Improving Network Configurations by the Analysis of Bipartite Geometric Network Structures in the Euclidean Vector Spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

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Abstract. This paper summarizes the analyses of specific bipartite geometric structures regarding the improvement of network and measurement configurations in the $\mathbb{R}^2$ and $\mathbb{R}^3$. The purposes are enhancements of methods to increase and to optimize the geometric stability and statistical reliability of bipartite geometric networks.

Bipartite geometric structures - here the bipartite frameworks in the $\mathbb{R}^n$ and bipartite directional networks in the $\mathbb{R}^2$ - show an outstanding characteristic concerning the quadrics: Quadrics function as critical location for these geometric structures. The question comes up if there is also a negative effect or deteriorating influence regarding combined bipartite geometric structures especially in the $\mathbb{R}^2$ and $\mathbb{R}^3$ caused by the quadrics. Synchrotrons like the SIS18 at the GSI and the future SIS100 for FAIR offer the possibility to research specific configurations related to the quadrics and other geometric properties.

Bipartite distantial networks in the $\mathbb{R}^3$ - e.g. used for calibration of measuring machines - presumably show characteristics that the target points are insufficiently controlled together depending on the amount of the stations. This can cause negative effects on the geometric stability and statistical reliability. The question arises, which range such a geometric instability can reach and which configuration of stations increases the geometric stability.

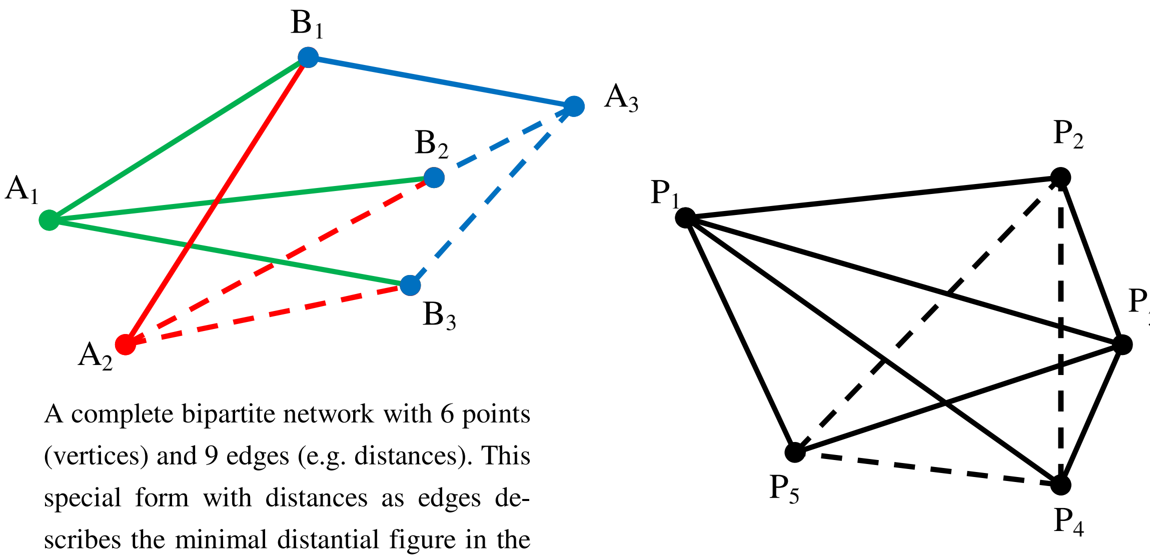
As a main method to analyze the described problems the homogeneous Plücker-Grassmann-coordinates for simulated and measured networks will be used. They are related to latent restrictions derived from the Jacobian matrix of partial derivatives. Further they are related to the partial redundancies, elements of the variance-covariance-matrix of the residuals.

Keywords: Bipartite geometric structures, bipartite distantial and directional networks, laser tracker networks, quadrics, critical locations, latent restrictions, geometry of particle accelerators, optimization, ideal figures, network adjustment.

1 Introduction

The progressive influence of tasks and principles from mechanical engineering in the fields of geodesy and industrial measurements leads to a rethinking concerning evolved structures in the classical geodesy and therefore geometry. This influence of mechanical engineering can be shown by the use of high precise measuring instruments like the laser tracker with accuracies of some micrometers. Laser trackers enable the possibility to determine objects and structures in three-dimensional space. These objects, which can have a dimension of some decimeters up to some kilometers long particle accelerators, are determined by using small, medium and large network structures, whereupon now non-triangular geometric structures have been employed. These non-triangular structures uniquely correspond to the bipartite graphs known from graph theory. Due to this their naming will be modified to "bipartite geometric structures/networks" respectively "bipartite geodetic networks".

The main characteristic regarding this bipartite geometric structure is the existing of two disjoint point groups A and B, where the vertices of group A are only connected with the vertices of group B. The group A can be considered as the stations of the instruments. The group B consists of the target points, observed by the instrument. To comprehend the structure of bipartite networks in comparison to the well-known triangular networks a contrasting juxtaposition of both types is shown in the following figure.



A complete bipartite network with 6 points (vertices) and 9 edges (e.g. distances). This special form with distances as edges describes the minimal distantial figure in the $\mathbb{R}^2$. The minimal closed path/closed polygon in the structure is a tetragon (dashed lines).

A complete triangular network with 5 points (vertices) and 10 edges (e.g. distances). Here the minimal closed polygon is the triangle (dashed lines).

Fig. 1. The comparison between bipartite and triangular networks

GSI Helmholtz Centre for Heavy Ion Research in Darmstadt, Germany, operates a unique large-scale accelerator for heavy ions. GSI maintains several particle accelerators. Presently, a large future project, the Facility for Antiproton and Ion Research (FAIR), is under construction which is intended to provide particle beams with a high intensity and quality. Many accelerators have a simple geometric size, like straight lines – the linacs – or circular respectively elliptic machines – the synchrotrons. The accuracy to align such complete coherent machines upon their construction often requires tolerance ranges of less than 0.1 millimeters. Therefore laser trackers and bipartite geodetic networks, with their nonreciprocal observations and with their specific geometric properties, come into play. Thereby, recent on-site work revealed that this methodology leads to unexpected effects regarding the adjustment and statistical analyses, which seem to influence and even interfere with the geometric stability and reliability of the measurements obtained.

1.1 Problem description

The bipartitedness of geometric networks, further the possible influences of their network substructures and the negative role of the quadrics was recognized by a test measurement in the former work for a provider for industrial measurements. During this test measurement with an instrument for calibrations of precise measuring machines – the LaserTRACER from the company ETALON AG – a major difficulty with the adjustment occurred. All distance residuals were exact 0.0000 mm, but the confidence region of all unknowns, the coordinates, had values of several centimeters! The network itself, which had no triangle in its structure, was built from distances and directions in such a specific manner, that a non-congruent and non-similar distortion occurred. Further investigations showed that this structure in an abstract sense was equivalent to the bipartite graphs from the graph theory. An investigation into the separation as a single distantial and a single directional network showed that the distantial structure was under-determined, but the directional structure over-determined. The variance-covariance-component-estimation where the a-priori standard deviations for the distances were set with a value by 0.5 micrometer – from manufacturer's data – and for the directions with several degrees – because the directions were measured to obtain approximated values and should have no influence in the adjustment – led to this described distortion. Thus, the over-determined directional network was forced into the under-determined distantial network, where its distances were restricted by the high accuracy of the observations and which itself was highly flexible. This phenomenon, developed from an erroneous and unknown consideration of this type of network geometry and its influence to the adjustment, was the trigger to analyze and research bipartite geodetic networks respectively bipartite geometric structures.

1.2 Relevance for bipartite geodetic networks in the R^3

Many bipartite geodetic networks show exactly this phenomenon that after a separation in single distantial and directional structures these substructures are not balanced regarding its rigidity. In their combination as a complete and combined network the rigidity is given. But how this network is biased by the geometry of the bipartite substructures, influenced by a-priori standard deviations, influenced further by additional unknowns – the unknowns of orientation occurring at directional structures – and biased by spatial approaches – from $\mathbb{R}^3$ nearly to $\mathbb{R}^2$ – is an open question and one main part of this dissertation. The particle accelerator at the GSI Helmholtz Centre provides the frame to build such bipartite networks as paradigm like similar network structures which were established at other particle accelerators in the world and for the network determination in the field of industrial measurements.

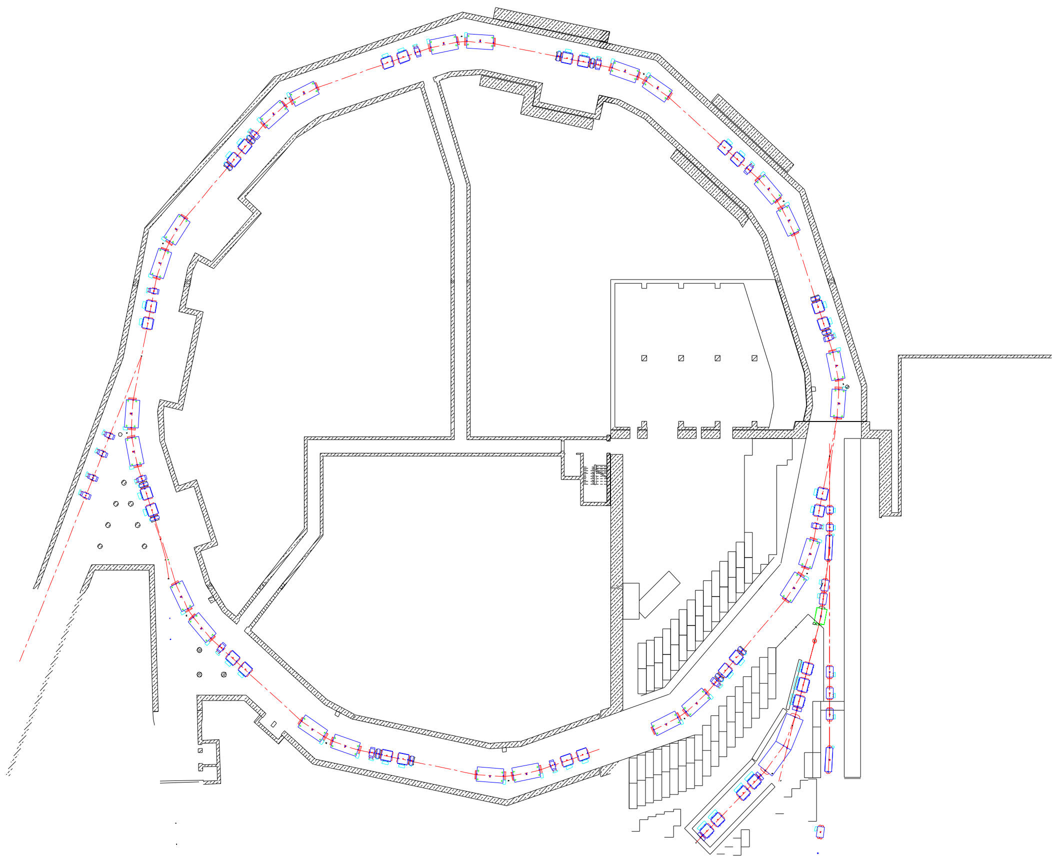


Fig. 2. The heavy ion synchrotron SIS18 at the GSI Helmholtz Centre in Darmstadt

2 Background and State of the Art

The given research outlined here combines knowledge from the fields of geodetics, geometry, algebra and graph theory.

2.1 Bipartite frameworks and bipartite directional networks in the $\mathbb{R}^2$

One part of bipartite geometric structures is analyzed and known under the name of bipartite frameworks. The functional relation between the vertices is the existence of a distance or length, which can be considered as a rod, bar, strut or a measured length. The measured length can be observed electro-optical, per laser, interferometric and via sound techniques. Important criteria for these distantial structures were the assumption of complete bipartitedness and the analysis for critical locations and degenerations. Bipartite frameworks degenerate when all points, which belong to the object, are located on a quadric (Bolker und Roth 1980). In the 19th century in the fields of statics the degeneration of such bipartite frameworks in the $\mathbb{R}^2$ was observed. Their relation to the conics was assumed but not directly proven (Wunderlich 1977a). Due to the development of electro-optical distance measuring instruments in the 60s/70s of the 20th century bipartite distantial structures were investigated for the $\mathbb{R}^2$ and $\mathbb{R}^3$. Here the minimal figures, the complete bipartite structures $K_{3,3}$ and $K_{4,6}$ were analyzed geometrically, especially for the quadric problematic (Wunderlich 1977a, 1977b; Rinner et al. 1969). In the 70s of the 20th century geometric structures corresponding to the bipartite frameworks were analyzed in the field of satellite geodetics, where the ground stations and the satellites function as the two disjoint point groups (Blaha 1971; Tsimis 1973; Grafarend und Sanso 1985). The rigidity of bipartite frameworks especially in the statics and for tensegrity frameworks in the $\mathbb{R}^n$ was investigated from the 80s of the 20th century. Here one important issue was the mathematical proof, that bipartite distantial structures degenerate when all vertices are located on a quadric (Bolker und Roth 1980; Whiteley 1984; Connelly und Gortler 2015).

A second part of bipartite geometric structures, the bipartite directional networks in the $\mathbb{R}^2$, were analyzed early in the 18th and 19th century. It is known as the 8-Point-Problem of Lambert and Clausen. These objects were rediscovered in the late 20th century. The main points of analyzing were the critical curves, where such structures degenerate. These curves are the quadrics too but also the circular cubic curves. The most well-known examples of such cubics are the conchoids of de Sluze, with Maclaurins trisectrix and the cissoid of Diocles as examples (Wunderlich 1976, 1978).

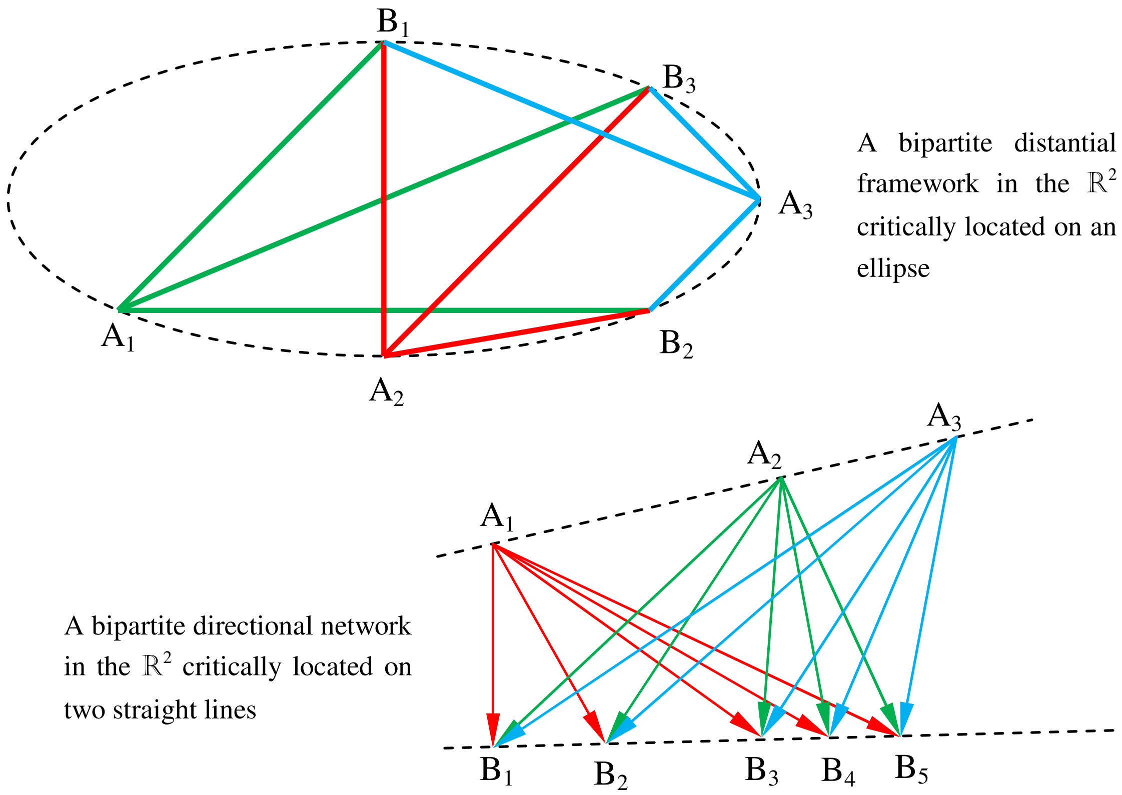


Fig. 3. The critical locations on two conics

2.2 The alternative solutions for the minimal bipartite frameworks $K_{3,3}$ in the R^2

Triangles with given side lengths can be constructed and embedded into the $\mathbb{R}^2$ in two ways, first regarding their correct respectively real location and second with its mirror image. The reason for these two possibilities results from the missing information of orientation for the triangle.

But the minimal bipartite framework $K_{3,3}$ in the $\mathbb{R}^2$ with given side lengths can offer 8 possible independent geometric figures and additional their 8 mirror images, altogether 16 figures. Presumably for the minimal bipartite frameworks in the $\mathbb{R}^n$ for $n \geq 3$ there exist also a great number of such alternative solutions, but the exact number and real examples with given side lengths are an open question. For the minimal bipartite frameworks in the $\mathbb{R}^2$ the first assumption was given by Wunderlich. Walter and Husty made a deeper investigation and confirmed Wunderlich's assumption of 8 possible alternative figures and gave some real examples (Wunderlich 1977a, 1977b; Walter and Husty 2007).

2.3 Graph theory and topology

Geodetic networks can be abstractly interpreted as graphs, whereupon the classical triangular net structures show relations to the complete graphs. Bipartite graphs itself play an important role to establish special forms of networks used in informatics (Petri nets), in mechanical engineering, in logistics and for processes in the economy (Diestel 2012). Special topological issues are embedding bipartite graphs on surfaces like spheres, n-tori and projective planes (Mohar und Thomassen 2001).

Graph structures are often related to polytopes, surfaces, n-manifolds (Ziegler 1994). Polytopes itself can topologically be classified through the decomposition into combinatorial surfaces to get invariants like the Euler characteristic, homology groups and to see the homeomorphism to orientable and non-orientable compact 2-manifolds without any boundary (Stöcker und Zieschang 1994; Kinsey 1997; Kühnel 2008).

3 Aim of work and contributions

One main part is the geometric and statistical analysis of specific bipartite geodetic networks in general and which are designed, measured, adjusted and optimized in the surroundings of particle accelerators at the GSI Helmholtz Centre. The main types are the combined structures and pure distantial networks. A further task which is related to the network design is the research on regular, ideal and optimal configurations and on alternative solutions regarding bipartite distantial figures. Another very important point is the analysis and the classification of bipartite directional structures, which are geodetically relevant in the $\mathbb{R}^2$ and $\mathbb{R}^3$. The analysis of geometric properties (minimal resp. basic figures, degenerations, optimal figures) is here also an integral part.

The circumstance that bipartite structures can represent static unique objects leads to the question. Does there exist a superior structure analogue to the n-simplices for distantial and directional bipartite geometries? Such a generalization and the development of other forms (e.g. non-complete or partial bipartite objects) will be transferred into the mathematical frame. One further task shall be the disclosure of possible relations between triangular structures (n-simplices) and bipartite structures, the possible bipartite equivalents. A first topological approach is the investigation and definition of polygonal manifolds respectively combinatorial surfaces with their embedding into orientable and non-orientable compact 2-manifolds.

In the course of the research several findings are expected:

1. Geometrical issues in general

- The analysis and classification of bipartite directional structures (minimal and basic figures, possible degenerations, regularities in the $\mathbb{R}^n$ for $n \geq 2$) → and the comparison to triangular directional structures
- The analysis and classification of combined bipartite geometric structures → distantial and directional type in the $\mathbb{R}^2$ and $\mathbb{R}^3$ with an outlook to the $\mathbb{R}^n$ for $n \geq 4$
- The investigation and deeper analysis of ideal, optimal and regular figures and concomitant curves for bipartite distantial structures especially for figures in the $\mathbb{R}^2$ and further $\mathbb{R}^3$ and $\mathbb{R}^4$
- The investigation of rigid partial bipartite structures of the distantial type
- The development of a consistent nomenclature for bipartite geometric structures of all types and elements connected with the bipartite structures

2. Issues regarding the optimization and network adjustment

- The extension of design models for geodetic networks
- The deeper analysis of the inner geometry of over-determined bipartite geometric structures with the help of latent restrictions, the associated normal form of a matrix and Plücker coordinates, which are related to the partial redundancies of the variance-covariance-matrix of the deviations. The normal form of a matrix correlates to the Grassmannian of a differentiable manifold, which consists the homogeneous Plücker-Grassmann-coordinates (Jurisch and Kampmann 1999, 2002)
- The deeper analysis of the interdependence between the stations and target points at over-determined bipartite distantial networks and its significant effect on the geometric stability and statistical reliability
- The continuation of analysis and the extension of further alternative solutions for bipartite frameworks in the $\mathbb{R}^2$ in general and especially for the over-determined case. This also leads to different pseudoinverse matrices and partial redundancies for one set of given distances and therefore to an extension of considerations for the network adjustments in general

3. Topological issues

- The topological classification of complete bipartite geometric figures by their decomposition into combinatorial surfaces and their embedding into compact orientable and non-orientable 2-manifolds without any boundary

4 Methodology

The GSI Helmholtz Centre with its existing particle accelerator and the planned machine for FAIR provides the possibility to measure and to analyze bipartite geodetic networks in a manifold way. Networks built from distances, directions and their combinations and spatial approaches (from $\mathbb{R}^3$ nearly to $\mathbb{R}^2$) are objects of inquiry. The analysis of possible effects caused by the quadrics and the substructures on the adjustment of actual measured bipartite geodetic networks especially in the $\mathbb{R}^3$ will be the main part of this dissertation. The results of the examination of actual measured and adjusted bipartite geodetic networks provide the basis of theoretical examination and optimization in general and concerning similar parts for FAIR like the SIS100 and other complex areas.

The geometry of the machines itself (circular, linear, elliptic forms) and the evaluation of networks with different instruments (mainly the laser tracker and further the theodolite and tachymeter) also provides the frame to examine and investigate regular and special situations how such a network can be built. The distinctiveness of the machines (synchrotrons, linacs, storage rings and so on) is mirrored directly in the geometry and the adjustment of bipartite geodetic networks. The adjustment of bipartite geodetic networks with all of their characteristics, its distinctiveness, the influence of the quadrics and the statistical analysis will be an essential part of the entire dissertation. Networks which are built to determine large facilities provide the basis as object of investigation.

The methodology can be categorized with the following tasks:

- Measurement and analysis concerning the influence of the quadrics and the substructures of simulated and real measured combined networks in the surroundings of the existing particle accelerator at the GSI (e.g. the SIS18), for the future machine FAIR (e.g. the SIS100) and with experimental networks in general; preferentially measured with the laser tracker
- Evaluation and analysis of measured and simulated distantial networks (bipartite frameworks) to improve conditions regarding geometric stability
- Analysis of simulated combined networks in the whole $\mathbb{R}^n$ to show geometric properties
- Analysis of bipartite directional networks in the $\mathbb{R}^n$ for $n \geq 3$, their classification, the depiction of their geometric resistance comparing the quadrics
- The development of a logic taxonomy and therefore a consistent nomenclature for bipartite geometric structures of any known type
- The search for ideal figures of bipartite distantial structures and the comparison to ideal triangular figures

The findings of the analyses regarding latent restrictions will be implemented in the network design and optimization issues. This also comprises the analyses on possible ideal configurations and results from the variance-covariance-component estimation at combined geodetic networks. And further the research on the consolidation of the reciprocal control of the target points via the stations will help to ensure the geometric stability and statistical reliability.

These described results shall also help to understand and comprehend the difficult nature of bipartite geometric structures in comparison to their well-known and well-analyzed triangular counterparts.

5 State of investigations and next steps

Several preliminary studies regarding bipartite geometric structures were carried out. This comprised detailed analyses of small bipartite distantial structures in the whole $\mathbb{R}^n$ and the discovery of rigid partial bipartite frameworks in the $\mathbb{R}^3$. A first approach regarding the research on ideal figures was done with bipartite frameworks in the $\mathbb{R}^2$ – especially analyzing the minimal figure $K_{3,3}$ – and with bipartite frameworks in the $\mathbb{R}^3$ and $\mathbb{R}^4$. A future work regarding bipartite frameworks will be an extended investigation on the alternative solutions for the minimal figure $K_{3,3}$ in the $\mathbb{R}^2$, taking in account further the over-determined case and the attempt to find alternative solutions in the $\mathbb{R}^3$.

Bipartite directional networks were analyzed especially in the $\mathbb{R}^2$ and $\mathbb{R}^3$ but also in the $\mathbb{R}^4$ and higher dimensions. Regarding this an important characteristic can be stated:

Hypothesis 1: *Bipartite directional networks in the $\mathbb{R}^n$ for n ≥ 3 show a geometric resistance comparing to the quadrics. They don't degenerate, when all network points are located on the corresponding quadrics in the $\mathbb{R}^n$.*

Further analyses were concerned with simple, small and regular combined bipartite networks in the $\mathbb{R}^2$. First analyses concerning the location on conics were also done with these geometric figures. As a next step these investigations will be extended on larger complete and partial bipartite structures in the $\mathbb{R}^2$ and $\mathbb{R}^3$. Therefore the conics – especially the circle and the straight lines – are the specific objects of investigation.

To facilitate the handling with naming a nomenclature for bipartite geometric structures was developed and will be developed further, for any known type of structure and for specific figures and characteristics.

Future works will concentrate on the simulation and evaluation of experimental networks, which later will be measured with the laser tracker at the GSI. Other important campaigns will be the measurement, adjustment and analysis of networks concerning synchrotrons like the SIS18 at the GSI. Here its specific circular shape and the spatial approach from $\mathbb{R}^3$ to $\mathbb{R}^2$ come into play. Considerations to analyze network measurements of large machines from other institutes in the world are also conceivable.

To analyze the geometric stability of over-determined bipartite frameworks in the $\mathbb{R}^3$ the next steps are the simulation and geometric analysis of such networks. Later the theoretical examination will be substantiated with distantial networks measured with laser trackers or the LaserTRACER.

6 Research Background

Working in the fields of automotive and aircraft industries, the engineering itself and especially in the field of particle accelerators I came in contact with small and large bipartite geodetic networks, their measurements, design and evaluation. The main tools were instruments like different types of laser trackers, the LaserTRACER – where the bipartite geometry was first recognized – and classical instruments like the theodolite and tachymeter which were embedded in bipartite network considerations.

At the GSI Helmholtz Centre I am responsible, among other things, for planning, operating and evaluating of different measurement campaigns for the machine areas and also for the geometric determination of normal and superconducting magnets. All these measurements – for small and large facilities – show as a network frame exactly such bipartite structures.

In the year 2014 I enrolled at the Graduate School at the Darmstadt University of Applied Sciences to prepare my doctorate. My first supervisor in Darmstadt at this time was Professor Dr Torsten-Karl Strempel from the faculty of mathematics and natural sciences, who advised me especially in the fields of geometry, topology and combinatorial geometry. Since August 2018 I am supervised by Professor Dr Klaus Habermehl, who advises me in the fields of geodesy and surveying.

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