Polynomial-Time Satisfiability Tests for 'Small' Membership Theories*†

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Abstract. We continue here our investigation aimed at the identification of 'small' fragments of set theory that are potentially useful in automated verification with proof-checkers based on the set-theoretic formalism, such as ÆtnaNova. More specifically, we provide a complete taxonomy of the polynomial and the NP-complete fragments comprising all conjunctions that may involve, besides variables intended to range over the von Neumann set-universe, the Boolean set operators $\cup, \cap, \setminus$ and the membership relators $\in$ and $\notin$. This is in preparation of combining the aforementioned taxonomy with one recently developed for similar fragments, but involving, in place of the membership relators $\in$ and $\notin$, the Boolean relators $\subseteq, \not\subseteq, =, \neq$, and the predicates '$\cdot = \varnothing$' and '$\mathsf{Disj}(\cdot,\cdot)$' (respectively meaning 'the argument set is empty' and 'the arguments are disjoint sets'), along with their opposites '$\cdot \neq \varnothing$' and '$\neg\mathsf{Disj}(\cdot,\cdot)$'.

Keywords: Satisfiability problem, Computable set theory, NP-completeness, Proof verification

1 Introduction

Since the late seventies, the satisfiability problem for fragments of set theory, namely the problem of algorithmically determining for any formula in a given fragment whether or not there exists an assignment of sets in the von Neumann universe to its free variables that makes the formula true, has been thoroughly studied within the research field named *Computable Set Theory* (see [4,6,15,14,9] for a fairly comprehensive account). The initial goal envisaged an automated proof verifier, based on the set-theoretic formalism, capable to carry out the formalization of extensive parts of classical mathematics (e.g., the Cauchy integral theorem of complex analysis). To dispense users with the very tiny details, typical of low-level logic-oriented proofs, such a proof verifier should have been able to automatically guess the 'obvious' deduction steps left as tacit, through an inferential core embodying an extensive library of decision procedures.

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However, a foundational quest to locate the precise frontier between the decidable and the undecidable in set theory (and in other fundamental mathematical theories as well) began long before the proof verifier came into being, and inspired much of the subsequent work. The progenitor fragments in this quest, whose decision problems were addressed in the seminal paper [11], are the so-called Multi-Level Syllogistic (MLS, for short) and its extension MLSS with the singleton operator $\{\cdot\}$. We recall that MLS is the quantifier-free fragment consisting of the propositional combination of literals of the following types:

$$x = y \cup z, \quad x = y \cap z, \quad x = y \setminus z, \quad x \in y.$$

As shown in [5], the satisfiability problem for MLS is NP-complete, even when restricted to conjunctions of flat literals of the above types, plus negative literals of type $x \notin y$. Then, *a fortiori*, all decidable extensions of MLS have an NP-hard satisfiability problem (even hyperexponential in some cases; see [2,7,8]).

Nevertheless, despite its NP-hardness, the decision procedure for a slightly extended variant of MLSS, implemented in the form of a decidable tableaux calculus (see [10]), has become the core of the most central inference primitive of the ÆtnaNova/Ref proof-checker [15], viz. the ELEM rule. Optimization of the MLSS-decision test, at least in favourable cases, is therefore of utmost importance to limit occasional poor performances of ÆtnaNova originating from the full-strength decision test.

For such reason, we recently undertook an investigation aimed at identifying 'small', yet useful, decidable fragments of set theory (which so far are all subfragments of MLS) endowed with efficient *polynomial-time* decision tests. In [3], we have recently reported about the complexity taxonomy of all subfragments of the set theory fragment denoted $\mathbb{BST}$ and consisting of the conjunctions of literals involving, besides set variables, the Boolean set operators $\cup, \cap, \setminus$, the Boolean relators $\subseteq, \not\subseteq, =, \neq$, and the predicates (both affirmed and negated) '$\cdot = \varnothing$' and '$\mathsf{Disj}(\cdot,\cdot)$', expressing respectively that a specified set is empty and that two specified sets are disjoint.

Here we examine the sublanguages of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$,[1] the fragment of set theory consisting of the conjunctions of literals involving, besides set variables, the Boolean set operators $\cup, \cap, \setminus$ and the relators $\in$ and $\notin$.

Of a fragment of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$, we say that *it is* NP*-complete* if it has an NP-complete satisfiability problem (see [12]). Likewise, we say that *it is polynomial* if its satisfiability problem has polynomial complexity. As in [3], as a first approximation, at least as far as polynomial fragments are concerned, it is enough to discover the *minimal* NP-complete fragments (namely the NP-complete fragments of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$ that do not strictly contain any NP-complete fragment) and the maximal polynomial fragments (namely the polynomial fragments of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$ that are not strictly contained in any polynomial fragment of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$). Indeed, any $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$-fragment either is contained in some maximal polynomial fragment or contains some minimal NP-complete fragment.

[1] The acronym MST stands for 'membership set theory', whereas $\mathbb{BST}$ stands for 'Boolean set theory'.

The paper is organized as follows. Preliminarily, in Section 2, we introduce the syntax and semantics of the fragment $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$ of our interest. Then, in Section 3, we provide polynomial decision procedures for its maximal polynomial subfragments $\mathsf{MST}(\cup,\in,\notin)$, $\mathsf{MST}(\cup,\in,\notin)$, and $\mathsf{MST}(\cup,\cap,\setminus,\notin)$, whereas in Section 4 we prove the NP-completeness of its minimal NP-complete subfragments $\mathsf{MST}(\cup,\cap,\in)$ and $\mathsf{MST}(\setminus,\in)$. Section 5 contains some closing considerations and hints for future investigations.

2 Syntax and Semantics

The fragments of set theory of which in this paper we are investigating the satisfiability problem are parts, syntactically delimited, of the quantifier-free language $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$. This is the collection of all conjunctions of literals of the two types $s \in t$ and $s \notin t$, where s and t stand for terms built up from a denumerably infinite supply of set variables $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \ldots$ by means of the Boolean set operators of union $\cup$, intersection $\cap$, and set difference $\setminus$.

More generally, we shall denote by $\mathsf{MST}(\mathsf{op}_1, \ldots, \mathsf{pred}_1, \ldots)$ the subtheory of $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$ involving exactly the set operators $\mathsf{op}_1, \ldots$ (drawn from the set $\{\cup,\cap,\setminus\}$) and the predicate symbols $\mathsf{pred}_1, \ldots$ (drawn from the set $\{\in,\notin\}$).

For any $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$-conjunction φ, we shall denote by $\mathit{Vars}(\varphi)$ the collection of set variables occurring in φ; $\mathit{Vars}(\tau)$ is defined likewise, for any term τ.

A *set assignment* M is any function sending a collection of set variables V (called the *domain* of M and denoted $\mathsf{dom}(M)$) into the von Neumann universe $\mathcal{V}$ of well-founded sets. We recall that the *von Neumann universe* (see [13, pp. 95–102]) is built up in stages as the union $\mathcal{V} := \bigcup_{\alpha \in On} \mathcal{V}_\alpha$ of the levels $\mathcal{V}_\alpha := \bigcup_{\beta<\alpha} \mathscr{P}(\mathcal{V}_\beta)$, with α ranging over the class On of all ordinal numbers, where $\mathscr{P}(\cdot)$ is the powerset operator. For any set $S \in \mathcal{V}$, the least ordinal α such $S \subseteq \mathcal{V}_\alpha$ is the *rank* of S, denoted $\mathsf{rk}\,(S)$.

To shorten proofs, we shall sometimes make use of *urelements* in the definition of set assignments. These are objects that do not own any element and yet are distinct from the empty set (and, therefore, from any set) and distinct among them. However, in all cases under consideration, it will always be possible, without disrupting the correctness of any of the proofs, to replace all urelements by 'proper' sets, all of which either sharing the same sufficiently high rank, or having a conveniently large cardinality.

Natural designation rules attach recursively a value to every term τ of $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$ such that $\mathit{Vars}(\tau) \subseteq \mathsf{dom}(M)$, for any set assignment M; here is how:

$$M(s \cup t) := Ms \cup Mt, \quad M(s \cap t) := Ms \cap Mt, \quad \text{and} \quad M(s \setminus t) := Ms \setminus Mt.$$

We also put: $M(s \in t) = \mathsf{true} \leftrightarrow Ms \in Mt$ and $M(s \notin t) := \neg M(s \in t)$, and then, recursively, $M(\varphi \wedge \psi) := M\varphi \wedge M\psi$, when φ, ψ are $\mathsf{MST}(\cup,\cap,\setminus,\in,\notin)$-conjunctions such that $\mathit{Vars}(\varphi \wedge \psi) \subseteq \mathsf{dom}(M)$.

For convenience, we shall represent terms of the form $x_1 \cup \ldots \cup x_h$ and $y_0 \cap \ldots \cap y_k$ as $\bigcup\{x_1, \ldots, x_h\}$ and $\bigcap\{y_0, \ldots, y_k\}$, respectively. Thus, for a set assignment M and a finite nonempty collection of set variables $L \subseteq \mathsf{dom}(M)$, we shall have $M(\bigcup L) = \bigcup_{x \in L} Mx$ and $M(\bigcap L) = \bigcap_{x \in L} Mx$. We also put $ML = \{Mx \mid x \in L\}$, so that $M(\bigcup L) = \bigcup ML$ and $M(\bigcap L) = \bigcap ML$ hold.

Given a conjunction φ of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$ and a set assignment M such that $\mathit{Vars}(\varphi) \subseteq \mathsf{dom}(M)$, we say that M *satisfies* φ, and write $M \models \varphi$, if $M\varphi = \mathsf{true}$. When M satisfies φ, we say that M is a *model* of φ.

A conjunction φ is *satisfiable* if it has some model, otherwise it is *unsatisfiable*. Any two conjunctions φ and ψ are *equisatisfiable* if they are either both satisfiable or both unsatisfiable.

Since $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$ is a subtheory of MLS, it plainly has a solvable *satisfiability problem*, namely there is an algorithm (called a *decision procedure* or a *satisfiability test*) that, for any given conjunction φ of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$, establishes in an effective manner whether φ is satisfiable or not.

In the following sections, we shall find out the maximal polynomial and the minimal NP-complete fragments of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$.

Remark 1. Our complexity results will implicitly refer to *linearly bounded* subclasses of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$, whose conjunctions φ meet the condition

$$\max\{j \mid \mathsf{x}_j \in \mathit{Vars}(\varphi)\} - \min\{j \mid \mathsf{x}_j \in \mathit{Vars}(\varphi)\} = \mathcal{O}(|\varphi|). \tag{1}$$

For instance, it is immediate to see that the class of all $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$-conjunctions φ such that $\mathit{Vars}(\varphi) = \{\mathsf{x}_1, \ldots, x_{|\mathit{Vars}(\varphi)|}\}$ is linearly bounded. For lists $\mathcal{L}_\varphi$ of sets of variables occurring in any conjunction φ belonging to some linearly bounded subclass of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$, it turns out that the duplicates of $\mathcal{L}_\varphi$ can be detected in linear time $\mathcal{O}(|\varphi|)$.

3 The maximal polynomial fragments of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$

The maximal polynomial fragments of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$ are $\mathsf{MST}(\cup, \in, \notin)$, $\mathsf{MST}(\cap, \in, \notin)$, and $\mathsf{MST}(\cup, \cap, \setminus, \notin)$, whose satisfiability problems can be solved in linear, quadratic, and constant time, respectively.

3.1 The fragment $\mathsf{MST}(\cup, \in, \notin)$

We prove that the satisfiability problem for $\mathsf{MST}(\cup, \in, \notin)$-conjunctions admits a linear-time solution by first showing that $\mathsf{MST}(\cup, \in)$ is linear and then proving that the satisfiability problem for $\mathsf{MST}(\cup, \in, \notin)$ can be reduced in linear time to that for $\mathsf{MST}(\cup, \in)$.

A linear-time satisfiability test for $\mathsf{MST}(\cup, \in)$ To start with, we provide a decision procedure for $\mathsf{MST}(\cup, \in)$ and then prove that it runs in linear time in the size of the input formula.

Theorem 1. *Let φ be a* $\mathsf{MST}(\cup,\in)$*-conjunction of the form* $\bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i$*, where the* L_i*'s and the* R_i*'s are finite nonempty collections of set variables. Then:*

(a) if $\prec$ *is a linear ordering of* $\mathit{Vars}(\varphi)$ *satisfying the condition*
$$\max(L_i, \prec) \;\prec\; \max(R_i, \prec), \;\; \textit{for } i = 1, \ldots, p, \tag{2}$$
then φ *has a model* M *such that*
- *(a$_1$)* $Mx = \{\mathbf{u}_x\} \cup \{\bigcup ML_i \mid x = \max(R_i, \prec), \; \textit{for } i = 1, \ldots, p\}$*, where the* $\mathbf{u}_x$*'s are pairwise distinct urelements; and*
- *(a$_2$)* $\bigcup ML \notin Mx$*, for every* $L \subseteq \mathit{Vars}(\varphi)$ *and* $x \in \mathit{Vars}(\varphi)$ *fulfilling the condition*
$$\bigwedge_{i=1}^{p} (L = L_i \longrightarrow x \notin R_i); \tag{3}$$

(b) if φ *is satisfiable, then there is a linear ordering* $\prec$ *of* $\mathit{Vars}(\varphi)$ *such that condition* (2) *holds.*

Hence, the satisfiability problem for $\mathsf{MST}(\cup,\in)$*-conjunctions is solvable.*

Proof. Let us first assume that there is a linear ordering $\prec$ of $\mathit{Vars}(\varphi)$ such that (2) holds. Following the ordering $\prec$, for $x \in \mathit{Vars}(\varphi)$ we put
$$Mx \coloneqq \{\mathbf{u}_x\} \cup \big\{\bigcup ML_i \mid x = \max(R_i, \prec), \; i = 1, \ldots, p\big\}, \tag{4}$$
where the $\mathbf{u}_x$ are pairwise distinct urelements. For any literal $\bigcup L_i \in \bigcup R_i$ in φ, with $i \in \{1, \ldots, p\}$, setting $x_i \coloneqq \max(R_i, \prec)$, we have $\bigcup ML_i \in Mx_i \subseteq \bigcup MR_i$, so that $M \models \bigcup L_i \in \bigcup R_i$. Thus, M models correctly all conjuncts of φ, and it plainly satisfies condition (a$_1$). In fact, also condition (a$_2$) is true for M. Preliminarily, we observe that we clearly have
$$\bigcup ML = \bigcup ML' \text{ if and only if } L = L', \tag{5}$$
for $L, L' \subseteq \mathit{Vars}(\varphi)$, as the urelements in $\bigcup ML$ are the same as those in $\bigcup ML'$ if and only if $L = L'$. Next, let $L \subseteq \mathit{Vars}(\varphi)$ and $x \in \mathit{Vars}(\varphi)$ be such that $\bigwedge_{i=1}^{p}(L = L_i \longrightarrow x \notin R_i)$ holds, but assume, by way of contradiction, $\bigcup ML \in Mx$. Then, by (5) and (4), $L = L_{i_0}$ for some $i_0 \in \{1, \ldots, p\}$ such that $x = \max(R_{i_0}, \prec) \in R_{i_0}$, contradicting (3). Therefore M satisfies condition (a$_2$) too.

Concerning condition (b), let us now assume that our conjunction φ is satisfiable, and let $\overline{M}$ be a model of φ. Also, let $\prec$ be any linear ordering of $\mathit{Vars}(\varphi)$ such that
$$\mathsf{rk}\left(\overline{M}x\right) < \mathsf{rk}\left(\overline{M}y\right) \longrightarrow x \prec y, \qquad \text{for } x, y \in \mathit{Vars}(\varphi), \tag{6}$$

Let us check that (2) holds for the ordering $\prec$. Let $i \in \{1, \ldots, p\}$. Therefore $\bigcup \overline{M} L_i \in \bigcup \overline{M} R_i$, so that $R_i \neq \emptyset$. Hence if $L_i = \emptyset$, $\max(L_i, \prec) \prec \max(R_i, \prec)$ holds trivially. On the other hand, if $L_i \neq \emptyset$, we have:
$\max\{\mathsf{rk}\left(\overline{M}x\right) \mid x \in L_i\} = \mathsf{rk}\left(\bigcup \overline{M} L_i\right) < \mathsf{rk}\left(\bigcup \overline{M} R_i\right) = \max\{\mathsf{rk}\left(\overline{M}x\right) \mid x \in R_i\}$, so that, by (6), $\max(L_i, \prec) \prec \max(R_i, \prec)$ holds also in this case. Therefore, (2) holds, completing the proof of (b).

Conditions (a) and (b) yield that the conjunction φ is satisfiable if and only if there is a linear ordering $\prec$ of $\mathit{Vars}(\varphi)$ satisfying (2), from which the decidability of the satisfiability problem for $\mathsf{MST}(\cup,\in)$ readily follows. □

Towards a linear satisfiability test for $\mathsf{MST}(\cup, \in)$, we derive next two conditions, which can be tested in linear time and whose conjunction is equivalent to (2). To begin with, in connection with any formula φ of $\mathsf{MST}(\cup, \in)$, we define the *left-variables* (resp., *right-variables*) of φ as those variables in φ occurring in the left-hand side L (resp., right-hand side R) of some literal $\bigcup L \in \bigcup R$ in φ. Right-variables of φ that are not left-variables are called *proper right-variables*.

Let now φ be any satisfiable $\mathsf{MST}(\cup, \in)$-conjunction and let $\prec$ be any linear ordering of $\mathit{Vars}(\varphi)$ fulfilling condition (2). Plainly, no left-variable in φ can be $\prec$-maximal. Hence

(A) φ has some proper right-variable.

In addition, by letting φ^- be the result of dropping from φ all conjuncts that involve some proper right-variable, we clearly have

(B) φ^- is satisfiable.

It turns out that conditions (A) and (B) are also sufficient for the satisfiability of φ. Indeed, assume that conditions (A) and (B) hold for a given $\mathsf{MST}(\cup, \in)$-conjunction φ, and let $\prec^-$ be a linear ordering of $\mathit{Vars}(\varphi^-)$ fulfilling condition (2) of Lemma 1 as applied to φ^-. Then, any extension $\prec$ of $\prec^-$ to a linear ordering of $\mathit{Vars}(\varphi^-)$ such that $x \prec y$, for every left-variable x and proper right-variable y of φ, fulfils condition (2) of Lemma 1 as applied to φ, proving that the conjunction φ is satisfiable.

The above considerations readily yield the following satisfiability test for $\mathsf{MST}(\cup, \in)$:

Algorithm 1 Satisfiability tester for $\mathsf{MST}(\cup, \in)$

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while φ contains some proper right-variable do
    drop from φ all the conjuncts involving some proper right-variable;
if φ is the empty conjunction then
    return "satisfiable";
else
    return "unsatisfiable";
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A straightforward implementation of Algorithm 1 is quadratic. An alternative implementation consists in counting the number of left-occurrences of each variable $x \in \mathit{Vars}(\varphi)$, while also maintaining, for each right-variable y in φ, the list of the conjuncts containing y. Variables with a zero counter are exactly the proper right-variables. While the conjunction φ contains right-variables with a zero counter, drop from it all conjuncts containing some right-variable with a zero counter and, accordingly, decrement the counters of the left-occurrences in the dropped conjuncts. If eventually one ends up with the empty conjunction, then the initial conjunction φ is declared satisfiable. Otherwise, φ is declared unsatisfiable.

It is not hard to see that the above implementation has a linear-time complexity. Hence, we have:

Theorem 2. *The satisfiability problem for* $\mathsf{MST}(\cup, \in)$ *can be solved in linear time.*

A linear-time reduction of $\mathsf{MST}(\cup, \in, \notin)$ to $\mathsf{MST}(\cup, \in)$ We shall now prove that the satisfiability problem for $\mathsf{MST}(\cup, \in, \notin)$ can be reduced in linear time to that of $\mathsf{MST}(\cup, \in)$, thereby yielding a linear-time decision procedure for the satisfiability problem of the extended fragment $\mathsf{MST}(\cup, \in, \notin)$.

Lemma 1. *The satisfiability problem for* $\mathsf{MST}(\cup, \in, \notin)$ *can be reduced in linear time to that for* $\mathsf{MST}(\cup, \in)$*.*

Proof. Let φ be a conjunction of $\mathsf{MST}(\cup, \in, \notin)$ of the form

$$\bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i \;\wedge\; \bigwedge_{j=p+1}^{n} \bigcup L_j \notin \bigcup R_j,$$

where the L_i's and R_i's are finite collection of set variables. For each $i = 1, \ldots, p$, let

$$R_i' \coloneqq R_i \setminus \bigcup \{R_j \mid L_j = L_i, \text{ for } j = p+1, \ldots, n\}, \tag{7}$$

and put $\quad \varphi' \coloneqq \begin{cases} \bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i' & \text{if no } R_i' \text{ is empty} \\ x_0 \in x_0 & \text{otherwise,} \end{cases}$

for a fixed, but otherwise arbitrary, set variable x_0. It is an easy matter to verify that if φ is satisfiable, then ($\varphi' \equiv \bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i'$ and) any model for φ is also a model for φ', i.e., $\models \varphi \longrightarrow \varphi'$.

Conversely, if φ' is satisfiable (and therefore $\varphi' \equiv \bigwedge_{i=1}^{n} \bigcup L_i \in \bigcup R_i'$), then, by Theorem 1(b), there is a linear ordering $\prec$ of $\mathit{Vars}(\varphi')$ fulfilling condition (2) and in addition, by Theorem 1(a), the conjunction φ' has a model M satisfying conditions (a_1) and (a_2) of the same theorem. We extend M to the variables $x \in \mathit{Vars}(\varphi) \setminus \mathit{Vars}(\varphi')$, if any, by setting $Mx \coloneqq \{\mathbf{u}_x\}$, where the $\mathbf{u}_x$'s are pairwise distinct urelements new to the assignment M. Since $\bigcup MR_i' \subseteq \bigcup MR_i$, for $i = 1, \ldots, p$, we plainly have $M \models \bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i$, namely M satisfies all the positive conjuncts of φ.

Let us now show that M satisfies the negative part $\bigwedge_{j=p+1}^{n} \bigcup L_j \notin \bigcup R_j$ of φ as well. Indeed, if this were not the case, there would exist a $j \in \{p+1, \ldots, n\}$ and a variable $x \in R_j$ such that $\bigcup ML_j \in Mx$. Since $\bigcup ML_j$ is not a urelement, Mx would contain some set and therefore $x \in \mathit{Vars}(\varphi')$. Thus, by condition ($a_1$) of Theorem 1, $Mx = \{\mathbf{u}_x\} \cup \big\{\bigcup ML_i \mid x = \max(R_i', \prec), \text{ for } i = 1, \ldots, p\big\}$, and so we would have $\bigcup ML_j = \bigcup ML_i$, for some $i \in \{1, \ldots, p\}$ such that $x = \max(R_i', \prec)$, so that $x \in R_i'$. The pairwise distinctness of the urelements would yield $L_j = L_i$. Hence, by (7), $x \notin R_i'$, a contradiction.

In conclusion, M must also satisfy the negative part $\bigwedge_{j=p+1}^{n} \bigcup L_j \notin \bigcup R_j$ of φ, and thus, in view of $M \models \bigwedge_{i=1}^{p} \bigcup L_i \in \bigcup R_i$ seen above, M satisfies φ.

Summing up, we have proved that if φ' is satisfiable, so is φ, which, in view of $\models \varphi \longrightarrow \varphi'$ observed above, yields the equisatisfiability of φ and φ'.

To complete the proof, it is enough to observe that the conjunction φ' can be constructed in $\mathcal{O}(|\varphi|)$ time, where $|\varphi|$ denotes the size of φ, using a suitable linear time algorithm to detect the duplicates in the list of sets $L_1, \ldots, L_n$. $\square$

In view of Theorem 2, Lemma 1 yields the following complexity result:

Theorem 3. *The satisfiability problem for* $\mathsf{MST}(\cup, \in, \notin)$ *can be solved in linear time.*

3.2 The fragment $\mathsf{MST}(\cap, \in, \notin)$

Let φ be a $\mathsf{MST}(\cap, \in, \notin)$-conjunction of the form

$$\bigwedge_{i=1}^{p} \bigcap L_i \in \bigcap R_i \;\wedge \bigwedge_{j=p+1}^{\ell} \bigcap L_j \notin \bigcap R_j, \tag{8}$$

where the L_u's and the R_u's are *nonempty* finite collections of set variables, for $u = 1, \ldots, \ell$. We shall denote by φ^+ and φ^- the *positive part* $\bigwedge_{i=1}^{p} \bigcap L_i \in \bigcap R_i$ and the *negative part* $\bigwedge_{j=p+1}^{\ell} \bigcap L_j \notin \bigcap R_j$ of φ, respectively. For decidability purposes and without loss of generality, we may assume that the following conditions hold for φ:

(C0) the positive part φ^+ of φ is nonempty, namely $p \geq 1$,

(C1) $L_h \neq L_i$, for any two distinct $h, i \in \{1, \ldots, p\}$.

Indeed, as for (C0), if the positive part of φ were empty, then φ would be always satisfiable (e.g., φ would be satisfied by the null assignment $M_\emptyset$ over $\mathit{Vars}(\varphi)$, which maps every set variable in φ to the empty set $\emptyset$). In addition, without disrupting satisfiability, condition (C1) can be enforced by replacing, for each set of variables $L \in \{L_1, \ldots, L_p\}$, the collection of conjuncts $\bigcap L_i \in \bigcap R_i$ in φ^+ such that $L_i = L$ by the single conjunct $\bigcap L \in \bigcap R$, where $R \coloneqq \bigcup \{R_i \mid L_i = L, \text{ for } i = 1, \ldots, p\}$, since $\models \bigcap \big\{ \bigcap R_i \mid L_i = L, \text{ for } i = 1, \ldots, p \big\} = \bigcap R$.

Notice that the duplicates in the list of sets $L_1, \ldots, L_p$ can be suitably detected in linear time, and therefore condition (C1) can be enforced in time $\mathcal{O}(|\varphi|)$.

Theorem 4. *Let φ be a $\mathsf{MST}(\cap, \in, \notin)$-conjunction of the form* (8) *and fulfilling conditions (C0) and (C1). Then φ is satisfiable if and only if the following two conditions hold:*

(a) $L_i = L_j \;\longrightarrow\; R_j \nsubseteq R_i$,
for $i = 1, \ldots, p$ and $j = p+1, \ldots, \ell$;

(b) there is an indexing of the conjuncts of φ^+ such that, for $h, i = 1, \ldots, p$,

$$L_i \subseteq R_h \;\longrightarrow\; h < i. \tag{9}$$

Proof. (Necessity). Let us first assume that φ is satisfiable, and let $M \models \varphi$.

Concerning condition (a), let $L_i = L_j$, for some $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, \ell\}$. Hence we have $\bigcap ML_i \in \bigcap MR_i \setminus \bigcap MR_j$, so $R_j \nsubseteq R_i$ must hold.

As for condition (b), any indexing of the conjuncts of the positive part φ^+ of φ such that

$$\mathsf{rk}\,(\bigcap ML_h) < \mathsf{rk}\,(\bigcap ML_i) \;\longrightarrow\; h < i, \tag{10}$$

for $h, i = 1, \ldots, p$, satisfies (9), since, for $h, i = 1, \ldots, p$,

$$\begin{aligned} L_i \subseteq R_h \;&\longrightarrow\; \bigcap ML_h \in \bigcap MR_h \subseteq \bigcap ML_i \\ &\longrightarrow\; \mathsf{rk}\left(\bigcap ML_h\right) < \mathsf{rk}\left(\bigcap ML_i\right) \\ &\longrightarrow\; h < i. \end{aligned}$$

(Sufficiency). Conversely, let us assume that conditions (a) and (b) of the theorem hold for φ.

A pair of (distinct) indices $u, v \in \{1, \ldots, \ell\}$ such that $L_u \neq L_v$ is said to be R^+*-indistinguishable (w.r.t. φ)* if $L_u \subseteq R_i \longleftrightarrow L_v \subseteq R_i$, for every $i = 1, \ldots, p$. Otherwise, we say that the pair u, v is R^+*-distinguishable*.

For each R^+-indistinguishable pair u, v, we set

$$\overline{k}_{uv} \coloneqq \min\left(\{u \mid L_u \nsubseteq L_v\} \cup \{v \mid L_v \nsubseteq L_u\}\right),$$

and let k_{uv} be the index such that $\{k_{uv}, \overline{k}_{uv}\} = \{u, v\}$, so that $L_{\overline{k}_{uv}} \nsubseteq L_{k_{uv}}$. Also, we let K be the collection of all the k_{uv}'s so defined.

Next, with each $k \in K$ we associate a distinct urelement $\mathbf{u}_k$, and put recursively, for $v = 1, \ldots, \ell$:

$$\mathcal{I}_i \coloneqq \{\mathcal{I}_u \mid L_v \subseteq R_i,\ \text{for } i = 1, \ldots, p\} \cup \{\mathbf{u}_k \mid L_v \subseteq L_k,\ \text{for } k \in K\}. \tag{11}$$

We also put, for $x \in \mathit{Vars}(\varphi)$:

$$Mx \coloneqq \{\mathcal{I}_i \mid x \in R_i,\ \text{for } i = 1, \ldots, p\} \cup \{\mathbf{u}_k \mid x \in L_k,\ \text{for } k \in K\}. \tag{12}$$

The rest of the proof is devoted to showing that the assignment M just defined satisfies φ. We shall use the following claims, whose proofs are omitted for lack of space.

Claim 1. $L_u = L_v \longleftrightarrow \mathcal{I}_u = \mathcal{I}_v$, *for* $u, v = 1, \ldots, \ell$.

Claim 2. *If* $\mathcal{I}_u \in \bigcap MS$, *where* $u \in \{1, \ldots, \ell\}$ *and* $S \subseteq \mathit{Vars}(\varphi)$, *then* $S \subseteq R_{\overline{i}}$, *for some* $\overline{i} \in \{1, \ldots, p\}$ *such that* $L_{\overline{i}} = L_u$.

Claim 3. $\bigcap ML_v = \mathcal{I}_v$, *for* $v = 1, \ldots, \ell$.

We start with the positive conjuncts. Thus, let $\bigcap L_i \in \bigcap R_i$ be any positive conjunct in φ, with $i \in \{1, \ldots, p\}$. By Claim 3, $\bigcap ML_i = \mathcal{I}_i$. In addition, by (12), $\mathcal{I}_i \in Mx$ for every $x \in R_i$, and therefore $\bigcap ML_i = \mathcal{I}_i \in \bigcap MR_i$, proving that M models correctly the conjunct $\bigcap L_i \in \bigcap R_i$, and in turn all the positive conjuncts of φ.

Next, we show that also the negative conjuncts of φ are satisfied by M. To this purpose, let $\bigcap L_j \notin \bigcap R_j$ be any negative conjunct in φ, with $j \in \{p+1, \ldots, \ell\}$. Again by Claim 3, $\bigcap ML_j = \mathcal{I}_j$. By way of contradiction, assume that $\bigcap ML_j \in \bigcap MR_j$, namely $\mathcal{I}_j \in \bigcap MR_j$. Then, by Claim 2, $R_j \subseteq R_{\overline{i}}$ and $L_j = L_{\overline{i}}$ for some $\overline{i} \in \{1, \ldots, p\}$, contradicting condition (a) of the theorem. Thus $\bigcap ML_j \notin \bigcap MR_j$, proving that M satisfies also the negative conjuncts of φ. Hence, M satisfies the conjunction φ. □

Complexity issues The complexity of the satisfiability problem for $\mathsf{MST}(\cap, \in, \notin)$ can be estimated as follows. Given a $\mathsf{MST}(\cap, \in, \notin)$-conjunction of the form (8), condition (a) of Theorem 4 can be tested in $\mathcal{O}(\varphi)$ time by detecting the duplicates in the list of sets $L_1, \ldots, L_p$. Next, we observe that condition (b) of Theorem 4 is equivalent to the aciclicity of the oriented graph $G_{\varphi^+} = (V_{\varphi^+}, E_{\varphi^+})$, where $V_{\varphi^+} \coloneqq \{1, \ldots, p\}$ and $E_{\varphi^+} \coloneqq \{(h, i) \mid L_i \subseteq R_h,\ \text{for } i, h \in V_{\varphi^+}\}$. The graph G_{φ^+} can be constructed in time $\mathcal{O}(p \cdot |\varphi^+|)$ and its aciclicity tested in time $\mathcal{O}(p^2)$. Hence, condition (b) can be tested in time $\mathcal{O}(p \cdot |\varphi^+|)$, yielding an overall time complexity $\mathcal{O}(p \cdot |\varphi^+| + |\varphi|)$ for the satisfiability problem of $\mathsf{MST}(\cap, \in, \notin)$.

Summing up:

Lemma 2. *The satisfiability problem for* $\mathsf{MST}(\cap, \in, \notin)$*-conjunctions can be solved in quadratic time.*

3.3 The trivial fragment $\mathsf{MST}(\cup, \cap, \setminus, \notin)$

Since any $\mathsf{MST}(\cup, \cap, \setminus, \notin)$-conjunction φ is trivially satisfied by the null assignment $M_\emptyset$ over $Vars(\varphi)$, defined by $M_\emptyset x = \emptyset$ for every $x \in Vars(\varphi)$, we readily have:

Theorem 5. *The satisfiability problem for the fragment* $\mathsf{MST}(\cup, \cap, \setminus, \notin)$ *can be solved in constant time.*

4 The minimal NP-complete fragments

4.1 The fragment $\mathsf{MST}(\cup, \cap, \in)$

We shall prove that the satisfiability problem for $\mathsf{MST}(\cup, \cap, \in)$-conjunctions is NP-complete by reducing the problem 3-SAT to it.

Let $F \coloneqq \bigwedge_{i=1}^{m}(L_{i1} \vee L_{i2} \vee L_{i3})$ be an instance of 3-SAT, where the L_{ij}'s are propositional literals, and let $P_1, \ldots, P_n$ be the distinct propositional variables occurring in F. Also, let $x, X_1, \overline{X}_1, \ldots, X_n, \overline{X}_n$ be $2n+1$ distinct set variables. For $i = 1, \ldots, m$, $j = 1, 2, 3$, and $k \in \{1, \ldots, n\}$ such that $L_{ij} \in \{P_k, \neq P_k\}$, put

$$T_{ij} \coloneqq \textbf{if } L_{ij} = P_k \textbf{ then } X_k \textbf{ else } \overline{X}_k \textbf{ endif} .$$

Finally, let

$$\Phi_F \coloneqq \bigwedge_{i=1}^{m}(x \in T_{i1} \cup T_{i2} \cup T_{i3}) \ \wedge \bigwedge_{k=1}^{n}(x \in X_k \cup \overline{X}_k \ \wedge \ X_k \cap \overline{X}_k \in x).$$

Theorem 6. *A 3-SAT instance F is propositionally satisfiable if and only if the corresponding* $\mathsf{MST}(\cup, \cap, \in)$*-conjunction Φ_F is satisfied by a set assignment.*

Proof. (Necessity). To begin with, let us assume that F is propositionally satisfiable, and let $\mathfrak{v}$ be a Boolean valuation that satisfies it. Let $M_\mathfrak{v}$ be the set assignment induced over $Vars(\Phi_F)$ by $\mathfrak{v}$ and defined as follows, for $k = 1, \ldots, n$: $M_\mathfrak{v} X_k \coloneqq \textbf{if } \mathfrak{v}(P_k) = \mathtt{true} \textbf{ then } \{\{\emptyset\}\} \textbf{ else } \emptyset \textbf{ endif}$ and $M_\mathfrak{v}\overline{X}_k \coloneqq \{\{\emptyset\}\} \setminus M_\mathfrak{v} X_k$, and such that $M_\mathfrak{v} x \coloneqq \{\emptyset\}$. Hence, $M_\mathfrak{v}(X_k \cup \overline{X}_k) = \{\{\emptyset\}\}$ and $M_\mathfrak{v}(X_k \cap \overline{X}_k) = \emptyset$ for each k, so that $M_\mathfrak{v} \models \bigwedge_{k=1}^{n}(x \in X_k \cup \overline{X}_k \ \wedge \ X_k \cap \overline{X}_k \in x)$.

Let $i \in \{1, \ldots, m\}$. Since, by hypothesis, $\mathfrak{v}$ satisfies F, then $\mathfrak{v}(L_{i1} \vee L_{i2} \vee L_{i3}) = \mathbf{t}$, so that $\mathfrak{v}(L_{ij_i}) = \mathbf{t}$ for some $j_i \in \{1, 2, 3\}$. Let $k \in \{1, \ldots, n\}$ be such that $L_{ij_i} \in \{P_k, \neg P_k\}$. If $L_{ij_i} = P_k$, then $\mathfrak{v}(P_k) = \mathbf{t}$ and $T_{ij_i} = X_k$, so that $M_\mathfrak{v} T_{ij_i} = \{\{\emptyset\}\}$. On the other hand, if $L_{ij_i} = \neg P_k$, then $\mathfrak{v}(P_k) = \mathbf{f}$ and $T_{ij_i} = \overline{X}_k$; hence, again, $M_\mathfrak{v} T_{ij_i} = \{\{\emptyset\}\}$. Thus, $M_\mathfrak{v} x = \{\emptyset\} \in \{\{\emptyset\}\} = M_\mathfrak{v}(T_{ij_i}) \subseteq M_\mathfrak{v}(T_{i1} \cup T_{i2} \cup T_{i3})$, proving that $M_\mathfrak{v}$ satisfies the literal $x \in T_{i1} \cup T_{i2} \cup T_{i3}$.

The arbitrariness of $i \in \{1, \ldots, m\}$, together with $M_\mathfrak{v} \models \bigwedge_{k=1}^{n}(x \in X_k \cup \overline{X}_k \ \wedge \ X_k \cap \overline{X}_k \in x)$ proved before, yields that the set assignment $M_\mathfrak{v}$ satisfies the conjunction Φ_F corresponding to F.

(Sufficiency). Let us now assume that the $\mathsf{MST}(\cup, \cap, \in)$-conjunction Φ_F is satisfiable, and let M be a set assignment that satisfies it. Then, for each $k = 1, \ldots, n$, we have $Mx \in MX_k \cup M\overline{X}_k \ \wedge \ MX_k \cap M\overline{X}_k \in Mx$, and therefore $Mx \notin MX_k \cap M\overline{X}_k$. Hence, either $Mx \in MX_k$ or $Mx \in M\overline{X}_k$, but not both. We now define a Boolean valuation $\mathfrak{v}_M$ induced on $P_1, \ldots, P_n$ by the set assignment M, by putting, $\mathfrak{v}_M(P_k) \coloneqq M(x \in X_k)$, for $k = 1, \ldots, n$. Since $M \models \Phi_F$, then

$M \models \bigwedge_{i=1}^{m} x \in T_{i1} \cup T_{i2} \cup T_{i3}$. Let $i \in \{1, \ldots, m\}$. Then $Mx \in MT_{ij_i}$ for some $j_i \in \{1, 2, 3\}$. Let $k \in \{1, \ldots, n\}$ be such that $T_{ij_i} \in \{X_k, \overline{X}_k\}$. If $T_{ij_i} = X_k$, then $L_{ij_i} = P_k$ and $\mathfrak{v}_M(P_k) = \mathbf{t}$. On the other hand, if $T_{ij_i} = \overline{X}_k$, then $L_{ij_i} = \neg P_k$ and $\mathfrak{v}_M(P_k) = \mathbf{f}$. In any case, $\mathfrak{v}_M(L_{ij_i}) = \mathbf{t}$, and so $\mathfrak{v}_M(L_{i1} \vee L_{i2} \vee L_{i3}) = \mathbf{t}$. The arbitrariness of $i \in \{1, \ldots, m\}$ yields that the Boolean valuation $\mathfrak{v}$ satisfies the 3-SAT instance F. □

From the previous theorem, and since the conjunction Φ_F corresponding to F can be constructed in time $\mathcal{O}(|F|)$, we can conclude that

Lemma 3. *The satisfiability problem for* $\mathsf{MST}(\cup, \cap, \in)$*-conjunctions is* NP*-complete.*

4.2 The fragment $\mathsf{MST}(\setminus, \in)$

We shall prove that the satisfiability problem for $\mathsf{MST}(\setminus, \in)$-conjunctions is NP-complete by reducing the 3-SAT problem to it.

Thus, let $F \coloneqq \bigwedge_{i=1}^{m}(L_{i1} \ \vee \ L_{i2} \ \vee \ L_{i3})$ be an instance of 3-SAT, where the L_{ij}'s propositional literals, and let $P_1, \ldots, P_n$ be the distinct propositional variables occurring in F. Also, let $x, X, X_1, \ldots, X_n$ be $n+2$ distinct set variables. For $i = 1, \ldots, m$, $j = 1, 2, 3$, and $k \in \{1, \ldots, n\}$ such that $L_{ij} \in \{P_k, \neg P_k\}$, let $T_{ij} \coloneqq$ **if** $L_{ij} = P_k$ **then** X_k **else** $X \setminus X_k$ **endif**. Finally, put

$$\Phi_F \coloneqq \bigwedge_{i=1}^{m}(X \setminus T_{i1} \setminus T_{i2} \setminus T_{i3}) \in x \ \wedge x \in X.$$

Theorem 7. *A 3-SAT instance F is propositionally satisfiable if and only if the* $\mathsf{MST}(\setminus, \in)$*-formula Φ_F is satisfied by a set assignment.*

Proof. (Necessity). First, let us assume that F is propositionally satisfiable, and let $\mathfrak{v}$ be a Boolean valuation over $P_1, \ldots, P_n$ that satisfies F. We define the set assignment $M_{\mathfrak{v}}$, induced over $\mathit{Vars}(\Phi_F)$ by $\mathfrak{v}$, by setting:

$$M_{\mathfrak{v}}X \coloneqq \{\{\emptyset\}\}, \qquad M_{\mathfrak{v}}x \coloneqq \{\emptyset\}, \qquad M_{\mathfrak{v}}X_k \coloneqq \begin{cases} \{\{\emptyset\}\} & \text{if } \mathfrak{v}(P_k) = \mathbf{t} \\ \emptyset & \text{otherwise,} \end{cases}$$

for $k = 1, \ldots, n$. Plainly $M_{\mathfrak{v}}x \in M_{\mathfrak{v}}X$. Let $i \in \{1, \ldots, m\}$. As, by hypothesis, the Boolean valuation $\mathfrak{v}$ satisfies F, then $\mathfrak{v}(L_{i1} \ \vee \ L_{i2} \ \vee \ L_{i3}) = \mathbf{t}$, so that there exists a $j_i \in \{1, 2, 3\}$ such that $\mathfrak{v}(L_{ij_i}) = \mathbf{t}$. Let $k \in \{1, \ldots, n\}$ be such that $L_{ij_i} \in \{P_k, \neg P_k\}$. If $L_{ij_i} = P_k$, then $\mathfrak{v}(P_k) = \mathbf{t}$ and $T_{ij} = X_k$, so that $M_{\mathfrak{v}}T_{ij_i} = \{\{\emptyset\}\}$. On the other hand, if $L_{ij_i} = \neg P_k$, then $\mathfrak{v}(P_k) = \mathbf{f}$ and $T_{ij_i} = X \setminus X_k$ and $M_{\mathfrak{v}}X_k = \emptyset$, and again $M_{\mathfrak{v}}T_{ij_i} = \{\{\emptyset\}\}$. Hence,

$$M_{\mathfrak{v}}X \setminus M_{\mathfrak{v}}T_{i1} \setminus M_{\mathfrak{v}}T_{i2} \setminus M_{\mathfrak{v}}T_{i3} = M_{\mathfrak{v}}X \setminus (M_{\mathfrak{v}}T_{i1} \cup M_{\mathfrak{v}}T_{i2} \cup M_{\mathfrak{v}}T_{i3}) = \emptyset \in \{\emptyset\} = M_{\mathfrak{v}}x.$$

Thus, the arbitrariness of $i \in \{1, \ldots, m\}$ together with $M_{\mathfrak{v}}x \in M_{\mathfrak{v}}X$ yields that the induced set assignment $M_{\mathfrak{v}}$ satisfies Φ_F.

(Sufficiency). Next, let us assume that the $\mathsf{MST}(\setminus, \in)$-conjunction Φ_F corresponding to F is satisfiable, and let M be a set assignment over its variables

that satisfies it, so that $Mx \in MX$ holds. We define the Boolean valuation $\mathfrak{v}_M$, induced on $P_1, \ldots, P_n$ by M, by setting $\mathfrak{v}_M(P_k) \coloneqq M(x \in X_k)$ and prove that it satisfies the 3-SAT instance F. Thus, let $i \in \{1, \ldots, m\}$. Since $M \models \Phi_F$, it holds that $M \models X \setminus T_{i1} \setminus T_{i2} \setminus T_{i3}$ and therefore

$$MX \setminus (MT_{i1} \cup MT_{i2} \cup MT_{i3}) = MX \setminus MT_{i1} \setminus MT_{i2} \setminus MT_{i3} \in Mx \in MX.$$

Hence, the well-foundedness of $\in$ yields that $Mx \notin MX \setminus (MT_{i1} \cup MT_{i2} \cup MT_{i3})$ must hold, so that $Mx \in MT_{i1} \cup MT_{i2} \cup MT_{i3}$ must hold as well. From the latter, it follows that $\mathfrak{v}_M(L_{i1} \lor L_{i2} \lor L_{i3}) = \mathbf{t}$. Finally, by the arbitrariness of $i \in \{1, \ldots, m\}$, we have that $\mathfrak{v}_M(F) = \mathbf{t}$, proving that F is satisfiable. □

Since the formula Φ_F can be constructed in time $\mathcal{O}(|F|)$, for any given 3-SAT instance F, from the previous theorem we immediately conclude that

Lemma 4. *The satisfiability problem for* $\mathsf{MST}(\setminus, \in)$*, is* NP*-complete.*

$\setminus$	$\cup$	$\cap$	$\in$	$\notin$	Complexity
⋆			⋆		NP-complete
	⋆	⋆	⋆		NP-complete
	⋆		⋆	⋆	$\mathcal{O}(n)$
		⋆	⋆	⋆	$\mathcal{O}(n^2)$
⋆	⋆	⋆		⋆	constant

Table 1. Maximal polynomial and minimal NP-complete fragments of $\mathsf{MST}(\setminus, \cup, \cap, \in, \notin)$

5 Conclusions

With in mind applications in automated proof checking with verifiers based on the set-theoretic formalism, in this paper we identified the maximal polynomial and the minimal NP-complete sublanguages of the fragment of set theory $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$, as reported in Table 1. These allow one to easily pinpoint the 14 polynomial fragments (of which 2 are quadratic, 4 are linear, and 8 are constant) and the 10 NP-complete fragments among the 24 sublanguages of $\mathsf{MST}(\cup, \cap, \setminus, \in, \notin)$.

We plan to extend our analysis when also the singleton operator $\{\cdot\}$, the Boolean relators $\subseteq, \not\subseteq, =, \neq$, and the predicates '$\cdot = \emptyset$' and '$\mathsf{Disj}(\cdot, \cdot)$' (both affirmed and negated) are allowed. A library of the related polynomial decision tests will then be implemented and integrated within the inferential core of the ÆtnaNova proof-checker.

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