Finite Element Method for the Lamé System in Domain with a Crack

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Abstract

In the present paper a Dirichlet problem for the Lamé system in the 2D cracked domain is considered. Solution to this problem is defined as R_ν-generalized one in the weighted functional set. For determining of approximate solution, the weighted finite element method is built. Numerical analysis of the model problem showed that convergence rate of the approximate R_ν-generalized solution to the exact one in the norm of the weighted Sobolev space amount to $O(h)$, that is twice greater than for the classical finite element method.

1 Introduction

Cracks frequently appear during production and use of different materials, parts and building structures. Presence of cracks has a great influence on durability of the constructions and a catastrophic consequence could arise from appearance and growth of cracks.

In many cases, a Lamé system is used as a foundation for mathematical modeling of the linear elastostatics in the cracked domain with different types of boundary conditions. It is well known that in the case of Dirichlet or Neumann boundary conditions on both sides of the crack the classic finite element method converges with the rate $O(h^{1/2})$, and in the case of Dirichlet conditions on one side and Newmann on the other one convergence rate is $O(h^{1/4})$.

In recent 30 years, on the base of classic works (see, for example, [1-4]) many researches of properties of solutions to the Lamé equations posed in the cracked or non-convex domains were performed, asymptotic expansion of the solutions near singularity points were obtained (see, for example, [5-7]). Taking into account these results and on the base of weak statement of the problem, many special numerical methods with the convergence speed $O(h)$ in the norm of Sobolev space $W_2^1(\Omega)$ were developed, such as: smoothed FEM [8] is based on the on the modification of the strain field; DPG FEM [9] suppose simultaneous approximation of the displacement and stress fields; extended FEM (Xfem) [10-11] is based on the addition of special function into the finite element space.

For the boundary value problems with singularity caused by different sources, such as degeneration of the input data or geometrical features of domain boundary, it was suggested to define the solution as R_ν-generalized one [12-16]. This approach allowed us to define suitable weighted space or set in which R_ν-generalized solution exists and is unique. Properties of R_ν-generalized solutions and weighted spaces where they are defined were deeply studied [17-22]. On the basis of obtained theoretical results, various modifications of the weighted finite element method for different problems were designed and investigated [23-30].

In the present paper, a Dirichlet problem for the Lamé system posed in 2D cracked domain is considered. The solution to this problem is defined as R_ν-generalized one in the special weighted functional set. For the numerical treatment of the problem, the weighted finite element method is constructed. Numerical experiment for the model problem is performed and a comparison of the weighted FEM with the classic FEM is carried out. Results of the series of computations showed that the approximate R_ν-generalized solution converges to the exact one in the norm of the weighted Sobolev space with the rate $O(h)$, that is twice greater than for the classical finite element method.

2 Problem Statement

Let $\bar{\Omega} = [-1,1]\times[-1,1]$ be a homogeneous isotropic body with the straight crack $[0,1]\times\{0\}$, $\partial\Omega$ is a boundary of the Ω. In addition to the Cartesian coordinate system we also introduce polar coordinates with pole in the origin $(0,0)$ and polar axis codirectional with the Ox axis.

Assume that the strains are small. In Ω consider a boundary value problem for the Lamé system (see [10]):

$$-(2\,\mathbf{div}(\mu\varepsilon(\mathbf{u})) + \nabla(\lambda\,\mathrm{div}\,\mathbf{u})) = \mathbf{f}, \quad x \in \Omega, \tag{1}$$

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$$u_i = q_i, \quad i = 1,2, \quad x \in \partial\Omega, \tag{2}$$

Here $\mathbf{u} = (u_1, u_2)$ is the displacement field, $\varepsilon(\mathbf{u})$ is the deformation tensor.

Denote $\Omega' = \{x \in \overline{\Omega} : (x_1^2 + x_2^2)^{1/2} \le \delta < 1\}$ a δ-neighborhood of the $(0,0)$. We introduce the weight function $\rho(x)$ which coincides with the distance to the origin in Ω' and equals to δ for $x \in \Omega \setminus \Omega'$. Using weight function, we define the weighted spaces $L_{2,\alpha}(\Omega)$, $W^1_{2,\alpha}(\Omega)$ and sets $W^1_{2,\alpha}(\Omega,\delta)$, $\overset{\circ}{W}{}^1_{2,\nu}(\Omega,\delta)$, $W^{1/2}_{2,\alpha}(\partial\Omega,\delta)$ (see, for example, [19] and [22]). For the corresponding spaces and sets of vector-functions we use notations $\mathbf{W}^1_{2,\alpha}(\Omega,\delta)$, $\mathbf{L}_{2,\alpha}(\Omega,\delta)$, $\overset{\circ}{\mathbf{W}}{}^1_{2,\alpha}(\Omega,\delta)$.

Assume that the right hand sides of (1) and (2) meet the following conditions:

$$\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega,\delta), \quad q_i \in W^{1/2}_{2,\beta}(\partial\Omega,\delta), \quad i = 1,2, \quad \beta > 0. \tag{3}$$

Introduce bilinear and linear forms respectively:

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \sigma(\mathbf{u}) : \varepsilon(\rho^{2\nu}\mathbf{v})\,dx, \qquad l(\mathbf{v}) = \int_\Omega \rho^{2\nu}\mathbf{f}\mathbf{v}\,dx.$$

Here $\sigma(\mathbf{u})$ denotes the stress tensor.

We say that the function $\mathbf{u}_\nu \in \mathbf{W}^1_{2,\nu}(\Omega,\delta)$ is an R_ν *-generalized solution* to the problem (1),(2), if almost everywhere on $\partial\Omega$ it satisfies boundary conditions (2) and for any $\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}^1_{2,\nu}(\Omega,\delta)$ the integral identity

$$a(\mathbf{u}_\nu, \mathbf{v}) = l(\mathbf{v})$$

holds for any fixed value of ν, satisfying the inequality $\nu \ge \beta$.

3 Weighted Finite Element Method

For the problem (1),(2) on the base of notion of R_ν-generalized solution the weighted finite element method is constructed. For that, a quasi-uniform triangulation T^h of the $\overline{\Omega}$ is performed and a special weighted basis functions are introduced.

By means of strains $x = -1 + \dfrac{2i}{N}$, $y = -1 + \dfrac{2j}{N}$, $i, j = 0..N$ $\overline{\Omega}$ is divided on squares and each of them by the diagonal is divided into two triangles K called finite elements, with vertices P_k $(k = 1,\dots,n')$ are called triangulation nodes. Let $\Omega^h = \bigcup_{K \in T^h} K$, $h = \dfrac{2\sqrt{2}}{N}$ is a longest length of the finite elements sides. We denote the set of all triangulation nodes as $\mathrm{P} = \{P_k\}_{k=1}^{k=n'}$ among which $\{P_k\}_{k=1}^{k=n}$ is the set of internal nodes and $\{P_k\}_{k=n+1}^{k=n'}$ is the set of boundary nodes.

For each $P_k \in \{P_k\}_{k=1}^{k=n}$ a weighted basis function $\psi_k(x) = \rho^{\nu^*}(x)\phi_k(x), \quad k = 1,\dots,n,$ is introduced, where $\phi_k(x)$ is a linear on each finite element and $\phi_k(P_j) = \delta_{kj}$, $k, j = 1,\dots,n$, δ_{kj} denotes a Kronecker delta, ν^* is a real number.

We define the set $V^h = \operatorname{span}\{\psi_k\}_{k=1}^{k=n}$, $\mathbf{V}^h = [V^h]^2$. It $\mathbf{V}^h$, we separate a subset $\overset{\circ}{\mathbf{V}}{}^h = \left\{\mathbf{v} \in V^h, v_i(P_k)|_{P_k \in \partial\Omega} = 0,\ i = 1,2\right\}$.

We say that the function $\mathbf{u}^h_\nu \in \mathbf{V}^h$ is an approximate R_ν-generalized solution by the weighted finite element method to the problem (1),(2), if it satisfies boundary conditions (2) in nodes $P_k \in \{P_k\}_{k=n+1}^{k=n'}$ and for all $\mathbf{v}^h(x) \in \overset{\circ}{\mathbf{V}}{}^h$ and $\nu > \beta$ the integral identity

$$a(\mathbf{u}^h_\nu, \mathbf{v}^h) = l(\mathbf{v}^h)$$

holds.

Components of approximate solution $\mathbf{u}^h_\nu$ will be found in the form

$$u^h_{\nu,1} = \sum_{k=1}^{n} d_{2k-1}\psi_k, \quad u^h_{\nu,2} = \sum_{k=1}^{n} d_{2k}\psi_k, \quad d_j = \rho^{-\nu^*}(P_{\left[\frac{j+1}{2}\right]})c_j, \quad j = 1,\dots,2n.$$

The unknowns d_j are defined from the system of linear equations $a(\mathbf{u}^h_\nu, \psi_k) = l(\psi_k), \quad k = 1\cdots 2n$.

4 Numerical Experiment

In the present section, the results of numerical experiment for a model problem are presented. We used the vector-function $\mathbf{u} = \dfrac{1}{E}\sqrt{\dfrac{r}{2\pi}}(1+\gamma)\left(\cos\left(\dfrac{\theta}{2}\right)(\xi - \cos(\theta)), \sin\left(\dfrac{\theta}{2}\right)(\xi - \cos(\theta))\right)$ as a solution, $\gamma = \dfrac{\lambda}{\lambda + 2\mu}$, $E = \dfrac{4\mu(\lambda+\mu)}{\lambda + 2\mu}$, $\xi = 3 - 4\gamma$ (see [10]). Lamé parameters $\lambda = 3.0$, $\mu = 5.0$.

Computations were carried out on meshes with different partition numbers N. The errors of approximate generalized ($\nu = 0$, $\rho(x) \equiv 1$) and R_ν-generalized solutions were compared in the norms of spaces $\mathbf{W}_2^1(\Omega)$, $\mathbf{W}_{2,\nu}^1(\Omega)$ and in the triangulation nodes. To do this, for the derived approximate generalized $\mathbf{u}^h$ and R_ν-generalized $\mathbf{u}_\nu^h$ solutions a relative errors $\eta = \dfrac{\left\|\mathbf{u}_\nu - \mathbf{u}^h\right\|_{W_2^1(\Omega)}}{\left\|\mathbf{u}\ \right\|_{W_2^1(\Omega)}}$, $\eta_\nu = \dfrac{\left\|\mathbf{u}_\nu - \mathbf{u}_\nu^h\right\|_{W_{2,\nu}^1(\Omega)}}{\left\|\mathbf{u}_\nu\right\|_{W_{2,\nu}^1(\Omega)}}$ were computed, respectively. In the nodes $P_i, i = 1,\dots,n$ an absolute errors $\delta_{ij} = \left|u_j(P_i) - u_j^h(P_i)\right|$, $\delta_{ij}^\nu = \left|u_{\nu,j}(P_i) - u_{\nu,j}^h(P_i)\right|$ were calculated. A number and coordinates of the nodes P_i, where the absolute errors are less than the limit value $\overline{\Delta} = 10^{-5}$ were counted. Results of numerical experiments are presented in tables and figures. Optimal parameters δ, ν, ν^* were derived by the software complex [31].

It Table 1 for the different N the values of η, η_ν and the ratios between them on the adjacent meshes are presented.

Table 1: Dependence of relative errors η, η_ν for the approximate generalized and R_ν-generalized ($\delta = 0.015$, $\nu = 1.8$, $\nu^* = 0.0$) solutions, respectively, on the N.

N	32		64		128		256		512		1024
η	$1.130 \cdot 10^{-1}$	1.41	$8.006 \cdot 10^{-2}$	1.41	$5.666 \cdot 10^{-2}$	1.41	$4.009 \cdot 10^{-2}$	1.41	$2.835 \cdot 10^{-2}$	1.41	$2.005 \cdot 10^{-2}$
η_ν	$1.024 \cdot 10^{-1}$	1.22	$8.347 \cdot 10^{-2}$	2.10	$4.003 \cdot 10^{-2}$	2.18	$1.836 \cdot 10^{-2}$	2.04	$9.002 \cdot 10^{-3}$	1.99	$4.523 \cdot 10^{-3}$

On Figure 1 a convergence rates of the approximate generalized (left) and R_ν- generalized (right) solutions are presented in the logarithmic scales. Solid lines designate convergence rate $O(h)$.

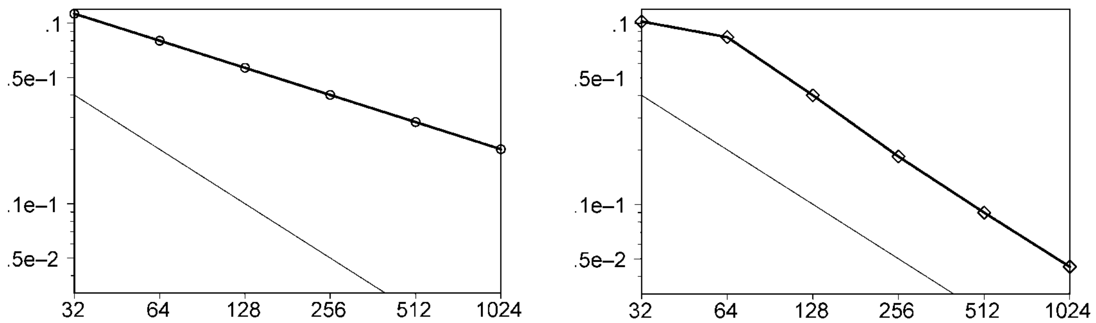

Figure 1: Dependence of relative errors η (left) for the approximate generalized and η_ν (right) for the approximate R_ν- generalized ($\delta = 0.015$, $\nu = 1.8$, $\nu^* = 0.0$) solutions on the N.

In Table 2 and Table 3 we present a number of nodes $P_i \in \{P_k\}_{k=1}^n$ in percentage of their total number, where the absolute errors δ_{ij}^ν for approximate R_ν-generalized solution and δ_{ij} for the approximate generalized solution, $i = 1,\dots,n$, $j = 1,2$ are less than the limit value $\overline{\Delta} = 2 \cdot 10^{-5}$. Distribution of the absolute error for component $u_{\nu,1}^h$ of the approximate R_ν-generalized solution and for component u_1^h of the approximate generalized solution for $N = 256$, $N = 512$ and $N = 1024$ are depicted on Fig. 2. Corresponding results for $u_{\nu,2}^h, u_2^h$ are depicted on Figure 3.

Table 2: The number of nodes $P_i \in \{P_k\}_{k=1}^n$ in percentage of their total number, where the absolute errors δ_{i1}^ν (n_1^ν, approximate R_ν-generalized solution) and δ_{i1} (n_1, approximate generalized solution), $i = 1,\dots,n$, are less than the limit value $\overline{\Delta} = 2 \cdot 10^{-5}$.

N	32	64	128	256	512	1024
n_1^ν	15.344%	14.148%	42.347%	86.736%	98.819%	99.740%
n_1	15.556%	29.286%	48.185%	74.523%	95.170%	99.449%

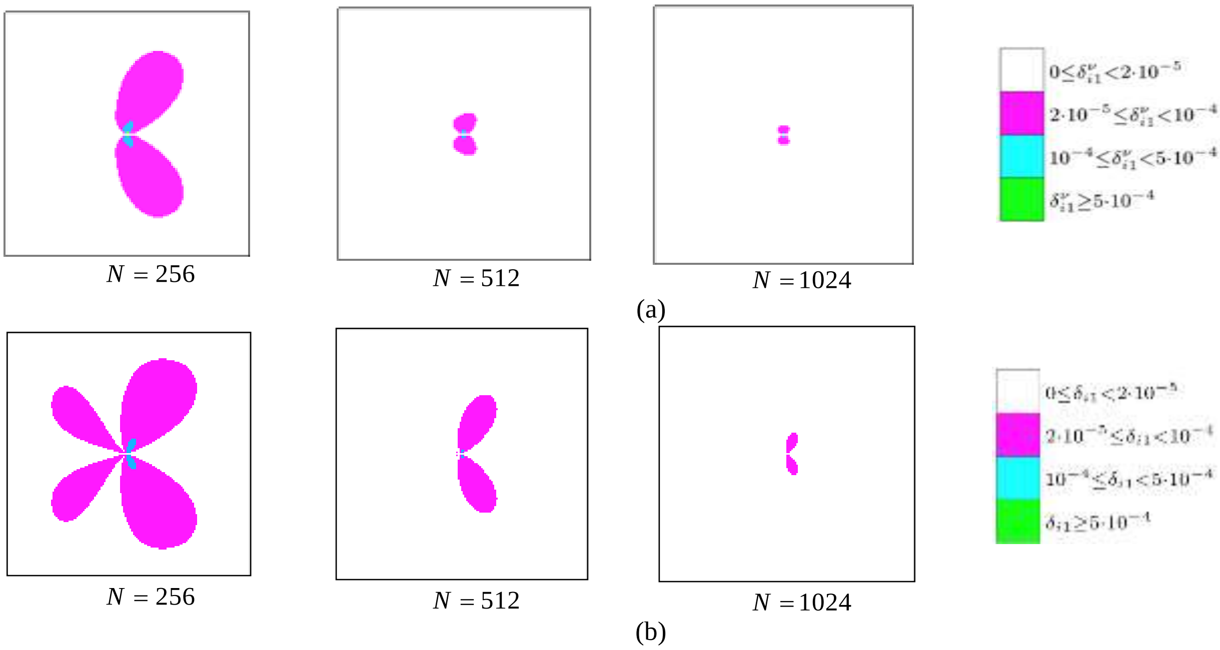

Figure 2: The errors δ_{i1}^{ν} and δ_{i1} for approximate R_{ν}-generalized $\delta = 0.015$, $\nu = 1.8$, $\nu^* = 0.0$ (a) and generalized (b) solutions, respectively, on meshes with different partition number N.

Table 3. The number of nodes $P_i \in \{P_k\}_{k=1}^{n}$ in percentage of their total number, where the absolute errors δ_{i2}^{ν} (n_2^{ν}, approximate R_{ν}-generalized solution) and δ_{i2} (n_2, approximate generalized solution), $i = 1, \cdots, n$, are less than the limit value $\overline{\Delta} = 2\cdot 10^{-5}$.

N	32	64	128	256	512	1024
n_2^{ν}	17.989%	13.665%	31.721	55.842%	94.834%	99.689%
n_2	17.143%	24.155%	34.790%	47.768%	63.128%	79.038%

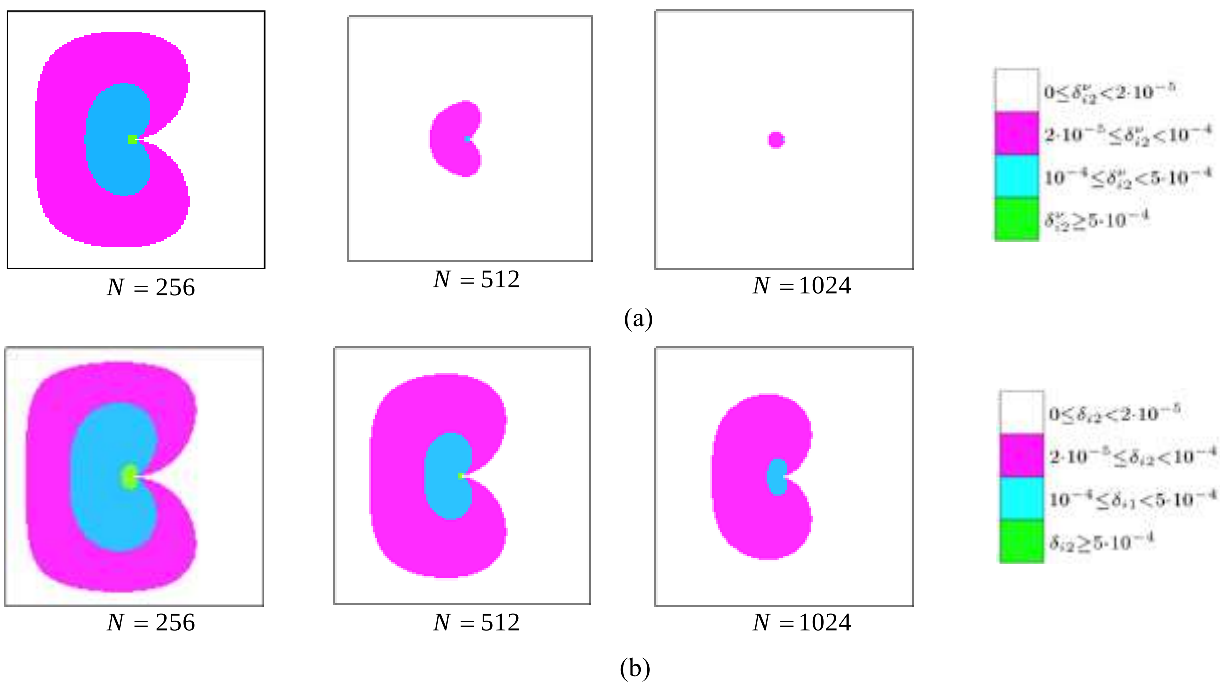

Figure 3: The errors δ_{i2}^{ν} and δ_{i2} for approximate R_{ν}-generalized $\delta = 0.015$, $\nu = 1.8$, $\nu^* = 0.0$ (a) and generalized (b) solutions, respectively, on meshes with different partition number N.

Remark. During numerical experiment for the current problem, in contrast to the problems in works [23-30], the founded optimal value $\nu^* = 0.0$. Convergence rate $O(h)$ is achieved thanks to the introduction of R_ν-generalized solution.

5 Conclusion

From the presented numerical results one can make the following conclusions:

- an approximate R_ν-generalized solution to the problem (1),(2) with the Dirichlet boundary conditions on both sides of the crack converges to the exact one with the rate $O(h)$ in the norm of the space $\mathbf{W}^1_{2,\nu}(\Omega)$, whereas approximate generalized one converges with the rate $O(h^{1/2})$ in the norm of the space $\mathbf{W}^1_2(\Omega)$;
- absolute error δ^ν_{ij} of the approximate R_ν-generalized solution in overwhelming majority of nodes is by one or two decimal orders less than absolute error δ_{ij} of the approximate generalized solution.

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