

Thus, the hard part to recognize the 3-colorable or 4-colorable planar graphs is when they contain triangles, because it exists planar graphs for both cases.

For example, any planar graph containing K_4 or odd wheels will request four colors to proper coloring those graphs. However, those patterns are no the unique 4-colorable cases. If we compose two (or more) wheels sharing one triangle and adding one edge to join the vertices of the last triangle of each wheel, we can form 3 or 4-colorables planar graphs. We show in the following section our proposal for the coloring of planar graphs

4.1 An algorithm for the coloring of planar graphs

We consider as input a planar graph $G = (V, E)$ whose draw has been already embedded in the plane. Let $Tres = \{1, 2, 3\}$ be the set of the three possible colors to use. To each vertex $x \in V(G)$ a set of prohibited colors is associated with it, denoted by $Tabu(x)$. The following lemma proposes a method for the 3-coloring of acyclic components whose vertices have at most one prohibited color.

Lemma 1 *An acyclic component where its vertices have at most one color as a restriction is 3-colorable.*

Proof 1 *The acyclic component is considered as a tree rooted in v_r. A pre-order coloring is made from v_r, where $Color(v_r) = MIN\{Tres - Tabu(v_r)\}$. When advancing in pre-order in each new level to be colored, all vertex y_i in the new* *level will have at most two restricted colors from its parent node and the color that could exist in $Tabu(y_i)$. Thus, it has been always available a color of the three possible in $Tres$. The 3-coloring process ends when all nodes of the tree have been visited in pre-order.*

We will call the above procedure for coloring acyclic graphs, as the $ISAT(G)$ process. A planar graph G has a set of closed non-intersected regions $R = \{r_1, \ldots, r_k\}$ called faces. Each face r_i is represented by the set of edges that bound its inside area. All edge $\{u, v\}$ in G that is not the border of some face from G, is represented by its vertices label uv, and they are called *acyclic edges.*

Two faces $r_i, r_j \in R$ are adjacents if they have common edges, this is, $E(r_i) \cap E(r_j) \neq \emptyset$. Otherwise, they are independent faces. Two acyclic edges are adjacents if they share a common endpoint. An acyclic edge is adjacent to a region r_i if they have just one common vertex. A set of faces is independent if each pair of them are independent.

Lemma 2 *Let $A = \{f_1, f_2, \ldots, f_n\}$ be a set of n faces where each face has at least one vertex that does not restrict color 3. If the set of faces is independent or all of them have a common vertex, then A is 3- colorable*

Proof 2 *This Lemma is shown by induction on the number of faces in the set.*

1. *For a single face this is 3-colorable, since every cycle with at least one unrestricted vertex is 3-colorable.*
2. *Suppose that the hypothesis on sets up to $n-1$ faces is validated.*
3. *Let A be a set of n faces where each face has at least one vertex that does not restrict color 3. If there is a face $f_a \in A$ that is independent with all other regions in A, since f_a is 3-colorable (as in the case 1), f_a can be removed from A. The remaining set in A has $n-1$ faces, and the inductive hypothesis is held.*
   Otherwise, the n faces in A share a common vertex. Let $x \in f_i, \forall f_i \in A$ be the common vertex. If $3 \notin Tabu(x)$ then by assigning the color 3 to x and removing it from the graph, all the faces in A become opened and form an acyclic graph, which is 3-colorable by Lemma 1.
   Assuming $Tabu(x) = 3$, but $\forall f_i \in A$, there is $y_i \in V(f_i)$ such that $Tabu(y_i) = \emptyset$. By assigning color 3 to each one of these $y_i's$, and eliminating them from each $V(f_i)$, an acyclic component is formed, that it is 3-colorable by Lemma 1.

Lemma 3 *For any vertex $v \in V(G)$, $N[v]$ is $\delta(v) + 1$-colorable.*

Proof 3 *Assume that all $y \in N(v)$ have different colors from each other. Then, v has $\delta(v)$ neighborhood colors and it can take only a different color from its neighborhood, so $N[v]$ takes $\delta(v) + 1$ colors. If there are repeated colors in the neighborhood of v, then it is needed the use of at most $\delta(v)$ colors, therefore, $N[v]$ is $\delta(v) + 1$-colorable.*

The previous lemma is applied in the sense that for any vertex of minimal degree (e.g less than 3), 3 colors is enough to be colored. Afterwards, it can be removed at the beginning of any 4-coloring algorithm in order to simplify the resulting graph. Also this Lemma justifies to keep only one edge between adjacent faces, because more than two edges between adjacent faces imply vertices of degree 2 that can be contracted to any of the extremal points of the common boundary between both faces.

We build an internal-face graph $G_f = (X, E(G_f))$ from G, in the following way:

1. Each face $r_i \in R$ has attached a node $x \in V(G_f)$ labeled by its composing edges.
2. Each acyclic edge from G has attached a node of G_f labeled by its vertices label.
3. There is an edge $\{u, v\} \in E(G_f)$ joining two adjacent nodes of G_f when its corresponding faces (or acyclic edges) are adjacent in G.
4. Each edge in G_f is labeled by the common elements (a vertex or edge) between the two adjacent nodes.

G_f is called the *internal-face graph* of G. Notice that G_f is not the dual graph of G, since in the construction of G_f the external face is not considered. Notice that G_f is a planar graph too, where its nodes represent faces or edges from G, but it is not neccesarily a tree graph. However, if G_f has a tree topology, then we achieve a relevant property for the coloring of G.

When G_f is a tree we call to G, its corresponding planar graph, a *polygonal tree* [6]. In this case, an order for visiting the faces of the planar graph provides efficient procedures for 3-coloring G, as the following thereom claims.

Theorem 1 *If the internal-face graph of a planar graph G has a tree topology, then G is 3-colorable.*

Proof 4 *Let G_f be the internal-face graph of a planar graph G. Each face of G represented by a node $x \in V(G_f)$ is 3-colorable since it is a simple cycle. Furthermore, all acyclic edge of G, represented by a node of G_f, is also 3-colorable since all acyclic graph is in fact 2-colorable. Now, we propose a 3-coloring for G based in traversing the nodes of G_f in pre-order, coloring first the face of the father node of G_f and after, the faces of its children. In each current level, the two adjacent faces (father and children in G) are considered. Both regions have two common extremal vertices x, y in its common boundaries. Those common vertices are colored first, and then, the remaining vertices in both faces have two prohibited colors at most. Notice that there is not a pair of adjacent vertices u and v in any of the both faces, such that $\{u, v\} \subseteq (N(x) \cap N(y))$, because then $\{x, y, u, v\}$ form K_4 and this subgraph can be not part of any polygonal tree. Thus, for all remaining vertices in both faces, it is available at least one color of the three possible in $Tres$. The 3-coloring process ends when all the nodes of the tree G_f have been visited in pre-order.*

The order in the traversing of the common edges between adjacent faces by the previous theorem, provides us maximal paths formed by the common edges and common vertices from the root node until a leaf node of G_f is achieved. This strategy is described in the following procedure.

1. To build the internal-face graph G_f from G.
2. If(G_f is a tree, or it is formed by independent faces, or all faces have a common vertex) then Return(G is 3-Colorable) /* by previous Lemmas and theorem */
3. Otherwise, let M be the set of common edges between each pair of adjacent faces.
4. To build the maximal independent sets from M.
5. It iterates over each one of the maximal independent sets from M. Let e.g. $S = \{s_1, s_2, \ldots, s_k\}$ be one of those maximal independent sets.
6. The color 3 is assigned to each $s_i \in S$, and $\forall y \in N(s_i), Tabu(y) = 3$.
7. $G = G - S$
8. The iteration on the maximal independent sets continues until all the sets are processed.
9. Only the new individual closed regions that were created by coloring the graph will remain in G.

At the end of this process we have as output: If G_f holds the conditions of the lemma or previous theorem, then G is 3-colorable. If the process returns a new graph G', where there is a face of odd length and all their adjacent vertices are restricted with the color 3, then G is 4-colorable.

4.2 Design of the proposal

We present now, a polynomial-time algorithm for the coloring of a planar graph G. Given an embedding of a planar graph G, the first step consists in reducing more than two edges between adjacent faces to only one common edge. It must be removed all $v \in V(G)$ such that $\delta(v) < 3$, since those vertices are 3-colorable (based on Lemma 3). Afterwards, a new planar graph G_a is formed from the remaining vertices and edges from G.

Furthermore, if G_a is 3-colorable then G is also 3-colorable. For example, if G_a is a polygonal tree, outerplanar, serial - parallel, or G_a is a triangle-free graph, then G is 3-colorable, based on the previous Lemmas and theorem.

The second step is to choose an initial region of G_a to start the process of vertices-coloring. For this, we look for the region (face) fr where the sum of the degrees of its vertices is maximal. When such region is identified, we search for the vertex $x_a \in V(fr)$ of maximum degree to be colored first, that is: $f(x_a) = color(x_a); \forall y \in N(x_a) : Tabu(y) = Tabu(y) \cup \{f(x_a)\};$

Once the first vertex x_a and a color c have been chosen, $S = S \cup \{x_a\}$; $G_a = G_a - \{x_a\}$, an iterative process is performed with the purpose to form a maximal independet set S from G_a. Thus, the main strategy in our proposal consists in calculating the maximal independent set S of G and assigning a color for every vertex $y \in S$. To the remaining graph $(G - S)$ is colored by using the ISAT procedure.

The following pseudo-codes apply the above described procedures.

Algorithm 1 3-*coloring*

Input: a planar graph G
Output: the graph G with a validate coloring

List the faces of the graph G
Form the graph G_f (internal-face graph of G)
$S = Max\_Ind\_Set(G_f)$; a maximal independent set of vertices of G
for each vertex $x \in S$ **do**
  assign color 3 to x
  for each $y \in N(x)$ **do**
    Assign $Tabu(y) = Tabu \cup \{3\}$
  end for
end for
$G = G - S$
if G contains an odd cycle whose vertices are restricted with color 3 **then**
  $X(G) = 4$; exit
end if
if G is a base case (acyclic or a polygonal tree) **then**
  Apply ISAT procedure to assign a valid coloring to G
else
  repeat
end if

Algorithm 2 Max_Ind_Set(G): Construction of a maximal independent set of G

Input: a planar graph G and his graph G_f(internal-face graph)
Output: S is a maximal independent set of G

Select $x \in V(G)$ the vertex that belongs to the largest number of odd cycles in G
Select $y \in V(G_f)$ the vertex of higher degree in G_f
for each $u \in N(y)$ **do**
  mark u as visited
end for
for each vertex $u \in G_f$ that is not visited **do**
  visit u and mark $N(u)$
end for
for each vertex u visited in the previous step **do**
  calculate the partial maximal independent set considering the edges of G
end for
S is the conjunction among the partial maximal independent sets already calculated

Example of planar graph G where the faces are labeled (Figure 1).

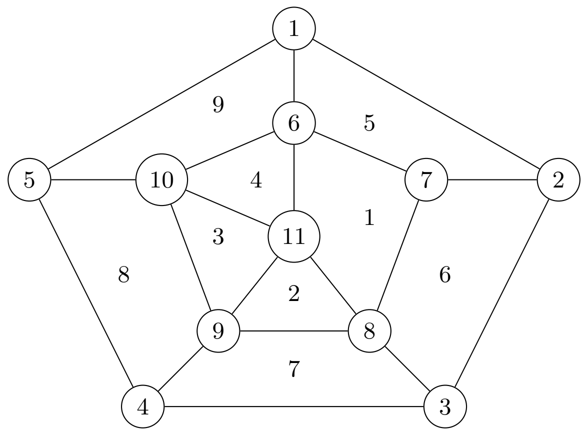

Fig. 1. Graph G with faces identified

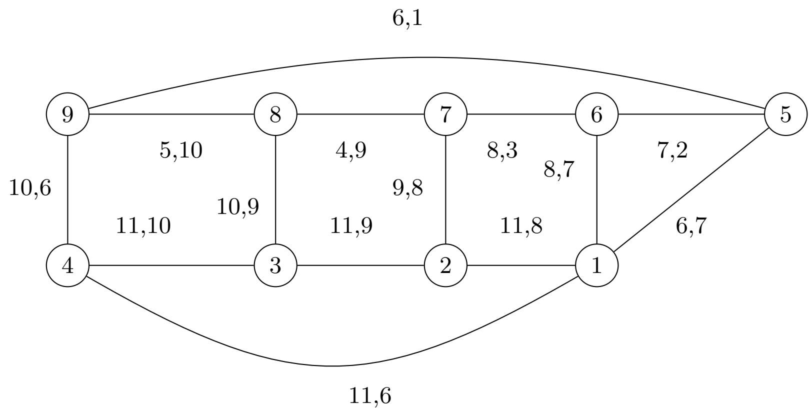

Fig. 2. The internal-face graph G_f of G

A relevant strategy in our algorithm is to consider the vertex appearing in the maximum number of odd cycles of G as part of the maximal independent S to be built. For example, according to the graphs in Figure 1 and 2, two maximal independent sets are: $S_1 = \{2, 4, 6, 8\}$, and $S_2 = \{1, 4, 7, 11\}$. However, S_2 contains the vertex 11 which appears in the maximum number of odd cycles of G, then S_2 is a better option than S_1.

Table 1. Odd cycles table

Vertex x	Odd cycles
1	0
2	0
3	0
4	0
5	0
6	1
7	0
8	1
9	2
10	2
11	3

The following tables show the results from step 1 through 7 of the algorithm. For example, Table 1 shows the list of all vertices of G in the first column, the second column represents the number of odd cycles in which this vertex belongs. Meanwhile, Table 2 shows the vertices obtained after executing the steps 2-8 of the algorithm.

Table 2. $N(x)$ table

Vertex visited x	$N(x)$
1	2,6,5,4
3	4,2,8
7	2,6,8
9	8,5,4

Table 3. Maximal independent sets: partial and total

Vertex	Edge label	Partial maximal independent set
1	(11,8),(8,7),(7,6),(6,11)	11,7
3	(11,10),(10,9),(9,11)	11
7	(4,9),(9,8),(8,3)	4
9	(1,6),(5,10),(10,6)	1
Maximal independent set		1,4,7,11

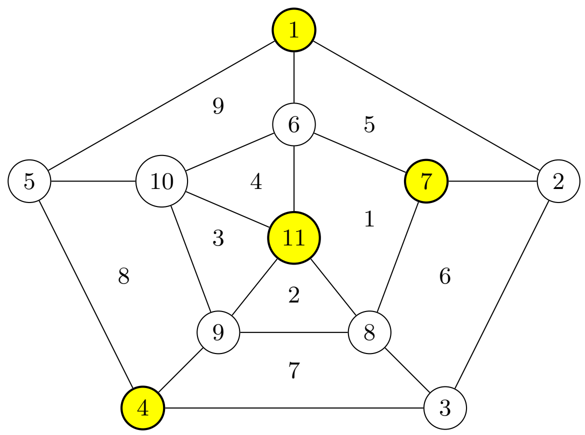

Fig. 3. Assigning color 3 to the vertices of S - a maximal independent set of G

After removing the vertices already colored, a base coloring case graph is obtained (applying ISAT), making it a 3-*colorable* instance.

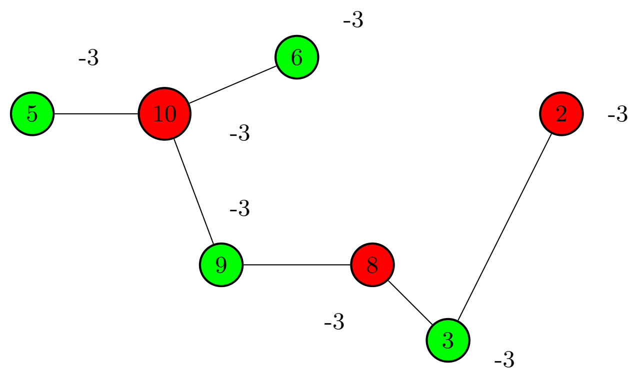

Fig. 4. A 2-coloring on the resulting graph $(G - S)$

4.3 Time-Complexity Analysis

The most expensive time-procedure in our proposal is the algorithm 2. The time-complexity analysis of this procedure follows in this section.

It's known that to build the internal-face graph G^f requires time $O(m+n)$. Into the algorithm 2, the step 1 can be performed in time $O(m+n)$, based in a depth-first search on G, in order to recognize the parity of the cycles in G. The step 2 requires time $O(f)$, that in a worst case, it is of order $O(m)$, when the faces in G has minimal lengths. The *for* in steps: $3 \rightarrow 5$, consists in visiting all vertex in G^f, that in a worst case, it is also of $O(m)$. The *for* in steps: $6 \rightarrow 8$, is similar to previous for, and it has also a time complexity of $O(m)$, in a worst case. The last *for* (steps: $9 \rightarrow 11$), consists in visiting all vertex in G, and it has a complexity time of $O(n)$. The most expensive step in algorithm 2 is the last step (step 12). It must to check the constraint for independence vertices in the arrays containing partial maximal independent sets, and in a worst case, it requires time of $O(n^2)$. Then, algorithm 2 has a polynomial time complexity of order $O(n^2)$.

5 Conclusion

A polynomial-time algorithm for coloring planar ghaphs has beeen shown. Our proposal is based on the computation of a maximal independent set S of the input graph G. The maximal independent set S has some characteristics. One of those is that it contains the vertex that appears in the maximum number of odd cycles of G. The construction of S considers the internal-face graph of G in order to select the vertices that decomposes a maximal number of initial faces of G.

A set of partial maximum sets on the internal-face of the graph are computed. The maximal union of those partial sets form the maximal independent set S, avoiding to visit all vertices of the original graph, and coloring first the vertices that are crucial for decomposing the input graph in a set of acyclic graphs.

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