Maximum entropy and principle of least action for electrotechnical systems in deterministic chaos mode

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Abstract

Entropy and its maximization are determined by different possible trajectories of a chaotic system moving in its phase space between two cells. The chaotic system, within the framework of this article, is understood as an electrotechnical system with deterministic chaos modes. The paths of the chaotic system in phase space are supposed to be differentiated by their actions, by the so-called principle of least action. It is shown that the maximization of entropy leads to the trajectory selection probability distribution as an action function from which one can easily obtain the probability of the electrotechnical system transition from one state to another. Of interest is the fact that the most probable trajectories are the paths of least action. This suggests that the principle of least action in a probabilistic situation is equivalent to the principle of maximum entropy or uncertainty associated with a particular probability distribution.

1 Introduction

The objective of this paper is to investigate distributions of probabilities attributed to different trajectories of a chaotic system moving between two points in phase space. The phase space of the system is usually defined so that the point in the system represents the system state. If the system has N-bodies and moves in a three-dimensional normal configuration space, the phase space has a dimension of $6N$ ($3N$-coordinates and $3N$-pulses) [1].

Let us now consider a non-equilibrium electrotechnical system moving in the phase space between two points a and b, which are located in two elementary cells of this phase space partition. If the movement of the electrical system is regular or if the n-dimensional state space has a positive or zero Riemannian curvature, there is only one possible trajectory between the two points. In the other case, there is only some unambiguous set of trajectories that track each other between the initial and final cells. These trajectories should be the ways of minimizing

In: Sergei S. Goncharov, Yuri G. Evtushenko (eds.): Proceedings of the Workshop on Applied Mathematics and Fundamental Computer Science 2020 , Omsk, Russia, 30-04-2020, published at http://ceur-ws.org

the action due to the principle of least action [2] and have some probability of occurrence. Any other set of trajectories must have a zero probability.

For an electrical system in chaotic motion, or when the Riemannian curvature of the phase space is negative, everything is different. Two points indistinguishable in the starting cell can be exponentially separated. Usually, these two points never meet in the final cell in the phase space after they leave the starting cell. However, they can pass through the same cell at two different points in time. Therefore, between two set points there can be multiple possible trajectories k ($k = 1, 2, ..., w$) with varying travel time $t_{ab}(k)$ through the electrotechnical system and varying probability $p_{ab}(k)$ of its choosing path k. This is called a trajectory probability distribution.

In this paper, the trajectory probability distribution due to dynamic instability is studied in terms of entropic instability theory and the principle of least action. First of all, we suppose that different trajectories of non-equilibrium electrotechnical systems moving between phase cells a and b are unambiguously differentiated by their action defined [3] as follows

$$A_{ab}(k) = \int_{t_{ab}(k)} L_k(t)dt, \tag{1}$$

where $L_k(t)$ is Lagrangian systems at time moment t on path k and is determined as $L_k(t) = U_k(t) - V_k(t)$, where $U_k(t)$ is total kinetic energy and $V_k(t)$ is the total potential energy of the electrotechnical system.

Integral $A_{ab}(k)$ is determined by path k at time $t_{ab}(k)$; $t_{ab}(k)$ is the time of system L_k traveling by path k. If paths k can be identified only by the value of their actions, then it is possible to study their probability distributions due to Jaynes entropic concept and maximum entropy method [4] taking into account the value of action $A_{ab}(k)$. This approach leads us to a probabilistic interpretation of Maupertuis's mechanical principle and probability distribution depending on the action.

2 Trajectory entropy

The entropy referred to is our lack of knowledge of the system in question. The more we know about the system, the less the entropy. According to Shannon [3], this entropy can be measured by formula $S = -\sum_i p_i \ln\ p_i$, where p_i is a certain probability attributed to situation i. We normalize $\sum_i p_i = 1$ as usual with a summation over all possible situations.

Now for an ensemble of w possible paths, Shannon's entropy can be defined as follows:

$$H(a, b) = -\sum_{k=1}^{w} p_{ab}(k) \ln\ p_{ab}(k). \tag{2}$$

Function $H(a, b)$ is the *entropy of path* and must be interpreted as the missing information needed to predict which path from a to b the system chooses from the ensemble. According to our initial assumption, the value that differentiates paths and their occurrence probability is a Lagrangian action.

3 Probability distribution of maximum entropy

An ensemble containing a large number of systems moving from a to b is considered. These systems are distributed among w paths according to $p_{ab}(k)$ in view of action $A_{ab}(k)$. The mathematical expectation of action on all possible paths can be calculated by means of

$$M(A_{ab}) = \sum_{k=1}^{w} p_{ab}(k)A_{ab}(k). \tag{3}$$

On the other hand, entropy $H(a, b)$ of path in formulae (2) is a convex function depending on the normed probability $p_{ab}(k)$. Due to Jaynes principle [4], to obtain an optimal distribution, $H(a, b)$ is to be maximized under the constrains imposed by our limited knowledge of the system and corresponding variables, i.e. with normalization $p_{ab}(k)$ and mathematical expectation A_{ab}

$$\delta \left[-H(a, b) + \alpha \sum_{k=1}^{w} p_{ab}(k) + \eta \sum_{k=1}^{w} p_{ab}(k)A_{ab}(k) \right] = 0. \tag{4}$$

This results in the following probability distribution

$$p_{ab}(k) = \frac{1}{Q} \exp\left[-\eta A_{ab}(k)\right]. \tag{5}$$

Putting this probability distribution (5) into $H(a,b)$ of ratio (2), we get

$$H(a,b) = \ln\ Q + \eta A_{ab} = \ln\ Q - \eta \frac{\partial}{\partial \eta} \ln\ Q, \tag{6}$$

where Q is determined as $Q = \sum_{k=1}^{w} \exp\left[-\eta A_{ab}(k)\right]$ and A_{ab} is determined by the expression

$$A_{ab} = -\eta \frac{\partial}{\partial \eta} \ln\ Q, \tag{7}$$

η is a Lagrange multiplier.

4 Stability of trajectory probability distribution

Now let us show that the specified probability distribution is stable concerning the action fluctuations. Suppose that each path is cut into two parts: part 1 (segments on the cell side a) and part 2 (segments on the side b). All segments of part 1 are contained in group 1 and all segments of part 2 are in group 2. Each group has trajectory entropy $H_1 = H_2 = H$ and mean action $A_1 = A_2 = A$. The total entropy is $H(a,b) = H_1 + H_2 = 2$ and total mean action is $A(a,b) = A_1 + A_2 = 2A$. Now consider a small variation in the division of the trajectories with virtual changes in the two groups, such that $\delta A_1 = \delta A = -\delta A_2$. As a result, the total entropy will be changed and can be written as

$$H'(a,b) = H(A + \delta A) + H(A - \delta A). \tag{8}$$

Because distribution (5) and ratio (6) result from the procedure of entropy maximization, the stability condition requires that entropy does not increase with virtual changes of these two groups:

$$\delta H = H'(a,b) - H(a,b) \leq 0, \tag{9}$$

i.e.

$$H(A + \delta A) + H(A - \delta A) - 2H(A) \leq 0, \tag{10}$$

which means

$$\frac{\partial^2 H}{\partial A^2} \leq 0. \tag{11}$$

Let us consider whether this condition of entropy stability is always fulfilled. As follows from equation (6), $\partial^2 H/\partial A^2 = \partial \eta/\partial A$. Then, given the definition of mean action (3), we calculate

$$\frac{\partial A}{\partial \eta} = -\delta^2, \tag{12}$$

which implies

$$\frac{\partial^2 H}{\partial A^2} = -\frac{1}{\delta^2} \leq 0, \tag{13}$$

where dispersion $\delta^2 = A_1^{-2} + A_2^{-2} \geq 0$ characterizes the fluctuation of action A.

This proves the stability of the maximum entropy distribution and ratio (5) relative to the action fluctuations on different trajectories.

5 Principle of maximum entropy and principle of least action

Now let us observe the connection between maximum trajectory entropy and the least action. It can be shown that the paths of least action are the most likely at $\eta = \partial H(a,b)/\partial A_{a,b} > 0$. In fact, due to expression (5), positive ? means that the trajectories of least action are statistically more likely than trajectories of most action. Thus, the most likely trajectories should minimize the action.

This property of probability distribution of expression (5) can be mathematically analyzed in the same way as the stability of probability distribution proved in section 4. Two groups 1 and 2 were considered before for the segments of the path with $H_1 = H_2 =$ and $_1 = _2 = A$. The total entropy is $H(a,b) = 2H$, and total mean action is $A(a,b) = 2A$. Now suppose that two groups are deformed so that $\delta H_1 = \delta H = -\delta H_2$. The total mean action after group deformation can be written as [6].

$$A'(a,b) = A_1(H_1 + \delta H_1) + A_2(H_2 + \delta H_2) = A(H + \delta H) + A(H - \delta H). \tag{14}$$

If the probability distribution of expression (5) and ratio (6) correspond to the least action, the total mean action after group deformation cannot decrease, $\delta A = A'(a, b) - A(a, b) \geq 0$, i.e.

$$A(H + \delta H) + A(H - \delta H) - 2A(H) \geq 0, \tag{15}$$

which means

$$\frac{\partial^2 A}{\partial H^2} \geq 0. \tag{16}$$

On the other hand, by means of ratio (6) we can prove that

$$\frac{\partial^2 A}{\partial H^2} = -\frac{1}{\eta^2}\frac{\partial \eta}{\partial H} = -\frac{1}{\eta^3}\frac{\partial \eta}{\partial A}. \tag{17}$$

Now in terms of $\partial\eta/\partial A = -1/\delta^2$, we get

$$\frac{\partial^2 A}{\partial H^2} = \frac{1}{\delta^2 \eta^3}. \tag{18}$$

It is seen that if equation (18) is true, we get

$$\eta \geq 0. \tag{19}$$

In other words, the positive value of η implies that the principle of entropy maximization is closely related to the principle of least action: *the most probable trajectories determined by the maximum entropy probability distribution are simply the paths of least action.*

6 Closing remarks

This work can contribute to studying the behaviour of chaotic systems. The more chaotic the system under consideration is, the more possible paths with different actions exist, and the greater the entropy is. Thus, it is assumed that the entropy of path $H(a, b)$ can be used as a chaos measure, as Kolmogorov-Sinai's entropy [7]. This is a very encouraging result for the method of entropy and action integrals calculation in discontinuous and undifferentiated space (e.g. strange attractors). The result of this work can be used to derive the method of the maximum entropy change for dynamic systems moving in fractal phase space [8].

To sum up, it can be stated that the entropy of trajectories is determined for many possible paths of chaotic systems moving between two cells in phase space. It is shown that different paths are physically identified by their actions, and the maximization of path entropy leads to the distribution of trajectory selection probability as a function of the action. In this case, we show that the most probable paths obtained from the maximum entropy probability distribution minimize the action. This indicates that the principle of least action in a probabilistic situation is equivalent to the principle of entropy or uncertainty maximization, associated with the probability distribution. This result can be considered as an argument to support this method of analysis for non-equilibrium systems.

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