Defining of Lyapunov Functions for the Generalized Linear Dynamical Object

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Abstract: **The paper deals with the developing method of determination of Lyapunov functions for a generalized linear dynamical object. This method is based on the shift and the rotation coordinate transformation which translates a motion of the considered object in a new virtual state space. One can perform such sort of transformation by using partial fraction decomposition. It is easy to define Lyapunov function in a new state space and then do inverse transformation. The proposed method can be used for a determination of signed Lyapunov functions, which can be used as basis for the analysis of system dynamic and the synthesis of desired motions**.

Keywords: ***stability analysis, Lyapunov function, coordinate transformation, dynamical system, partial fraction decomposition.***

I. Introduction

Nowadays Lyapunov functions are widely used to solve different control problems. One can find these functions very usable while synthesis and analysis problems are being solved. The great interest to Lyapunov functions can be explained by their unique properties and strong mathematical background which allows getting mathematically valid results. One can easily use these results while optimization problems are formulated for various dynamical systems.

Although one can find a lot of publications in recent scientific periodicals which describe definitions of non-quadratic Lyapunov functions [1], quadratic forms are still commonly used for defining candidate to Lyapunov function [2-4]. This fact can be simply explained by physical meaning of these quadratic functions which have an energetic background and show redundant energy of dynamical object.

Due to Lyapunov's theorem about stability of motion corresponding Lyapunov function must satisfy Sylvester criterion [5]. Since, there is an infinity number of quadratic functions which satisfy this criterion, their definition is a nontrivial problem of the control theory. This problem is solved by using Riccati and/or Lyapunov equations, which in common case depend on some cost function and can be solved only numerically [6,7].

In order to avoid above-mentioned drawbacks of Lyapunov function's we suggest define them by developing analytical method for defining form and coefficients of Lyapunov function only as functions on parameters of the considered object.

Our paper is organized as follows: first of all we consider the transformation of a generalized linear object into parallel form. Secondly we define Lyapunov function for the transformed object. Thirdly, we perform inverse transformation and write down Lyapunov function which depends only on parameters and coordinates of dynamical object. Lastly, we show the example of using proposed approach and make a conclusion.

II. Usage of Parallel Model for Simulation and Analysis of Dynamical System

A. Representation of object's dynamic in parallel way

Let us consider a linear single-input dynamical object which dynamic is given as follows

$$sx_j = \sum_{i=}^{n} b_{ij} x_i + m_n U , \quad (1)$$

where $s = d/dt$ is a derivative operator, x_i, x_j are state variables, U is a control input, b_{ij}, m_n are coefficients, n is an order of dynamical object.

Equations (1) can be rewriting into matrix form

$$s\mathbf{X} = \mathbf{BX} + \mathbf{M}U , \quad (2)$$

where $\mathbf{X} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$ is a state space vector, $\mathbf{M} = \begin{pmatrix} 0 & 0 & \cdots & m_n \end{pmatrix}^T$ is a n-th sized vector of input coefficients, and $\mathbf{B}$ is a n-th sized square matrix of coefficients

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} . \quad (3)$$

One can find a matrix transfer function of the control object (1) in an easy way by using equation (2)

$$\mathbf{W}(s) = (s\mathbf{E} - \mathbf{B})^{-1} \mathbf{M} , \quad (4)$$

here $\mathbf{E}$ is an identity matrix.

We assume that this matrix $\mathbf{B}$ has only real eigenvalues. In this case its characteristic polynomial can be written thus

$$D(s) = det(s\mathbf{E} - \mathbf{B}) = \prod_{i=1}^{n}(s + \lambda_i), \quad (5)$$

where λ_i are the eigenvalues of matrix $\mathbf{B}$.

Now we suggest to simplify transfer function (4) in the following way

$$\mathbf{W}(s) = \prod_{i=1}^{n}\frac{1}{(s + \lambda_i)}\mathbf{A}, \quad (6)$$

where

$$\mathbf{A} = adj(s\mathbf{E} - \mathbf{B})\mathbf{M}, \quad (7)$$

here $adj(s\mathbf{E} - \mathbf{B})$ is the matrix which is adjunct to matrix $s\mathbf{E} - \mathbf{B}$

It is clear that first cofactor of expression (6) contain n-fold multiplication of elementary fractions. This multiplication can be replaced with a sum of some elementary fractions as follows [8]

$$\prod_{i=1}^{n}\frac{1}{(s + \lambda_i)} = \sum_{i=1}^{n}\frac{\alpha_i}{(s + \lambda_i)}, \quad (8)$$

where

$$\begin{gathered} \sum_{i=1}^{n}\alpha_i = 0; \quad \sum_{i=1}^{n}\alpha_i\left(\sum_{j=1}^{i-1}\lambda_j + \sum_{j=1+1}^{n}\lambda_j\right) = 0 \\ \vdots \\ \sum_{i=1}^{n}a_i\left(\prod_{j=1}^{i-1}\lambda_j \cdots \prod_{j=i+1}^{n}\lambda_j\right) = 1. \end{gathered} \quad (9)$$

Expression (9) is obtained by using only characteristic polynomial (5) and it is independent of selected component of matrix $\mathbf{A}$. This fact allows to rewrite matrix transfer function (6) thus

$$\mathbf{W}(s) = \sum_{i=1}^{n}\frac{\alpha_i}{(s + \lambda_i)}\mathbf{A}. \quad (10)$$

Thereby, we replace the series transfer function (6) where the calculation can be performed only by using consecutive calculations with the parallel one (10) which can be calculated in a parallel way. One of the benefits of such an approach is increasing of the calculation speed while parallel simulation is implemented.

B. Direct and inverse coordinate transformations

Apart from above-mentioned calculation advantage, the proposed approach has significant methodological values. One can find these methodological benefits while performing stability analysis and considering energy transformation. In this case the proposed approach allows us to perform some coordinate transformation from normal phase space to some virtual one and in such a way simplify Lyapunov function.

Let us consider these transformations in detail.

First of all, we define square matrix $\mathbf{Y}$ as follows

$$\mathbf{Y} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} \quad (11)$$

and set the following interrelation

$$x_j = \sum_{i=1}^{n} y_{ji}, \quad (12)$$

where

$$y_{ji} = W_{ji}(s)U, \quad (13)$$

here a_i is the i-th component of matrix $\mathbf{A}$,

$$W_{ji}(s) = \alpha_j a_i / (s + \lambda_j). \quad (14)$$

Expression (12) allows us make the following statement.

Statement 1. For state variable x_j inverse coordinate transformation from virtual phase space into normal is performed by summing all relevant virtual space variable y_{ji}.

Now we consider determination of y_{ji} coordinates while direct transformation is being performed.

Let us take into account equation (15) and complete equation (12) with its first n-1 derivatives. In such a way we get a *n*-th order system of linear equations with n unknown variables y_{ji}

$$L_f^k x_j = \sum_{i=1}^{n} L_f^k y_{ji}, \quad k = 0,...,n-1, \quad (15)$$

where $L_f^k x_j, L_f^k y_{ji}$ are k-th order Lie derivatives.

Solution of this system for unknown virtual state variable y_{ji} allows us define them in such a way

$$y_{ji} = \sum_{k=1}^{n} \gamma_{kji} x_k + \kappa_{ji} U, \quad (16)$$

where $\gamma_{kji}, \kappa_{ji}$ are some numbers.

Expressions (12) and (16) allow us claim the following.

Statement 2. Direct and inverse transformations are described with simple algebraic expressions and it is defined with some family of shift and rotation transformations.

That is why the above-given transformation can be use for simplification of a dynamical system description and performing some actions like stability analysis.

C. Stability analysis

It is clearly understood that one can use expressions (12) and (16) for stability analysis. This analysis after performing transformation (8) comes down to analysis of stability every transfer function (14). This fact allows us formulate the following statement.

Statement 3. Considered dynamical object has stable dynamic on condition that every parallel channels has stable dynamics as well. In this case we can claim that all components of vector $\mathbf{Y}$ are bounded.

One can perform stability analysis for each channel in a different way. The simplest one is analysis of λ_i eigenvalues. The more complex one is based on usage of Lyapunov functions. In spite of its complexity Lyapunov function allows us not only to do stability analysis but consider energy conversion while dynamical object is operating, also define algorithms and structure of controller for the considered object.

D. Lyapunov function dedetrmination

The simplest Lyapunov function is the following quadratic expression

$$V_{ji} = k_{ji} y_{ji}^2, \quad (17)$$

where k_{ji} is a positive number.

So, while analysis of stability is being performed, one can use expression (12) and determine the following Lyapunov function

$$V_j = \sum_{i=1}^{n} k_{ji} y_{ji}^2 . \quad (18)$$

The function (18) can be written down as matrix expression

$$V_j = \mathbf{Y}_j \mathbf{K}_j \mathbf{Y}_j^T , \quad (19)$$

where

$$\mathbf{Y}_j = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}, \quad (20)$$

$$\mathbf{K}_j = \begin{pmatrix} k_{j1} & 0 & \cdots & 0 \\ 0 & k_{j2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{jn} \end{pmatrix}. \quad (21)$$

Quadratic form (19) depends only on diagonal matrix (21). It is quite clear that determinant of matrix (21) and its diagonal minors can be easily defined thus

$$det(\mathbf{K}_j) = \prod_{i=1}^{n} k_{ji} \quad (22)$$

$$M_k = \prod_{i=1}^{k} k_{ji}, \quad k = 1,\ldots,n . \quad (23)$$

Analysis of expressions (22) and (23) allows us to formulate the following statement.

Statement 4. According to Sylvester criterion [6] Lyapunov function (19) is positive if k_{ji} coefficients are positive.

Positiveness of j-th Lyapunov function (19) means stability of j-th channel dynamic.

One can substitutes interrelation (16) info function (18) and write down following expression for Lyapunov function in real state variables

$$V_j = \sum_{i=1}^{n} k_{ji} \left(\sum_{k=1}^{n} \gamma_{kji} x_k + \kappa_{ji} U \right)^2 . \quad (24)$$

Lyapunov function (24) is a positive function as well due to positivenes of function (18).

One can open brackets in function (24) and transform it into modification of well-known quadratic Lyapunov function

$$V_j = \mathbf{ZWZ}^T , \quad (25)$$

where

$$\mathbf{Z} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n & U \end{pmatrix}, \quad (26)$$

$$\mathbf{W} = \begin{pmatrix} w_{11} & \cdots & w_{1(n+1)} \\ \vdots & \ddots & \vdots \\ w_{(n+1)1} & \cdots & w_{(n+1)(n+1)} \end{pmatrix}, \quad (27)$$

here

$$\begin{aligned} & w_{ii} = k_{ij} \gamma_{kji}, \quad i = 1,\ldots,n; \quad w_{i(n+1)} = k_{ij} \kappa_{ji}; \\ & w_{ij} = 2k_{ji} \gamma_{kji} \gamma_{mji}, \quad i,m = 1,\ldots,n; \\ & w_{ij} = 2k_{ji} \gamma_{kji} \kappa_{ji}, \quad (i = n) or (j = n) \end{aligned} \quad (28)$$

The function (25) is a j-th component of matrix Lyapunov function

$$\mathbf{V} = \begin{pmatrix} V_1 & V_2 & \cdots & V_n \end{pmatrix}, \quad (29)$$

which describes redundant energy stored in each channel.

One can use Lyapunov function defined in such a way for stability analysis and design of closed-loop control system. Now let us consider example of defining Lyapunov functions for the speed and current loops of DC motor.

III. Example Use for Proposed Approach

Let us consider a dynamic of DC motor given by the following equations

$$sx_1 = b_{12} x_2 ; sx_2 = b_{21} x_1 + b_{22} x_2 + m_2 U, \quad (30)$$

where coefficients a_{ij} are defined thus

$$b_{12} = \frac{1}{T_m}; b_{21} = b_{22} = -\frac{1}{T_e}; m_2 = \frac{1}{T_e}; T_m = \frac{JR}{c^2}; T_e = \frac{L}{R}, \quad (31)$$

here J is a rotor inertia, R is a armature resictance, c is a back-emf constant, L is an armature inductance, y_1, y_2 are DC rotor speed and current respectively.

We assume that the rotor inertia has significant value and the following condition is true

$$T_m > 4T_e . \quad (32)$$

In this case both of eigenvalues of the characteristic polynomial

$$D(s) = s^2 - b_{22} s - b_{12} b_{21} . \quad (33)$$

are negative

$$\lambda_{1,2} = 0.5 b_{22} \pm 0.5 \sqrt{4 b_{12} b_{21} + b_{22}^2} . \quad (34)$$

This fact allows us represent dynamic DC motor in form (7)

$$\mathbf{W}(s) = \begin{pmatrix} x_1(s)/U(s) \\ x_2(s)/U(s) \end{pmatrix} = \frac{\mathbf{A}}{s^2 - b_{22} s - b_{12} b_{21}}, \quad (35)$$

where

$$\mathbf{A} = \begin{pmatrix} m_2 b_{12} & m_2 s \end{pmatrix}^T \quad (36)$$

or

$$\mathbf{W}(s) = \begin{pmatrix} \dfrac{m_2 b_{12}}{s^2 - b_{22} s - b_{12} b_{21}} & \dfrac{m_2 s}{s^2 - b_{22} s - b_{12} b_{21}} \end{pmatrix}^T . \quad (37)$$

Now we transform transfer function (35) into the form (10)

$$\frac{\mathbf{A}}{s^2 - b_{22} s - b_{12} b_{21}} = \frac{\boldsymbol{\alpha 1}}{s - \lambda_1} + \frac{\boldsymbol{\alpha 2}}{s - \lambda_2}, \quad (38)$$

where

$$\boldsymbol{\alpha 1} = \begin{pmatrix} \dfrac{b_{12} m_2}{\lambda_1 - \lambda_2} & \dfrac{m_2 \lambda_1}{\lambda_1 - \lambda_2} \end{pmatrix}^T ; \boldsymbol{\alpha 2} = \begin{pmatrix} \dfrac{-b_{12} m_2}{\lambda_1 - \lambda_2} & \dfrac{-m_2 \lambda_2}{\lambda_1 - \lambda_2} \end{pmatrix}^T . \quad (39)$$

The matrices (39) allow us write the following equations

$$\begin{aligned} & sy_{11} = \lambda_1 y_{11} + \alpha 1_1 U; \quad sy_{12} = \lambda_2 y_{12} + \alpha 2_1 U; \\ & sy_{21} = \lambda_1 y_{21} + \alpha 1_2 U; \quad sy_{22} = \lambda_2 y_{22} + \alpha 2_2 U. \end{aligned} \quad (40)$$

These equations allow us rewrite interrelations (12) and (16) between real and virtual coordinates if the output variable is the variable x_1

$$x_1 = y_{11} + y_{12}; x_2 = \frac{\lambda_1}{b_{12}} y_{11} + \frac{\lambda_2}{b_{12}} y_{12} + \frac{\alpha 1_1 + \alpha 2_1}{b_{12}} U \quad (41)$$

and if the output variable is the variable x_2

$$x_2 = y_{21} + y_{22};$$
$$x_1 = \frac{\lambda_1 - b_{22}}{b_{21}} y_{21} + \frac{\lambda_2 - b_{22}}{b_{21}} y_{22} + \frac{\alpha 1_2 + \alpha 2_2 - m_2}{b_{21}} U. \quad (42)$$

We consider expression (41) and (42) as inverse coordinate transformation. The expressions for the direct transformation can be obtained as solution equations (41) and (42) for state variables $y_{11}, y_{12}, y_{21}, y_{21}$

$$y_{11} = \frac{-\lambda_2 x_1 - x_2 b_{12} + U(\alpha 1_1 + \alpha 2_1)}{\lambda_1 - \lambda_2}; \quad (43)$$
$$y_{12} = \frac{-x_2 b_{12} + \lambda_1 x1 + U(\alpha 1_1 + \alpha 2_1)}{\lambda_1 - \lambda_2}$$

$$y_{21} = \frac{b_{21} x_1 + (-\lambda_2 + b_{22}) x_2 - (\alpha 1_2 + \alpha 2_2 - m_2) U}{\lambda_1 - \lambda_2};$$
$$y_{22} = \frac{-b_{21} x_1 + (\lambda_1 - b_{22}) x_2 + (\alpha 1_2 + \alpha 2_2 - m_2) U}{\lambda_1 - \lambda_2}. \quad (44)$$

One can use coordinates (43) and (44) to determine Lyapunov functions. The simplest ones can be written down for the speed loop

$$V_1 = y_{11}^2 + y_{12}^2 = w_{11} x_1^2 + 2 w_{12} x_1 x_2 + w_{22} x_2^2 + + 2 w_{13} x_1 U + 2 w_{23} x_2 U + w_{33} U^2, \quad (45)$$

where

$$w_{11} = \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2}; w_{12} = \frac{-b_{12}(\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2};$$
$$w_{22} = \frac{2 b_{12}^2}{(\lambda_1 - \lambda_2)^2}; w_{23} = \frac{2 b_{12}(\alpha 1_1 + \alpha 2_1)}{(\lambda_1 - \lambda_2)^2}; \quad (46)$$
$$w_{33} = \frac{2(\alpha 1_1 + \alpha 2_1)^2}{(\lambda_1 - \lambda_2)^2}; w_{13} = \frac{2(\lambda_1 + \lambda_2)(\alpha 1_1 + \alpha 2_1)}{(\lambda_1 - \lambda_2)^2}.$$

Lyapunov function for the current loop can be defined in a similar way but it has different coefficients

$$w_{11} = \frac{2 b_{21}^2}{(\lambda_1 - \lambda_2)^2}; w_{22} = \frac{(\lambda_2 - b_{22})^2 + (\lambda_1 - b_{22})^2}{(\lambda_1 - \lambda_2)^2};$$
$$w_{12} = -\frac{b_{21}(\lambda_2 - b_{22})}{(\lambda_1 - \lambda_2)^2} - \frac{b_{21}(\lambda_1 - b_{22})}{(\lambda_1 - \lambda_2)^2}; \quad (47)$$
$$w_{13} = \frac{-2 b_{21}(\alpha 1_2 + \alpha 2_2 - m_2)}{(\lambda_1 - \lambda_2)^2}; w_{33} = \frac{2(\alpha 1_2 + \alpha 2_2 - m_2)^2}{(\lambda_1 - \lambda_2)^2};$$
$$w_{23} = \frac{(\lambda_1 + \lambda_2 - 2 b_{22})(\alpha 1_2 + \alpha 2_2 - m_2)}{(\lambda_1 - \lambda_2)^2}.$$

If one takes into account matrices (39) and performs analysis of coefficients (46) and (47), it still can be possible define that coefficients w_{13}, w_{23}, w_{33} in expression (46) are equal to zero and coefficients w_{13}, w_{23}, w_{33} in expression (47) can be simplified as follows

$$w_{11} = \frac{2 b_{21}^2}{(\lambda_1 - \lambda_2)^2}; w_{22} = \frac{(\lambda_2 - b_{22})^2 + (\lambda_1 - b_{22})^2}{(\lambda_1 - \lambda_2)^2};$$
$$w_{12} = -\frac{b_{21}(\lambda_2 - b_{22})}{(\lambda_1 - \lambda_2)^2} - \frac{b_{21}(\lambda_1 - b_{22})}{(\lambda_1 - \lambda_2)^2}; w_{13} = \frac{-2 m_2 b_{21}}{(\lambda_1 - \lambda_2)^2}; \quad (48)$$
$$w_{33} = \frac{2 m_2^2}{(\lambda_1 - \lambda_2)^2}; w_{23} = \frac{-m_2(\lambda_1 + \lambda_2 - 2 b_{22})}{(\lambda_1 - \lambda_2)^2}.$$

This fact allows us formulate following statement.

Statement 5. One should define Lyapunov function (25) in an extended *n+1*-th order space state with space vector (26) if output variable is described with the differential equation which contains control input. It can be defined in a normal *n*-th order state space otherwise.

IV. Conclusion

The proposed approach based on decomposition of transfer function of linear dynamical object with elementary fractions can be used for simulation of considered object, the study of its dynamics and synthesis of its control system. This approach simplifies mathematical model of a linear object and transform this model into some virtual state space. The mentioned transformation allows us define Lyapunov function in an easy way. This function can be defined for both object and linear closed-loop control system. In this case its coefficients depend only on parameters of dynamical system.

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