Mathematical Model of Dynamics of Homomorphic Objects

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Abstract. The paper concerns topical problem of mathematical modeling of dynamics of heterogeneous groups with a logistic function as a basic one. Joint use of mathematical models of biological systems and computer-based simulation makes it possible to minimize time and save material resources while determining general tendencies of subpopulation progress; and to forecast state of the system as well as possible consequences of artificial intervention in the environment. Among other things, it concerns forecasting of genetic abnormalities. The paper proposes a model of dynamics of progress of a population consisting of *n* subpopulations. The model is represented in the form of differential equations with transition coefficients within their right sides. The transition coefficients mirror the share of species getting from i^{th} subpopulation to j^{th} one. The proposed system is not Voltairian one since its phase trajectories may cross coordinate axes. It has been proved that the system of differential equations is degenerated in the neighbourhood of equilibrium points. Analysis of the system of differential equations for n=2 has demonstrated a potential for three bifurcations. It has been proved that nine bifurcation types are possible for n=3. Numerical computer-based experiments have shown that the proposed model is stable as for the disturbance of its coefficients, and the obtained characteristics of the degenerated system are close to real ones.

Keywords: mathematical model, computer-based simulation, differential model, logistic function, bifurcation characteristics

1 Introduction

Environment is one of the most information-intensive objects. Essentially, it is multi-component, and experiences constant pressure of human business activities.

Outbreak of deep global ecological changes caused the necessity to analyze their dynamics, evaluate it, and forecast to make possible decisions aimed at strategy generation for future community advance. Thus, the development of complex models of dynamics of natural-industrial processes, and the development of systems for managerial decision support on their basis is the upcoming tendency to develop information technology, and to implement it.

Anthropogenic pressure results in the worsening of the global environmental impact as well as in the increased number of pathologies of biological objects (i.e. immune retrogression, decreased reproductive function etc.). In the context of unstable environmental factors and mosaicism of areas of species, genetic inhomogeneity of species and certain populations increase significantly. Features of the situation should be involved while planning environmental protection measures, performing ecological monitoring, and solving problems concerning forecast of future of the populations. It is extremely important to study a level of genetic heterogeneity of people. Accumulation of pathologic recessive genes may be latent for a long time, and starting from a certain moment it may be manifested in the form of rapid growth of the number of definite hereditary diseases.

Use of mathematical models of dynamics of heterogenic populations as well as their application for computer-based simulation makes it possible to identify efficiently and with minimum time consumption the common tendencies for subpopulation progress, to forecast the system state as well as possible consequences of artificial intervention in the process [1-4]. Among other things, mathematical modeling helps forecast formation of genetic abnormalities.

In its classic view, the system of dynamics of arbitrarily isolated populations is represented as $\dot{x}_i = f_i\left(\overline{x}\right), i = \overline{1,n}$ where x_i is a size of i th population, and $f_i\left(\overline{x}\right)$ determines its reproductive potential. The paper proposes to consider a model of subpopulation dynamics in the form of $\dot{x}_i = \sum_{j=1}^{n} A_{ij} f_j\left(\overline{x}\right), i = \overline{1,n}$, i.e. when $f_j\left(\overline{x}\right)$, $j = \overline{1,n}$ function is a part of a right-hand member of an equation with some coefficient $A_{ij} \in \left[0;1\right]$ which will be defined below as a transition coefficient. Similar problems were considered in [5], and in [6] they were further developed.

2 Mathematical model of dynamics of homomorphic objects with logistic function as a basic one

1. The research is based upon the concept of a population as a set of species which can be divided conventionally into n subpopulations; from the genetic viewpoint, they are more or less homogenous while differing from each other significantly. They are not isolated reproductively; thus, there is certain degree of probability that inheritors of species from i^{th} subpopulation will get to j^{th} population. In general, differential model of the system may be expressed as follows:

$$\frac{dx_j}{dt} = \sum_{i=1}^{n} A_{ji} \cdot f_i(x), \qquad j = \overline{1, n} \tag{1}$$

where x_j is a size of j^{th} subpopulation, $f_i(x)$ is a function, describing general reproductive potential of i^{th} subpopulation, and A_{ji} is a share of inheritors of i^{th} population getting to j^{th} one. Assume that $\sum_j A_{ij} = 1.$ in terms of any i. $f_i(x)$ function mirrors the commonly known logistic law

$$f_i(x) = a_i \cdot \left(1 - \frac{1}{K}\sum_{l=1}^{n} x_l\right) x_i \tag{2}$$

where a_i mirrors reproductive potential of subpopulation with i index, and K is a capacity of the population survival areal. According to (1), (2), population growth nears zero when its size nears zero or when total size of all subpopulations nears maximum possible ecological capacity of K environment.

(1), (2) system is not Voltairian one to the extent that its trajectories may cross coordinate axes and, for instance, local behaviour of the system in the neighbourhood of the reference point depends upon its characteristics not only in the first quarter. Such results have found their application in other industries [7], as well as in such works as [8-11].

3 Equilibrium points of the system

To study equilibrium points of the system (1), (2) apply standard analysis on Lyapunov [9]. It is seen easily that zero point (i.e. reference point) is one of the equilibrium points. Moreover, there is also endless number of equilibrium points lying in the plane

$$\sum_i x_i = K\ . \tag{3}$$

Character of arrangement of equilibrium points is rather natural from ecological viewpoint. It is known that if there are no representatives of the certain species, they cannot originate from nothing. If subpopulations inhabit one and the same ecological niche consuming the same resources, their random distribution within the niche in terms of their sizes is equivalent.

Theorem 1: (1), (2) system is degenerated system in the neighbourhood of specific points of stationary hyperplane (3).

Proving: general view of ij^{th} component of Jacobian matrix J of (1), (2) system is $J_{ij} = A_{ij} a_j \left(1 - \frac{\sum_{k=1}^{n} X_k}{K} \right) - \frac{\sum_{p=1}^{n} A_{ip} a_p x_p}{K}$. According to (3), general view of ij^{th} component of Jacobian matrix J of (1), (2) system within planes of stationary hyperplane will be $J_{ij} = -\frac{1}{K} \sum_{k=1}^{n} A_{ik} \cdot a_k \cdot X_k$. Since J_{ij} does not depend on j directly, $J_{pi} = J_{hi}$, $p,h = \overline{1,n}$ is true; it follows that vectorial columns of Jacobian matrix are linearly dependent, $Det(J) = 0$; hence, the system is degenerated within specific points of stationary hyperplane $\sum_{i=1}^{n} x_i = K$.

The theorem has been proved.

If n=2, then the system is represented as:

$$\begin{cases} \frac{dx_1}{dt} = (\lambda_1 a_1 x_1 + (1-\lambda_2) a_2 x_2) \left(1 - \frac{x_1 + x_2}{K} \right) \\ \frac{dx_2}{dt} = (\lambda_2 a_2 x_2 + (1-\lambda_1) a_1 x_1) \left(1 - \frac{x_1 + x_2}{K} \right) \end{cases}. \tag{4}$$

Following symbols were introduced for (4) system: parameter $\lambda_i = A_{ii}$ mirrors a part of subpopulation i growth belonging to parental one according to its phenotypic characteristics. In terms of circulation system, parameters are $\sum_{i=1, i \neq j}^{n} A_{ij} = 1 - \lambda_i$, $j = \overline{1,n}$.

Subpopulation may vary in reproductive coefficients a_i, initial size, and in transition coefficient λ_i (i.e. certain share of growth of a subpopulation belonging to “parental” one according to its phenotypic characteristics).

Figures 1 and 2 demonstrate samples of phase patterns in the context of two-dimensional case for two sets of parameters. From ecological point of view, we are interested mainly in the first quarter where subpopulation sizes are positive.

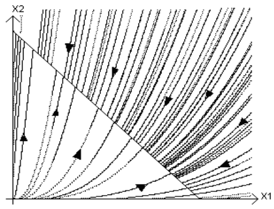


Fig. 1. Phase pattern of system (4) with following parameters: $\lambda_1 = 0.95; \lambda_2 = 0.95;$
$a_2 = 0.3; a_1 = 0.1.$

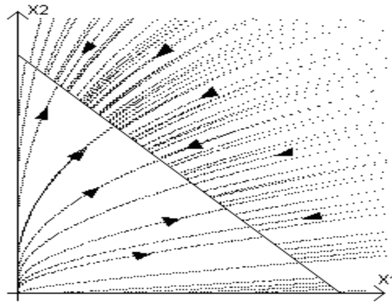


Fig. 2. Phase pattern of system (4) with following parameters:
$\lambda_1 = 0.95; \lambda_2 = 0.95;\ a_1 = 5; a_2 = 2.$

Taking into consideration the fact that we are interested in the first quarter, i.e. $x_1, x_1 \in [0, K]$, the system behaviuor near (3.3) line depends completely upon a_i indices: if $a_1 > 0$, $a_2 > 0$ then all points of the section are attracting; if $a_1 < 0$, $a_2 < 0$ they are repelling; if a_i have unlike signs, the section will consist of two parts – with attracting points, and with repelling ones.

Theorem 2: degenerated stationary line $x_1 + x_2 = K$ of (4) system consists of points of attracting ray (i.e. attractor) with $x_1 (a_2 - a_1)/K - a_2 < 0$ general view, and points of repelling ray (i.e. repeller) with $x_1 (a_2 - a_1)/K - a_2 > 0$ general view. Coordinates of a point, connecting rays of attractor and repeller are $\left(\frac{a_2}{a_2 - a_1} K; \frac{a_1}{a_1 - a_2} K \right)$; within the point, characteristic equation of Jacobian matrix has zero root of 2nd order.

Proving: according to theorem 1, system (4) is degenerated in the neighbourhood of points of stationary line $x_1 + x_2 = K$. Elements of Jacobian matrix system are:

$$J_{11} = \lambda_1 a_1 \left(1 - \frac{x_1 + x_2}{K} \right) - \frac{\lambda_1 a_1 x_1 + (1 - \lambda_2) a_2 x_2}{K};$$

$$J_{12} = (1-\lambda_2) a_2 \left(1 - \frac{x_1 + x_2}{K}\right) - \frac{\lambda_1 a_1 x_1 + (1-\lambda_2) a_2 x_2}{K};$$

$$J_{21} = (1-\lambda_1) a_1 \left(1 - \frac{x_1 + x_2}{K}\right) - \frac{(1-\lambda_1) a_1 x_1 + \lambda_2 a_2 x_2}{K};$$

$$J_{22} = \lambda_2 a_2 \left(1 - \frac{x_1 + x_2}{K}\right) - \frac{(1-\lambda_1) a_1 x_1 + \lambda_2 a_2 x_2}{K}.$$

Within points of stationary line $x_1 + x_2 = K$, Jacobian matrix is $J = \begin{pmatrix} -(\lambda_1 a_1 x_1 + (1-\lambda_2) a_2 x_2)/K & -(\lambda_1 a_1 x_1 + (1-\lambda_2) a_2 x_2)/K \\ -((1-\lambda_1) a_1 x_1 + \lambda_2 a_2 x_2)/K & -((1-\lambda_1) a_1 x_1 + \lambda_2 a_2 x_2)/K \end{pmatrix}$, and its characteristic equation is $t^2 + t(a_1 x_1 + a_2 x_2)/K = 0$ where due to degeneration of Jacobian matrix, $t_1 = 0$, and $t_2 = -(a_1 x_1 + a_2 x_2)/K$.

Within points of stationary line $x_1 + x_2 = K$, nonzero root of the characteristic equation can be represented as $t_2 = (a_2 - a_1) x_1/K - a_2$; then, if $(a_2 - a_1) x_1/K - a_2 < 0$ it is obvious that points of the stationary line are attracting, and if $(a_2 - a_1) x_1/K - a_2 > 0$, they are repelling. It is understood that the two rays will be combined within the point meeting $\begin{cases} (a_2 - a_1) x_1/K - a_2 = 0 \\ x_1 + x_2 = K \end{cases}$ condition also having $\left(\frac{a_2}{a_2 - a_1} K; \frac{a_1}{a_1 - a_2} K\right)$ coordinates. Within the point where attractor and repeller are combined, according to $x_1(a_2 - a_1)/K - a_2 = 0$ condition, root of the characteristic equation is $t_2 = 0$.

The theorem has been proved.

The system trajectory may be directed either to endlessness or to equilibrium points lying on a plane (3). In the latter case, finite point of the phase trajectory depends upon initial conditions. The dependence is not linear one; and while approaching attractor, trajectory density is not constant. In some specified sense, one can state that probability of getting to different points of the attractor is not similar.

4 Analysis of bifurcation characteristics of the system

Analysis of system (4) demonstrates possibility of three bifurcations. Suppose that equality of zero of a real part of no less than one of proper meaning of Jacobian matrix is the required bifurcation condition.

Theorem 3: system (4) is regenerated in the neighbourhood of singular point (i.e. reference point), if $a_1 = 0 \vee a_2 = 0 \vee \lambda_1 + \lambda_2 = 1 \vee \begin{cases} a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$.

Proving: Jacobian matrix of system (4) within singular point (i.e. reference point) is $J = \begin{pmatrix} \lambda_1 a_1 & (1-\lambda_2) a_2 \\ (1-\lambda_1) a_1 & \lambda_2 a_2 \end{pmatrix}$; its characteristic equation is $t^2 - Tr(J) t + Det(J) = 0$ where $Tr(J) = \lambda_1 a_1 + \lambda_2 a_2$, and $Det(J) = a_1 a_2 (\lambda_1 + \lambda_2 - 1)$. It is obvious that while fulfilling any of $\left[\begin{array}{l} a_1 = 0 \\ a_2 = 0 \\ \lambda_1 + \lambda_2 = 1 \end{array} \right.$ conditions, Jacobian is $Det(J) = 0$; thus, the coordinate system is degenerated in the neighbourhood of the reference point. If $\begin{cases} Det(J) = a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ Tr(J) = \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$, then reference point is elliptic point, and real part of the pair of complex roots of characteristic equation $Tr(J) = \lambda_1 a_1 + \lambda_2 a_2$ is equal to zero. Hence, the system is degenerated in the neighbourhood of reference point, if $\begin{cases} Det(J) = a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ Tr(J) = \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$.

The theorem has been proved.

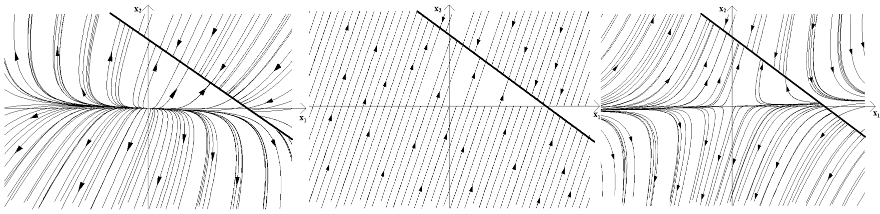
Theorem 4: in the neighbourhood of a reference point, system (4) is of a saddle type, if $a_1 a_2 (\lambda_1 + \lambda_2 - 1) < 0$. If $0 < a_1 a_2 (\lambda_1 + \lambda_2 - 1) < \frac{(\lambda_1 a_1 + \lambda_2 a_2)^2}{4}$, then reference point is a node; if $a_1 a_2 (\lambda_1 + \lambda_2 - 1) > \frac{(\lambda_1 a_1 + \lambda_2 a_2)^2}{4}$, then it is a focus. In the context of the two latter cases, point is stable if $\lambda_1 a_1 + \lambda_2 a_2 < 0$; it is unstable, if $\lambda_1 a_1 + \lambda_2 a_2 > 0$.

Proving: Jacobian matrix of system (4) is $J = \begin{pmatrix} \lambda_1 a_1 & (1-\lambda_2) a_2 \\ (1-\lambda_1) a_1 & \lambda_2 a_2 \end{pmatrix}$ at the origin, and its characteristic equations are $t^2 - (\lambda_1 a_1 + \lambda_2 a_2) t + a_1 a_2 (\lambda_1 + \lambda_2 - 1) = 0$. It is obvious, that if $Det(J) = a_1 a_2 (\lambda_1 + \lambda_2 - 1) < 0$, then discriminant of characteristic equation being $D(J) = Tr(J)^2 - 4Det(J)$ is more than zero, and the product of roots of the characteristic equation (according to Vieta theorem) is less than zero. Hence, the origin is less than zero and the origin is of a saddle type by definition. Therefore, consider that $Det(J) > 0$. Discriminant $D(J)$ of the characteristic equation is $D(J) = (\lambda_1 a_1 + \lambda_2 a_2)^2 - 4 a_1 a_2 (\lambda_1 + \lambda_2 - 1)$. It is understood that if

$a_1a_2(\lambda_1+\lambda_2-1) > \frac{(\lambda_1a_1+\lambda_2a_2)^2}{4}$, then singular point (i.e. reference point) is elliptic one and real part of a pair of complex roots has stable focus; if $\lambda_1a_1+\lambda_2a_2 > 0$, it is unstable. If $a_1a_2(\lambda_1+\lambda_2-1) < \frac{(\lambda_1a_1+\lambda_2a_2)^2}{4}$, then the singular point is hyperbolic one. According to the assumption, $Det(J) > 0$. Hence, roots of the characteristic equation (as a result of Vieta theorem) have similar sign; consequently, if $\lambda_1a_1+\lambda_2a_2 < 0$, then the origin has a type of a stable node; if $\lambda_1a_1+\lambda_2a_2 > 0$ it is of unstable node type. The theorem has been proved.

Below, the results, represented in theorems 3 and 4, are illustrated. Fig. 4 demonstrates phase patterns in the process of reproductive coefficient a_1 transition through zero value. Other parameters have been selected as follows:

$$\lambda_1 = 0,8;\ \ \lambda_2 = 0,8;\ \ a_2 = 5;\ \ K = 100 \tag{5}$$



a b c (5)

Fig. 3. Phase pattern of system (4) with parameters and **a)** $a_1 = 1$**, b)** $a_1 = 0$**, and c)** $a_1 = -1$**.**

In Fig. 3, the origin is unstable nodes and all points of $x_1 + x_2 = K$ line are attractive. Such behavior is typical for the system under standard conditions when two progressing subpopulations complement each other by a certain share of their inheritors while increasing system-wide population biomass. When overall size of the series achieves maximum acceptable edge *K*, the subpopulation growth comes to an end in the sense that the number of newborn species is equal to the number of died ones.

In the context of the degenerated case (Fig. 3, b) all the phase trajectories are straight ones, zero equilibrium point almost decays and line (3) stays to be attractor. The case is realistic from the practical point of view since when subpopulation one has zero coefficient of reproductive coefficient, its species are available owing to the growth of subpopulation two. In this context, certain share of species of subpopulation x_2 belonging to parental one according to their phenotypical characteristics, is λ_2 and share of inheritors will belong to x_1; thus, $(1-\lambda_2)$. It is obvious that in such a case, ratio of size of two subpopulations will be stable, i.e. phase trajectories will be straight lines.

In the case represented in Fig. 3, c, saddle is a reference point. Certain share of points within a line (3) are repelling; certain share of trajectories tend to infinity despite the fact that in the neighborhood of the reference point, where realistic values of initial size of the subpopulations, the system remains unstable as before. Such a "scenario" for the system development is widely used under the real conditions since in the context of one system degradation it can support sufficient level of its size owing to the subpopulation being developed; when total biomass achieves its critical value, the former can preserve the ratio.

Fig. 4 represents another bifurcation case, when λ_1 parameter goes through critical value 0.8. Other system parameters were selected as follows:

$$\lambda_2 = 0.2; \; a_1 = -1; \; a_2 = 1.5; \; K = 100 \tag{6}$$

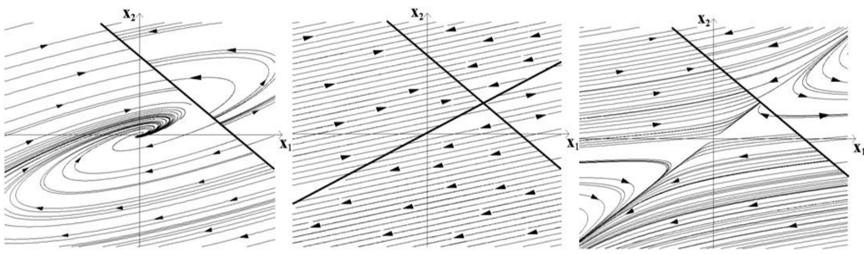


Fig. 4. Phase pattern of system (4) with parameters (6) and **a)** $\lambda_1 = 0.7$ **; b)** $\lambda_1 = 0.8$ **; c)** $\lambda_1 = 0.9$ **.**

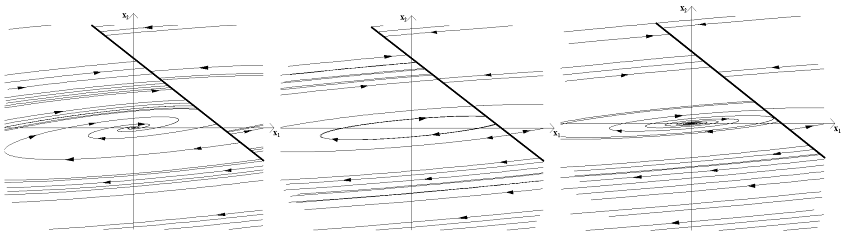
The case is not realistic from biological point of view but it is interesting mathematically. In terms of the selected coefficients, subpopulation one is at a disadvantage: its reproductive coefficient is negative and it exists owing to subpopulation two producing mainly species of subpopulation one. In Fig. 4, reference point is a stable focus; if the initial number of species is insufficient, both subpopulations die out. In the context of other initial conditions, the trajectories tend to the upper part of a line (3). Fig. 4, b represents transition degenerated case when two lines (i.e. attractor and repeller) are available. Boundary trajectory points are focused on two half-lines and certain trajectories tend to infinity. After subsequent magnification of λ_1 in Fig. 4, it becomes obvious that reference point becomes a saddle. The majority of trajectories tend to the upper part of a line (3); moreover, their density is maximal within lower part of the half-line. As before, in the context of sufficiently large size of population two, trajectories may tend to infinity.

In addition to the considered cases, another bifurcation when $(\lambda_1 + \lambda_2 - 1)a_1 a_2 > 0$ may take place. In this context, critical values of parameters are determined using the equation

$$\lambda_1 a_1 + \lambda_2 a_2 = 0 \tag{7}$$

It is understood that in the neighborhood of point (7), discriminant of characteristic equation is negative, i.e. singular point is a focus. In this context, bifurcation is a change of a stable focus for unstable one or vice versa. Fig. 5 shows such a bifurcation when condition (7) is fulfilled and discriminant of characteristic equation is negative:

$$\lambda_1 = 0.8; \;\; a_1 = -0.25; \;\; \lambda_2 = 0.1; \;\; K = 100 \tag{8}$$



a b c

Fig. 5. Phase pattern of system (4) with parameters (8) and **a)** $a_2 = 1$**; b)** $a_2 = 2$**; c)** $a_2 = 3$**.**

When reference point is a stable focus, in terms of small x_2 the system cannot achieve its equilibrium within a line (3), sizes of the both subpopulations tend to zero. Taking into consideration the fact that in the context of ecological system, x_i coordinates cannot have negative values, the case of unstable focus cannot be considered as essentially different though area of initial values x_2 , in terms of which the system is in equilibrium, is quite broader.

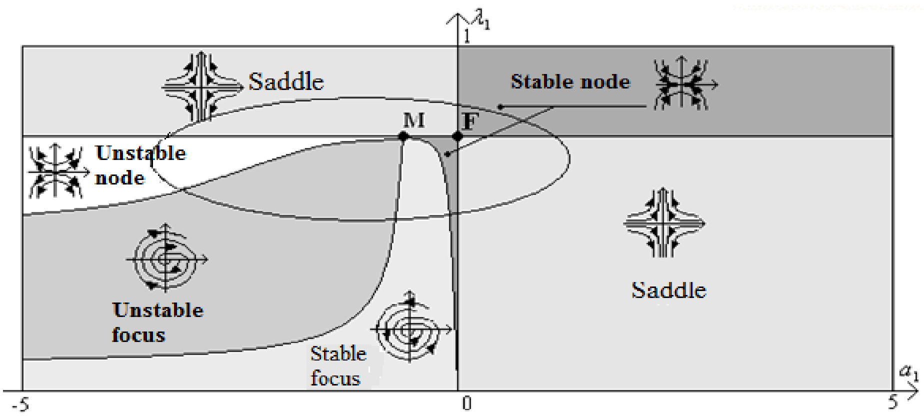


Fig. 6. Bifurcation diagram of system (4) in terms of fixed λ_2 =0.1 and a_2 =2

Bifurcation diagram represents general information concerning potential bifurcations of system (4) (Fig. 6). As it has been demonstrated (Fig. 3-5), bifurcations of *node-saddle, focus-saddle, and stable focus-unstable focus* types are observed within the system. Fig. 7 demonstrates a segment of bifurcation diagram (Fig. 6); namely, it is the segment where density of bifurcational curves is maximal.

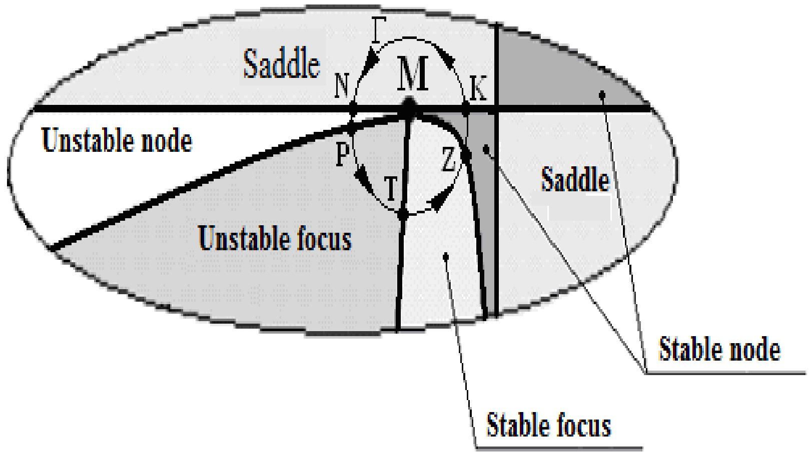


Fig. 7. Segment of bifurcation diagram demonstrated in Fig. 6.

As it is seen, degenerated point *M*, within which bifurcation of type three originates, is available within parametric space of system (4). Bifurcation diagram (Fig. 6) proves graphically the results, represented in theorems 5 and 6.

Features of such a point are as follows: ordinary system bifurcations are possible within the point as well as such more complicates transitions as “stable focus-unstable node”, “stable focus-unstable focus”,“focus-saddle”, “unstable focus-stable node”, “node-centre”, and “centre-saddle”.

Specify full-circle trajectory Γ within certain neighbourhood of point *M* in bifurcation diagram represented by Fig. 7. The trajectory, just as a part of the bifurcation diagram in the neighbourhood of point *M*, is shown schematically. Actually, radius of trajectory Γ is 0.01. Trajectory Γ passes through all segments of the bifurcation diagram; thus, the system behaviour in the neighbourhood of point M is quite sufficient to describe in full every possible state graph of variations of proper values as well as a discriminant of the characteristic equation of system (4) while anticlockwise move is taking place along the trajectory Γ. For definiteness, locate K, N, P, T, and Z points within the diagram.

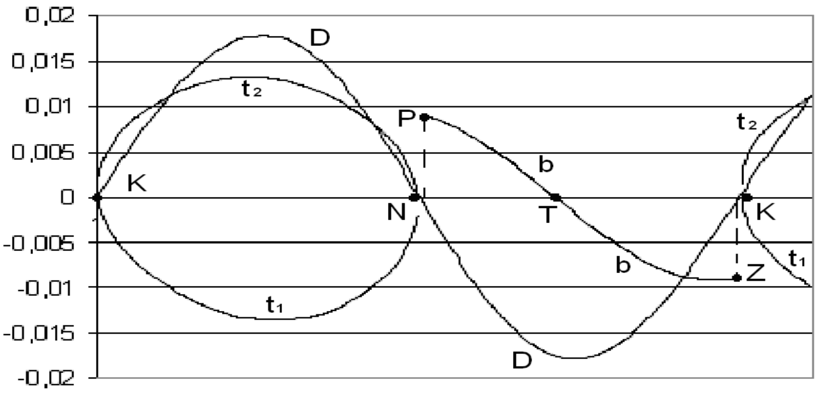


Fig. 8. Dynamics of characteristic numbers and Jacobian matrix discriminant of the system (4) in the context of movement along trajectory Γ (Fig. 7).

For the diagram in Fig. 8 following symbols were used: D - discriminant of the characteristic equation; t_1, and t_2 - proper numbers of Jacobian matrix of the system (4); b - real part of a pair of complex proper values of Jacobian matrix of system (4). Discretization interval applied for the diagram is 0.01.

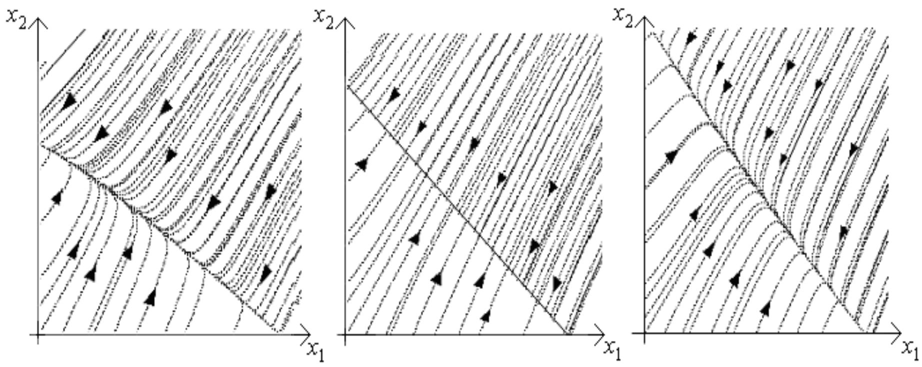
Every of K, N, P, T, Z points of trajectory Γ of the bifurcation diagram shown in Fig. 7 is separate bifurcation. KN arch of trajectory Γ is within the part of parametric field, where topology of phase space on the neighbourhood of singular point is a saddle, which corresponds to a positive value of a discriminant of characteristic value as well as to negative production of proper values of the characteristic matrix. NP arch corresponds to a type of a singular point (i.e. reference point) *unstable node* depending upon the variation from negative proper parameter to positive one. Bifurcation transition through point P is stipulated by the transition of proper values of Jacobian matrix to imaginary plane. Analytical condition for such a bifurcation origin is $Tr^2(J) - 4Det(J) = 0$. Discriminant of characteristic equation takes negative values when proper values are found out within imaginary axis. In this context, the graph mirrors only the real part of a pair of the complex characteristic numbers of Jacobian matrix being $b = -Tr(J)$. $Tr(J) = 0$ condition is fulfilled within point T of the system (4) stipulating loss of stability of elliptic equilibrium point as well as variation of the phase space topology in the neighbourhood of the singular point (i.e. reference point) from stable focus to unstable one.

System (4) is degenerated system in the sense that it involves continuous set of equilibrium points of measurability 1. Generally, such systems are unstable structurally. To analyze characteristics of the system from the point of view, it is possible to introduce to it additional parameter ΔK which reflects certain difference in ecological capacity for the two subpopulations. Express system (4) for two-dimensional case in such a way:

$$\begin{cases} \dfrac{dx_1}{dt} = \lambda_1 \cdot a_1 \cdot \left(1 - \dfrac{x_1 + x_2}{K}\right) \cdot X_1 + \left(1 - \lambda_2\right) \cdot a_2 \cdot \left(1 - \dfrac{x_1 + x_2}{K + \Delta K}\right) \cdot x_2 \\ \dfrac{dx_2}{dt} = \lambda_2 \cdot a_2 \cdot \left(1 - \dfrac{x_1 + x_2}{K + \Delta K}\right) \cdot X_2 + \left(1 - \lambda_1\right) \cdot a_1 \cdot \left(1 - \dfrac{x_1 + x_2}{K}\right) \cdot x_1 \end{cases} \tag{9}$$

Fig. 9 shows phase patterns of system (9) in terms of different values of coefficient ΔK ; other parameters were selected as follows:

$$\lambda_1 = 0.3; \quad a_1 = 0.5; \quad \lambda_2 = 0.5; \quad a_2 = 0.3; \quad K = 100. \tag{10}$$



a b c (5)

Fig. 9. Phase pattern of the system (4) with parameters (10), and **a)** $\Delta K = -30$**, b)** $\Delta K = 0$**, c)** $\Delta K = 30$**.**

Computer simulation demonstrates significant stability of the system. As it is seen from the figures, even in the context of substantial disturbance of *ΔK* nature of the trajectories does not experience essential changes within the major part of the phase space. Topology variation, taking place objectively, is seen in the slow motion of the system along a line (3) after its arriving towards (*K*,*0*) point or (*0*,*K*) point.

Another disturbance type is connected with discretization interval effect being always available in the context of computer simulation. Actually, discrete model form is more adequate to the reality since population size cannot vary continuously. In the context of the degenerated systems, discretization may result in the breakdown trajectories and the system behaviour may experience qualitative variations. System (4) demonstrates stability to such disturbances as well. In terms of rather high coefficients of reproductive functions, cases are possible when phase trajectories pass attractor (3); generally, the trajectory returns to its stationary line during the next interval. Fig. 10 shows a fragment of a line (3) and trajectories with sufficiently large discretization interval in its neighbourhood.

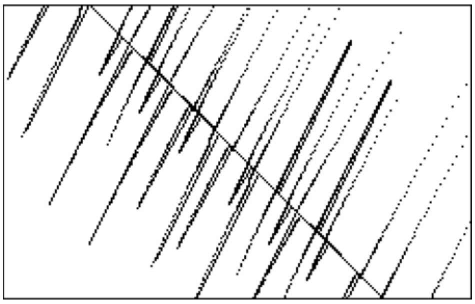


Fig. 10. Trajectories of the system (4) with Δt = 5 interval in the neighbourhood of attractor (3).

5 Bifurcations of comeasurability >1 system

Theorem 5: in the neighbourhood of the singular point (i.e. reference point), system (4) may have only three bifurcations of comeasurability 2 arising if $\begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases} \cup \begin{cases} a_1 = 0 \\ \lambda_1 + \lambda_2 = 1 \end{cases} \cup \begin{cases} a_2 = 0 \\ \lambda_1 + \lambda_2 = 1 \end{cases}$.

Proving: according to the results of theorem 3, system (4) is degenerated in the neighbourhood of a reference point, if $a_1 = 0 \cup a_2 = 0 \cup \lambda_1 + \lambda_2 = 1 \cup \begin{cases} a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$. It is obvious that in the context general type of bifurcations of comeasurability 2 of (3.4) system should be $C_4^2 = 6$, namely

$$\begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}, \begin{cases} a_1 = 0 \\ \lambda_1 + \lambda_2 = 1 \end{cases}, \begin{cases} a_2 = 0 \\ \lambda_1 + \lambda_2 = 1 \end{cases} \quad (11)$$

$$\begin{cases} a_1 = 0 \\ a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}, \begin{cases} a_1 = 0 \\ a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}, \begin{cases} \lambda_1 + \lambda_2 = 1 \\ a_1 a_2 (\lambda_1 + \lambda_2 - 1) > 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}. \quad (12)$$

As it is understood, zero-dimensional set is the solution for systems (12); thus, only three bifurcations (11) of comeasurement 2 of system (4) within the singular point (i.e. reference point). The theorem has been proved.

Theorem 6: in the neighbourhood of the singular point (i.e. reference point), system (4) experiences the only bifurcation of comeasurement 3 arising, if $\begin{cases} a_1 a_2 (\lambda_1 + \lambda_2 - 1) = 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$.

Proving: topological structure of a phase pattern in the neighbourhood of a reference point of the system (4) may be in such nonequivalent states as “stable node”, “unstable node”, “saddle”, “stable focus”, “unstable focus”. It is commonly supposed that “node” and “focus” are equivalent topologically. We emphasize that in this context, two-dimensional set “saddle” is transitional state between “stable node” and “unstable node”; one-dimensional set “centre” is transitional state between “stable focus” and “unstable focus”. Relying upon the abovementioned, differ conditionally topological structures of a phase pattern of “node” type and “focus” type. Analytical condition to transfer from “stable node” to “stable focus”, and from “unstable node” to “unstable focus” zero equality of a discriminant of characteristic equation of the system (4) being $D(J) = Tr^2(J) - 4Det(J) = (\lambda_1 a_1 + \lambda_2 a_2)^2 - 4a_1 a_2(\lambda_1 + \lambda_2 - 1)$. Basing upon general view of determinant of characteristic matrix $Det(J) = a_1 a_2(\lambda_1 + \lambda_2 - 1)$ as well as real part of complex roots of a characteristic equation $Tr(J) = \lambda_1 a_1 + \lambda_2 a_2$, we can see that if the conditions of the theorem $\begin{cases} Det(J) = 0 \\ Tr(J) = 0 \end{cases}$ are fulfilled, then discriminant of the characteristic equation is equal to zero; hence, if $\begin{cases} a_1 a_2(\lambda_1 + \lambda_2 - 1) = 0 \\ \lambda_1 a_1 + \lambda_2 a_2 = 0 \end{cases}$ then system (3.4) will experience bifurcation of comeasurement 3 at the reference point. The theorem has been proved.

6 Conclusions

The paper represents results of analysis of mathematical model of dynamics of heterogeneous groups with logistic function as a basic one for $n = 2$ cases. Such a model is not Voltairian one to compare with classic model; i.e. its trajectories may cross coordinate axes. Moreover, it depends heavily on transition coefficients proposed by the paper. Analysis of a model of metapopulation dynamics, including several subgroups competiting for common resource, has helped demonstrate rather diverse potential system dynamics. Notwithstanding the degenerated, in some specified sense, nature of the model, its dynamics is not trivial since in the context of variation of the system parameters, three bifurcation types are possible if $n = 2$ and nine bifurcation types are possible if $n = 3$. As the numerical experiments have shown, the system is stable sufficiently stable in relation to its coefficient disturbance; and characteristics of nondegenerated systems, which can be obtained in such a way, are close to the system under analysis. However, in the case of stability of nontrivial equilibrium, finite state of the system depends upon initial conditions and it cannot be considered as absolutely random one. Following of the system trajectories towards equilibrium has certain regularities which will be described in detail during analysis of bifurcation characteristics of the system.

Models of metapopulation dynamics and regularities of directivity of the system trajectories to equilibrium may be used by IT while developing systems to support managerial decision making.

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