# Inference Methods for Mamdani-Type Systems Based on Fuzzy Truth Value

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**Abstract**

The article introduces inference methods for Mamdani-type fuzzy systems, which can be implemented with polynomial computational complexity for any t-norms and multiple fuzzy inputs. Center average and center of gravity defuzzification methods were used for case of multiple rules in rule base. Network architectures of systems corresponding to inference methods introduced in the article are provided.

## 1. Introduction

Mamdani's approach addresses the question of interpretation of the expression "if $X$ is $A$ then $Y$ is $B$", where $X$ and $Y$ are linguistic variables, $A$ and $B$ are linguistic values of $X$ and $Y$ respectively. The source of uncertainty consists in the fact that "if $X$ is $A$ then $Y$ is $B$" can be interpreted in two different ways. First, the most obvious way is to consider this expression as "$X$ is $A$ and $Y$ is $B$", or as $(x, y)$ is $A \times B$, where $A \times B$ is a Cartesian product of fuzzy sets $A$ and $B$. Hence, with this interpretation "if $X$ is $A$ then $Y$ is $B$" is a joint constraint on $X$ and $Y$. An alternative way consists in understanding "if $X$ is $A$ then $Y$ is $B$" as a conditional constraint or, equivalently, an implication. Many different implications are known. This way was considered in [Mik18] for systems with multiple inputs. The subject of this article is the development of Mamdani's approach.

For systems with multiple fuzzy inputs, which represent a formalization of terms of linguistic variables or inaccurate measurements, inference methods based on max-min and max-product composition operations are known [Rut10]. Operators min (taking minimum) and product (arithmetical product) are t-norms [Als06] that correspond to Mamdani's [Mam74] and Larsen's [Lar80] inference rules respectively. But for other t-norms, replacement of which can be necessary for learning of fuzzy systems, implementation of inference for multiple fuzzy inputs with polynomial computational complexity is impossible. In this article, methods that solve this problem are considered.

The statement of the problem and estimation of complexity of fuzzy inference is made in section 2. In section 3, an inference method using a measure of possibility for each input of a multiple-input system is considered. Section 4 introduces an inference method based on fuzzy

*Russian Advances in Artificial Intelligence: selected contributions to the Russian Conference on Artificial intelligence (RCAI 2020), October 10-16, 2020, Moscow, Russia*

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truth value. Sections 5 and 6 consider inference for a rule base with use of center average and center of gravity methods respectively.

## 2. Statement of the problem

A linguistic model is represented by a fuzzy rule base $R_k,\ k = \overline{1, N}$ of the form:

$$R_k \ :\ \text{If } x_1 \text{ is } A_{1k} \text{ and } x_2 \text{ is } A_{2k} \text{ and } \dots \text{ and } x_n \text{ is } A_{nk} \text{ then } y \text{ is } B_k, \tag{1}$$

where $N$ is the number of fuzzy rules, $A_{ik} \subseteq X_i$, $i = \overline{1, n}$, $B_k \subseteq Y$ are fuzzy sets, defined by membership functions $\mu_{A_{ik}}(x_i)$ and $\mu_{B_k}(y)$ respectively; $x_1,\ x_2,\ \dots,\ x_n$ are input variables of the linguistic model, while

$$[x_1,\ x_2,\ \dots,\ x_n]^{\mathrm{T}} = \boldsymbol{x} \in X_1 \times X_2 \times \dots \times X_n.$$

Symbols $X_i,\ i = \overline{1, n}$ and $Y$ stand for input and output variables spaces respectively.

Let us denote $\boldsymbol{X} = X_1 \times X_2 \times \dots \times X_n$ and $\boldsymbol{A_k} = A_{1k} \times A_{2k} \times \dots \times A_{nk}$, whereas

$$\mu_{\boldsymbol{A_k}}(\boldsymbol{x}) = \underset{i=\overline{1,n}}{\mathrm{T}_1}\, \mu_{A_{ik}}(x_i), \tag{2}$$

where $\mathrm{T}_1$ is an arbitrary t-norm, then rule (1) can be represented in the form of fuzzy implication

$$R_k \ :\ \boldsymbol{A_k} \longrightarrow B_k, \quad k = \overline{1, N}.$$

Rule $R_k$ can be formalized as a fuzzy relation, defined over set $\boldsymbol{X} \times Y$, i.e. $R_k \subseteq \boldsymbol{X} \times Y$ is a fuzzy set with membership function

$$\mu_{R_k}(\boldsymbol{x},\ y) = \mu_{\boldsymbol{A_k} \to B_k}(\boldsymbol{x},\ y).$$

Mamdani's model defines the function $\mu_{\boldsymbol{A_k} \to B_k}(\boldsymbol{x},\ y)$ based on known membership functions $\mu_{\boldsymbol{A_k}}(\boldsymbol{x})$ and $\mu_{B_k}(y)$ as follows [Rut10, Peg09]:

$$\mu_{\boldsymbol{A_k} \to B_k}(\boldsymbol{x},\ y) = \mathrm{T}_2\big(\mu_{\boldsymbol{A_k}}(\boldsymbol{x}),\ \mu_{B_k}(y)\big) = \mu_{\boldsymbol{A_k}}(\boldsymbol{x}) \overset{\mathrm{T}_2}{*} \mu_{B_k}(y),$$

where $\mathrm{T}_2$ is an arbitrary t-norm.

The problem consists in defining fuzzy inference $B'_k \subseteq Y$ for a system, represented in the form (1), if the inputs are assigned fuzzy sets $\boldsymbol{A'} = A'_1 \times A'_2 \times \dots \times A'_n \subseteq \boldsymbol{X}$ or "$x_1$ is $A'_1$ and $x_2$ is $A'_2$ and ... and $x_n$ is $A'_n$" with the corresponding membership function $\mu_{\boldsymbol{A'}}(\boldsymbol{x})$, which is defined as

$$\mu_{\boldsymbol{A'}}(\boldsymbol{x}) = \underset{i=\overline{1,n}}{\mathrm{T}_3}\, \mu_{A'_i}(x_i), \tag{3}$$

where $\mathrm{T}_3$ is an arbitrary t-norm.

According to *fuzzy modus ponens* rule [Zad73], fuzzy set $B'_k$ is defined by the composition of fuzzy set $A'$ and relation $R_k$, i.e.

$$B'_k = \boldsymbol{A'} \circ (\boldsymbol{A_k} \longrightarrow B_k),$$

or, at the level of membership functions,

$$\mu_{B'_k}(y) = \sup_{x \in X} \left\{ \mu_{\boldsymbol{A'}}(\boldsymbol{x}) \overset{\mathrm{T}_4}{*} \left( \mu_{\boldsymbol{A_k}}(\boldsymbol{x}) \overset{\mathrm{T}_2}{*} \mu_{B_k}(y) \right) \right\}, \tag{4}$$

where $\mathrm{T}_4$ is an arbitrary t-norm. Computational complexity of expression (4) equals $O(|\boldsymbol{X}| \times |Y|)$.

## 3. Inference method based on possibility measure for each input

Let us consider the inference (4) when

$$\mathrm{T}_1 = \mathrm{T}_2 = \mathrm{T}_3 = \mathrm{T}_4 = \mathrm{T}, \tag{5}$$

then

$$\mu_{B'_k}(y) = \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \mu_{\boldsymbol{A}'}(\boldsymbol{x}) \overset{\mathrm{T}}{*} \left(\mu_{\boldsymbol{A}_k}(\boldsymbol{x}) \overset{\mathrm{T}}{*} \mu_{B_k}(y)\right) \right\}. \tag{6}$$

Due to associativity of t-norms, the expression (6) can be transformed into

$$\mu_{B'_k}(y) = \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \mu_{\boldsymbol{A}'}(\boldsymbol{x}) \overset{\mathrm{T}}{*} \mu_{\boldsymbol{A}_k}(\boldsymbol{x}) \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y). \tag{7}$$

Using (2) and (3) we can further transform (7):

$$\mu_{B'_k}(y) = \sup_{\substack{x_1\in X_1\\ x_2\in X_2\\ \dots\\ x_n\in X_n}} \left\{ \mu_{A'_1}(x_1) \overset{\mathrm{T}}{*} \mu_{A'_2}(x_2) \overset{\mathrm{T}}{*} \dots \overset{\mathrm{T}}{*} \mu_{A'_n}(x_n) \overset{\mathrm{T}}{*} \mu_{A_{1k}}(x_1) \overset{\mathrm{T}}{*} \mu_{A_{2k}}(x_2) \overset{\mathrm{T}}{*} \dots \overset{\mathrm{T}}{*} \mu_{A_{nk}}(x_n) \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y).$$

Associativity and commutativity of t-norms enables us to rearrange $\mu_{A'_i}(x_i)$ and $\mu_{A_{ik}}(x_i)$, which allows us to obtain

$$\mu_{B'_k}(y) = \sup_{\substack{x_1\in X_1\\ x_2\in X_2\\ \dots\\ x_n\in X_n}} \left\{ (\mu_{A'_1}(x_1) \overset{\mathrm{T}}{*} \mu_{A_{1k}}(x_1)) \overset{\mathrm{T}}{*} (\mu_{A'_2}(x_2) \overset{\mathrm{T}}{*} \mu_{A_{2k}}(x_2)) \overset{\mathrm{T}}{*} \dots \overset{\mathrm{T}}{*} (\mu_{A'_n}(x_n) \overset{\mathrm{T}}{*} \mu_{A_{nk}}(x_n)) \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y),$$

and, since t-norms are non-decreasing,

$$\mu_{B'_k}(y) = \sup_{x_1\in X_1} \{\mu_{A'_1}(x_1) \overset{\mathrm{T}}{*} \mu_{A_{1k}}(x_1)\} \overset{\mathrm{T}}{*} \sup_{x_2\in X_2} \{\mu_{A'_2}(x_2) \overset{\mathrm{T}}{*} \mu_{A_{2k}}(x_2)\} \overset{\mathrm{T}}{*} \dots \overset{\mathrm{T}}{*} \sup_{x_n\in X_n} \{\mu_{A'_n}(x_n) \overset{\mathrm{T}}{*} \mu_{A_{nk}}(x_n)\} \overset{\mathrm{T}}{*} \mu_{B_k}(y),$$

what can be written as

$$\mu_{B'_k}(y) = \underset{i=\overline{1,n}}{\mathrm{T}} \left\{ \sup_{x_i\in X_i} \{\mu_{A'_i}(x_i) \overset{\mathrm{T}}{*} \mu_{A_{ik}}(x_i)\} \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y) = \underset{i=\overline{1,n}}{\mathrm{T}} \left\{ \Pi_{A_{ik}|A'_i} \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y), \tag{8}$$

where

$$\Pi_{A_{ik}|A'_i} = \sup_{x_i\in X_i} \{\mu_{A'_i}(x_i) \overset{\mathrm{T}}{*} \mu_{A_{ik}}(x_i)\}$$

is a scalar value which, according to its definition in [Dub90], is a measure of possibility for $i$-th input, meaning how much $A'_i$ corresponds to $A_{ik}$ (or vice versa).

Thus, we have proved that inference method (8) is possible if all four t-norms are similar (5). In contrast to [Rut10], this t-norm may be arbitrary.

## 4. Inference method based on fuzzy truth value

Applying the truth modification rule [Bor82]

$$\mu_{\boldsymbol{A}'}(\boldsymbol{x}) = \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}\left(\mu_{\boldsymbol{A}_k}(\boldsymbol{x})\right),$$

where $\tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(\cdot)$ denotes the fuzzy truth value of a fuzzy set $\boldsymbol{A}_k$ with respect to $\boldsymbol{A}'$, representing a compatibility membership function $CP(\boldsymbol{A}_k,\ \boldsymbol{A}')$ of $\boldsymbol{A}_k$ relatively to $\boldsymbol{A}'$, while $\boldsymbol{A}'$ is considered as true [Zad78, Dub90]:

$$\tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v) = \mu_{CP(\boldsymbol{A}_k,\,\boldsymbol{A}')}(v) = \sup_{\substack{\mu_{\boldsymbol{A}_k}(\boldsymbol{x})=v \\ x\in X}} \{\mu_{\boldsymbol{A}'}(\boldsymbol{x})\}, \quad v \in [0;1],$$

let us denote $v = \mu_{\boldsymbol{A}_k}(\boldsymbol{x})$. Then we get:

$$\mu_{\boldsymbol{A}'}(\boldsymbol{x}) = \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}\left(\mu_{\boldsymbol{A}_k}(\boldsymbol{x})\right) = \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v).$$

Hence *fuzzy modus ponens* rule for systems with $n$ inputs can be represented as follows:

$$\mu_{B'_k}(y) = \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v) \overset{\mathrm{T}_4}{*} \left(v \overset{\mathrm{T}_2}{*} \mu_{B_k}(y)\right) \right\}. \tag{9}$$

Computational complexity of expression (9) has order of $O(|v| \times |Y|)$. As proven in [Kut15, Sin16]:

$$\begin{aligned}\mu_{CP(\boldsymbol{A}_k,\,\boldsymbol{A}')}(v) &= \underset{i=\overline{1,n}}{\tilde{\mathrm{T}}_1}\, \mu_{CP(A_{ki},\,A'_i)}(v_i) = \\ &= \left(\mu_{CP(A_{k1},\,A'_1)}(v_1)\ \tilde{\mathrm{T}}_1\ \mu_{CP(A_{k2},\,A'_2)}(v_2)\right)\ \tilde{\mathrm{T}}_1\ \mu_{CP(A_{k3},\,A'_3)}(v_3)\ \tilde{\mathrm{T}}_1\ \ldots\ \tilde{\mathrm{T}}_1\ \mu_{CP(A_{kn},\,A'_n)}(v_n),\end{aligned}$$

where $\tilde{\mathrm{T}}_1$ is an extended according to the extension principle $n$-ary t-norm [Dub90] and

$$\mu_{CP(A_{ki},\,A'_i)}(v_i) = \sup_{\substack{\mu_{A_{ki}}(x_i)=v_i \\ x_i\in X_i}} \{\mu_{A'_i}(x_i)\}.$$

Particularly, if $n = 2$, then

$$\mu_{CP(\boldsymbol{A}_k,\,\boldsymbol{A}')}(v) = \underset{i=\overline{1,2}}{\tilde{\mathrm{T}}_1}\, \mu_{CP(A_{ki},\,A'_i)}(v_i) = \sup_{\substack{v_1\ \mathrm{T}_1\ v_2\ =\ v \\ (v_1,v_2)\in[0;1]^2}} \{\mu_{CP(A_{k1},\,A'_1)}(v_1)\ \mathrm{T}_3\ \mu_{CP(A_{k2},\,A'_2)}(v_2)\}.$$

Computational complexity of the latter expression has order of $O(|v|^2)$. In case $\mathrm{T}_4 = \mathrm{T}_2 = \mathrm{T}$, then associativity of t-norms allows us to transform (9) into

$$\mu_{B'_k}(y) = \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v) \overset{\mathrm{T}}{*} \left(v \overset{\mathrm{T}}{*} \mu_{B_k}(y)\right) \right\} = \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v) \overset{\mathrm{T}}{*} v \right\} \overset{\mathrm{T}}{*} \mu_{B_k}(y) = \Pi_{\boldsymbol{A}_k|\boldsymbol{A}'} \overset{\mathrm{T}}{*} \mu_{B_k}(y), \tag{10}$$

where $k = \overline{1,N}$ and

$$\Pi_{\boldsymbol{A}_k|\boldsymbol{A}'} = \sup_{v\in[0;1]} \{\tau_{\boldsymbol{A}_k|\boldsymbol{A}'}(v) \overset{\mathrm{T}}{*} v\} \tag{11}$$

is a scalar value which represents a generalization of an expression defined in [Yag83] and means how much terms $\boldsymbol{A_k}$ of rule $k$ correspond to input values $\boldsymbol{A'}$ (or vice versa).

This means that using fuzzy truth values in (9) makes its computational complexity polynomial and does not impose restrictions onto t-norms (5).

In case $\boldsymbol{A_k} = \boldsymbol{A'}$, then $\tau_{\boldsymbol{A_k}|\boldsymbol{A'}}(v) = v$, i.e. $CP(\boldsymbol{A_k}, \boldsymbol{A'})$ is "true". Hence

$$\mu_{B'_k}(y) = \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A_k}|\boldsymbol{A'}}(v) \overset{T}{*} v \right\} \overset{T}{*} \mu_{B_k}(y) = \sup_{v\in[0;1]} \left\{ v \overset{T}{*} v \right\} \overset{T}{*} \mu_{B_k}(y) = 1 \overset{T}{*} \mu_{B_k}(y) = \mu_{B_k}(y),$$

what indicates the fulfillment of the first criterion of correspondence of an inference method to approximate reasoning [Rut10].

Let us consider inference based on (10), which belongs to so-called FITA-approaches (First Inference, Then Aggregate), i.e. when inference for every rule is performed prior to aggregation of the result. Aggregation for Mamdani model is implemented by means of S-norms [Rut04]. For example, let us use the Lukasiewicz t-norm [Als06] in (10), which could not be used in inference before due to the computational complexity:

$$\Pi_{\boldsymbol{A_k}|\boldsymbol{A'}} \overset{T}{*} \mu_{B_k}(y) = \{0, \Pi_{\boldsymbol{A_k}|\boldsymbol{A'}} + \mu_{B_k}(y) - 1\}. \tag{12}$$

FITA-fuzzy process based on (12) is illustrated in figure 1, where three fuzzy sets $B_k$, $k = \overline{1,3}$ with Gaussian membership functions are depicted subsequently. Here we assume that these fuzzy sets are normal, i.e. $\sup_y\{\mu_{B_k}(y)\} = 1$. Each of $B'_k$ is derived from a particular rule according to formula (11) from fuzzy set $B_k$ by pushing it down. The membership function obtained as union of fuzzy sets $B'_k$, $k = \overline{1,3}$ using maximum operation is depicted at the bottom of the figure. The maximum operation is an example of S-norms.

Let us compare the shapes of fuzzy sets $B'_k$, derived with the use of Lukasiewicz's t-norm, to ones that were obtained using minimum and arithmetical product operations. In the first case, membership functions are being "truncated", in the second case they are being "scaled" [Kru01].

## 5. Fuzzy system based on center average defuzzification method

Let us consider the systems introduced in section 4 having fuzzy inputs and using the center average defuzzification method [Rut04]. In this case, the crisp output value is defined by the following formula:

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot \mu_{B'_k}(\overline{y}_k)}{\sum_{k=\overline{1,N}} \mu_{B'_k}(\overline{y}_k)}, \tag{13}$$

where $\overline{y}$ is the crisp output of a system, consisting of $N$ rules; $\overline{y}_k$ are centers of membership functions $\mu_{B_k}(y)$, $k = \overline{1,N}$, i.e. points, for which

$$\mu_{B_k}(\overline{y}_k) = \sup_{y\in Y}\{\mu_{B_k}(y)\} = 1 \tag{14}$$

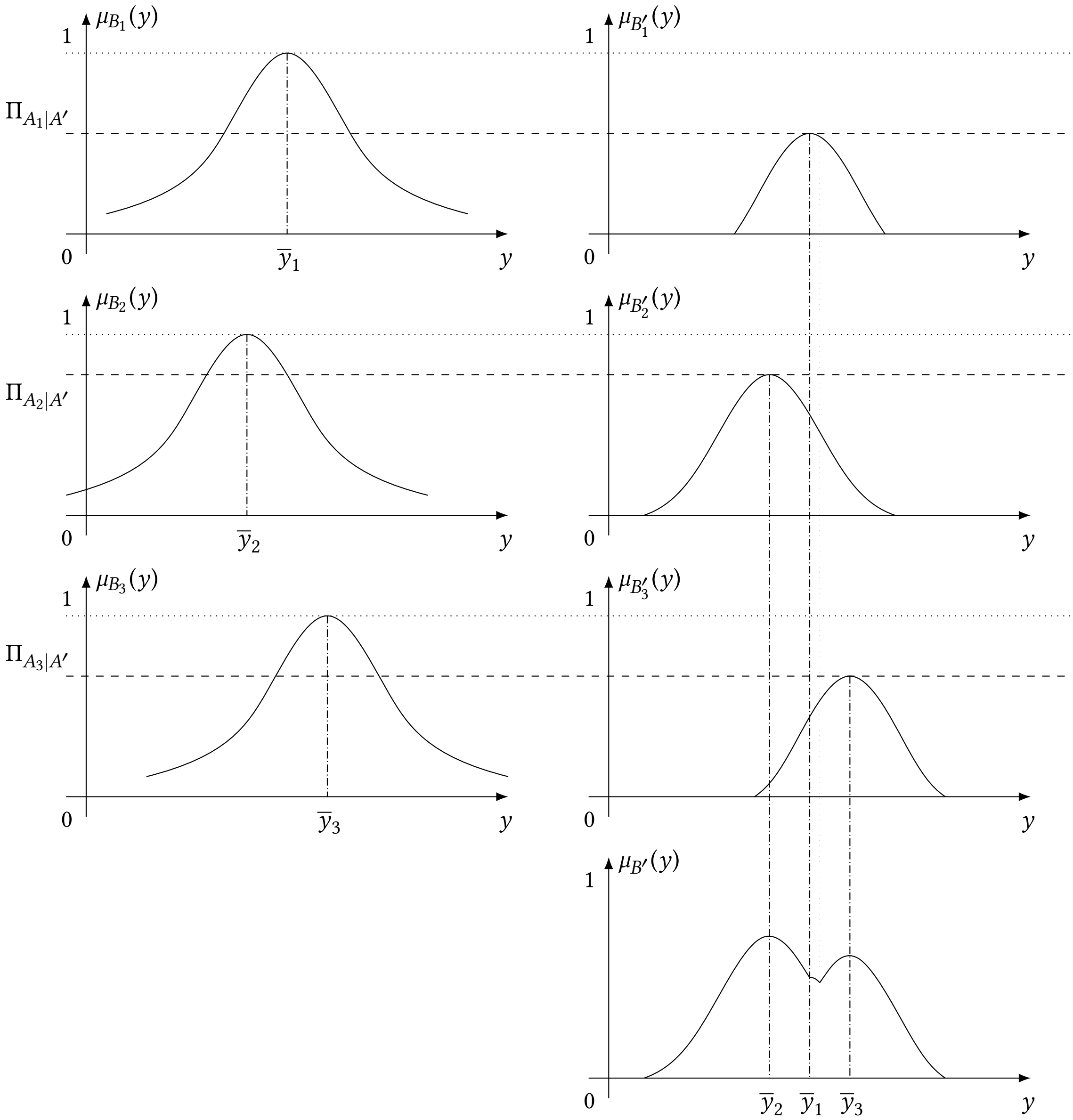



**Figure 1:** Graphical representation of inference based on (12) and the Lukasiewicz t-norm

is true. According to expressions (9) and (13) we get

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A_k}|\boldsymbol{A'}}(v) \overset{\mathrm{T}_4}{*} \left( v \overset{\mathrm{T}_2}{*} \mu_{B_k}(\overline{y}_k) \right) \right\}}{\sum_{k=\overline{1,N}} \sup_{v\in[0;1]} \left\{ \tau_{\boldsymbol{A_k}|\boldsymbol{A'}}(v) \overset{\mathrm{T}_4}{*} \left( v \overset{\mathrm{T}_2}{*} \mu_{B_k}(\overline{y}_k) \right) \right\}}. \tag{15}$$

From (14) follows

$$\sup_{v\in[0;1]} \left\{ \tau_{A_k|A'}(v) \overset{T_4}{*} \left(v \overset{T_2}{*} 1\right) \right\} = \sup_{v\in[0;1]} \left\{ \tau_{A_k|A'}(v) \overset{T_4}{*} v \right\} = \Pi_{A_k|A'}, \tag{16}$$

because a t-norm meets boundary condition $T(a; 1) = a$ by definition. Substituting (16) into (15), we get

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot \Pi_{A_k|A'}}{\sum_{k=\overline{1,N}} \Pi_{A_k|A'}}. \tag{17}$$

Therefore the result $\overline{y}$ does not depend on the specific t-norm $T_2$ when using the center average defuzzification method for systems with fuzzy inputs.

Let us consider the inference with crisp input data, hence

$$\tau_{A_k|A'}(v) = \delta(v - v_k) = \begin{cases} 1, & \text{if } v = v_k, \\ 0, & \text{if } v \neq v_k, \end{cases}$$

where

$$v_k = \underset{i=\overline{1,n}}{T_1} \mu_{A_{ik}}(\overline{x}_i), \quad k = \overline{1, N},$$

in which $\overline{x}_i$, $i = \overline{1, n}$ are crisp input values, and $T_1$ is a t-norm formalizing the conjunction in $k$-th rule's antecedent. Then

$$\Pi_{A_k|A'} = \sup_{v\in[0;1]} \left\{ \delta(v - v_k) \overset{T_2}{*} v \right\} = v_k,$$

considering that $T_2(1; v_k) = v_k$. Therefore, the output value is defined as follows:

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot v_k}{\sum_{k=\overline{1,N}} v_k},$$

what turns out to be the zero order Takagi-Sugeno's fuzzy inference algorithm [Kru01]. Thus, system output does not depend on t-norms $T_2$ and $T_4$ in the case of crisp input data and the center average defuzzification method. The structure of a fuzzy system that is described by expression (17) is shown in figure 2.

## 6. Fuzzy system based on the center of gravity defuzzification method

Let us consider those systems introduced in section 4 having fuzzy inputs and using a discrete variant of the center of gravity defuzzification method [Rut04]

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot \mu_{B'_k}(\overline{y}_k)}{\sum_{k=\overline{1,N}} \mu_{B'_k}(\overline{y}_k)}, \tag{18}$$

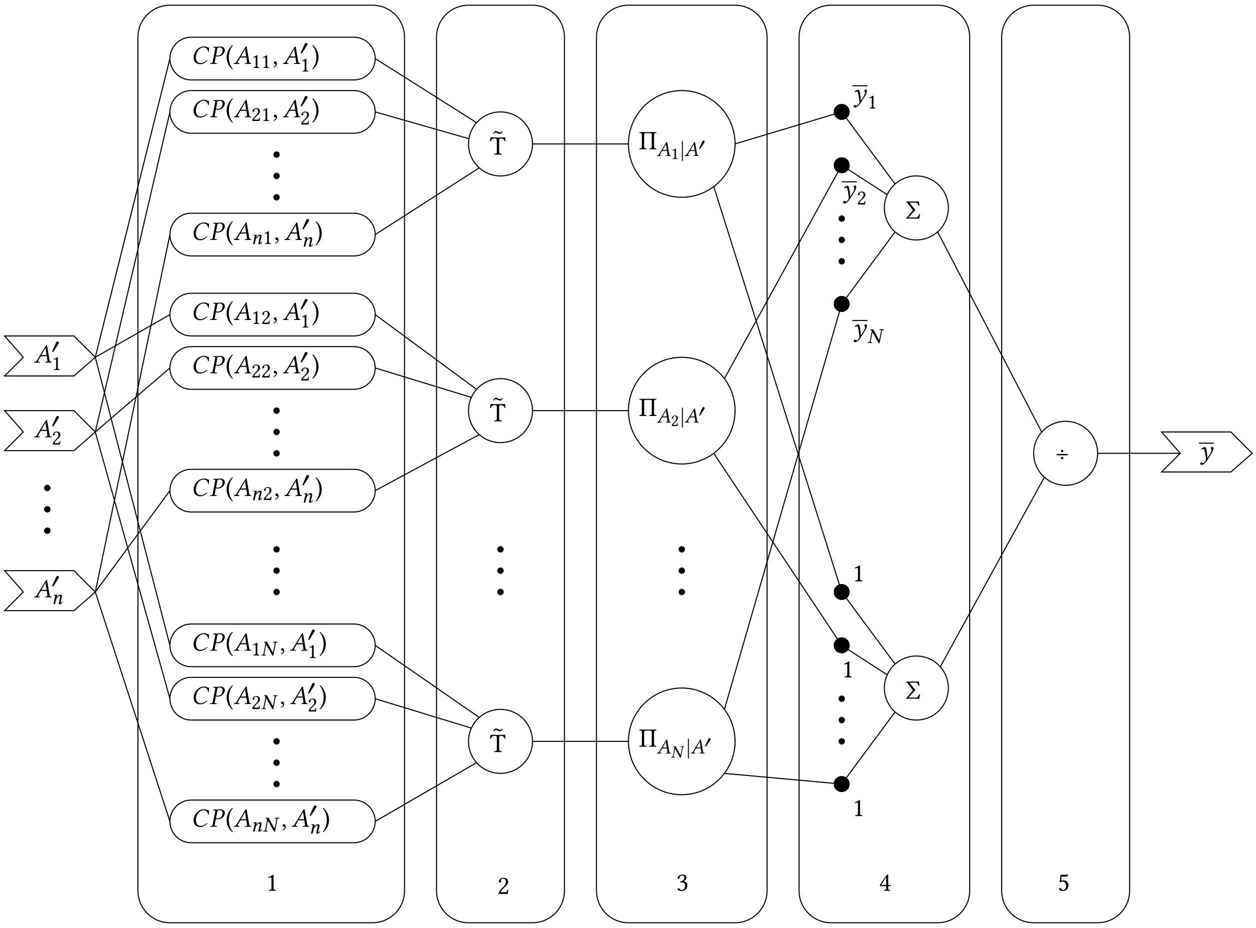



**Figure 2:** Network structure of inference process based on (17)

where $\overline{y}$ is the crisp output value, and $\overline{y}_k$ are the centers of membership functions $\mu_{B_k}(y)$, $k = \overline{1, N}$, defined by expression (14). Fuzzy set $B'$ is derived by the union of fuzzy sets $B'_k$, $k = \overline{1, N}$ using the maximum operator or any other S-norm, i.e.

$$\mu_{B'}(y) = \underset{k=\overline{1,N}}{\mathrm{S}} \mu_{B'_k}(y). \tag{19}$$

From (18), (9) and (19) we get

$$\overline{y} = \frac{\sum_{k=\overline{1,N}} \overline{y}_k \cdot \underset{j=\overline{1,N}}{\mathrm{S}} \left\{ \sup\limits_{v \in [0;1]} \left\{ \tau_{\boldsymbol{A}_j|\boldsymbol{A}'}(v) \overset{\mathrm{T}_4}{*} \left( v \overset{\mathrm{T}_2}{*} \mu_{B_j}(\overline{y}_k) \right) \right\} \right\}}{\sum_{k=\overline{1,N}} \underset{j=\overline{1,N}}{\mathrm{S}} \left\{ \sup\limits_{v \in [0;1]} \left\{ \tau_{\boldsymbol{A}_j|\boldsymbol{A}'}(v) \overset{\mathrm{T}_4}{*} \left( v \overset{\mathrm{T}_2}{*} \mu_{B_j}(\overline{y}_k) \right) \right\} \right\}}. \tag{20}$$

Let us denote $\mu_{B_j}(\overline{y}_k) = b_{jk}$. From (14) follows $b_{kk} = \mu_{B_k}(\overline{y}_k) = 1$. According to (12), the S-norm can be written as follows:

$$\underset{j=\overline{1,N}}{\mathrm{S}}\left\{\sup_{v\in[0;1]}\left\{\tau_{A_j|A'}(v) \overset{\mathrm{T}_4}{*} \left(v \overset{\mathrm{T}_2}{*} b_{jk}\right)\right\}\right\} = \mathrm{S}\left(\Pi_{A_j|A'}, \underset{\substack{j=\overline{1,N}\\ j\neq k}}{\mathrm{S}}\left\{\sup_{v\in[0;1]}\left\{\tau_{A_j|A'}(v) \overset{\mathrm{T}_4}{*} \left(v \overset{\mathrm{T}_2}{*} b_{jk}\right)\right\}\right\}\right). \tag{21}$$

The network architecture corresponding to expression (20) with substitution (21) is represented in figure 3. If $\mathrm{T}_4 = \mathrm{T}_2 = \mathrm{T}$, then

$$\underset{j=\overline{1,N}}{\mathrm{S}}\left\{\sup_{v\in[0;1]}\left\{\tau_{A_j|A'}(v) \overset{\mathrm{T}}{*} \left(v \overset{\mathrm{T}}{*} b_{jk}\right)\right\}\right\} = \mathrm{S}\left(\Pi_{A_j|A'}, \underset{\substack{j=\overline{1,N}\\ j\neq k}}{\mathrm{S}}\left\{\Pi_{A_j|A'} \overset{\mathrm{T}}{*} b_{jk}\right\}\right). \tag{22}$$

In this case the network architecture of the system takes the form represented in figure 4. If

$$b_{jk} \approx 0 \quad \text{for} \quad j,\ k = \overline{1, N}, \quad j \neq k, \tag{23}$$

then expressions (21) and (22) will take the form of (17), and the network architectures given in figures 3 and 4 take the form of the architecture depicted in figure 2. Figure 5 provides an example of fuzzy sets $B_k,\ k = \overline{1, N}$ that meet condition (23). Therefore, the center average and center of gravity (defined by expression (13)) defuzzification methods lead to same results for the same input data.

## 7. Conclusion

Inference based on fuzzy truth value enables us to spread Mamdani's approach onto systems with multiple fuzzy inputs regardless of the t-norms used, thereby eliminating exponential computational complexity.

Moreover, the most important advantage of using the concept of fuzzy truth value is the fact that the relation between the premise and fact is represented as a fuzzy set, in contrast to methods [Rut10, Als06], which reduce this relation to a scalar value.

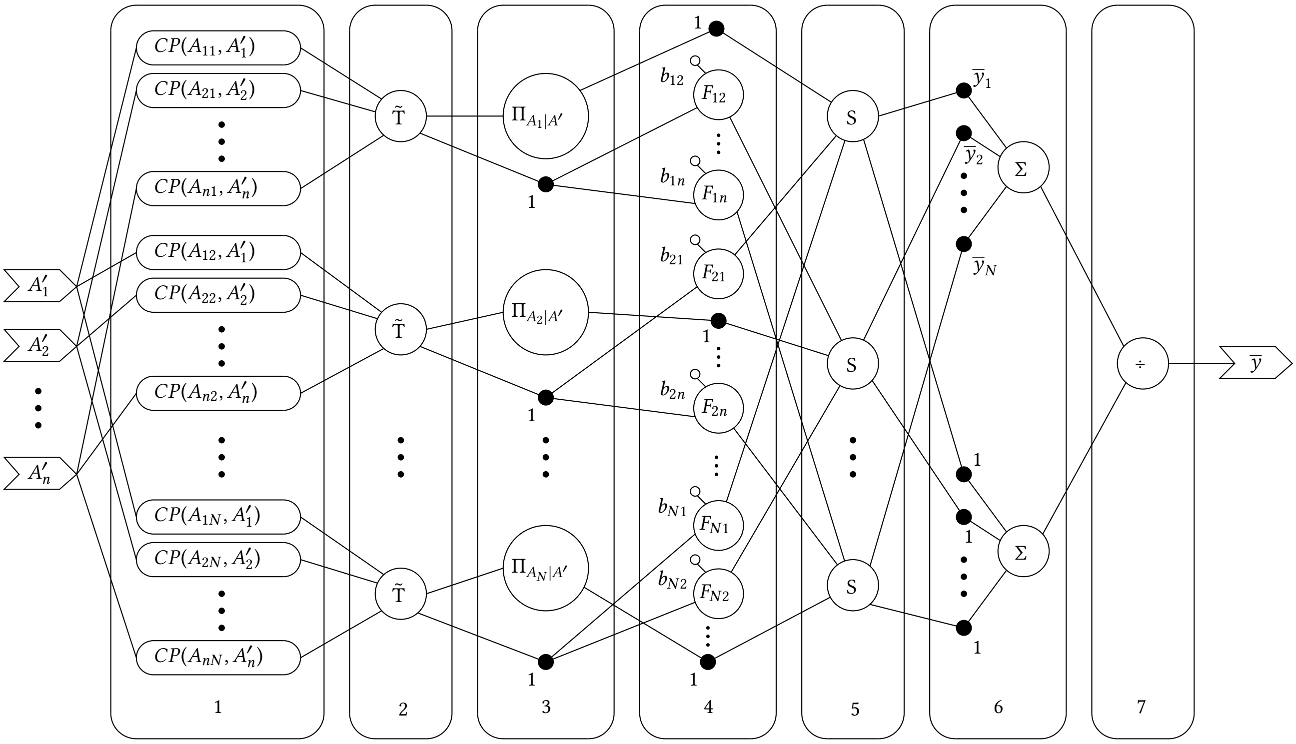
Representing all the relationships between the premises and facts within the same space of truthfulness reduces the computational complexity of the inference result from exponential to polynomial.

Expressions of output values for fuzzy systems utilizing measure of possibility generalization (11) with the use of center average and center of gravity defuzzification methods were introduced in the article.

Formulas (17), (20), (21), (22) were used to build network structures. Using learning algorithms for their parameters they can be transformed into neuro-fuzzy systems.

## Acknowledgments

This work is partially supported by RFBR (grant №20-07-00030).




where $F_{jk} = \sup\limits_{v\in[0;1]} \{\tau_{\boldsymbol{A}_j|\boldsymbol{A}'}(v) \overset{T_4}{*} (v \overset{T_2}{*} b_{jk})\}$

**Figure 3:** Network structure of inference process based on (21)

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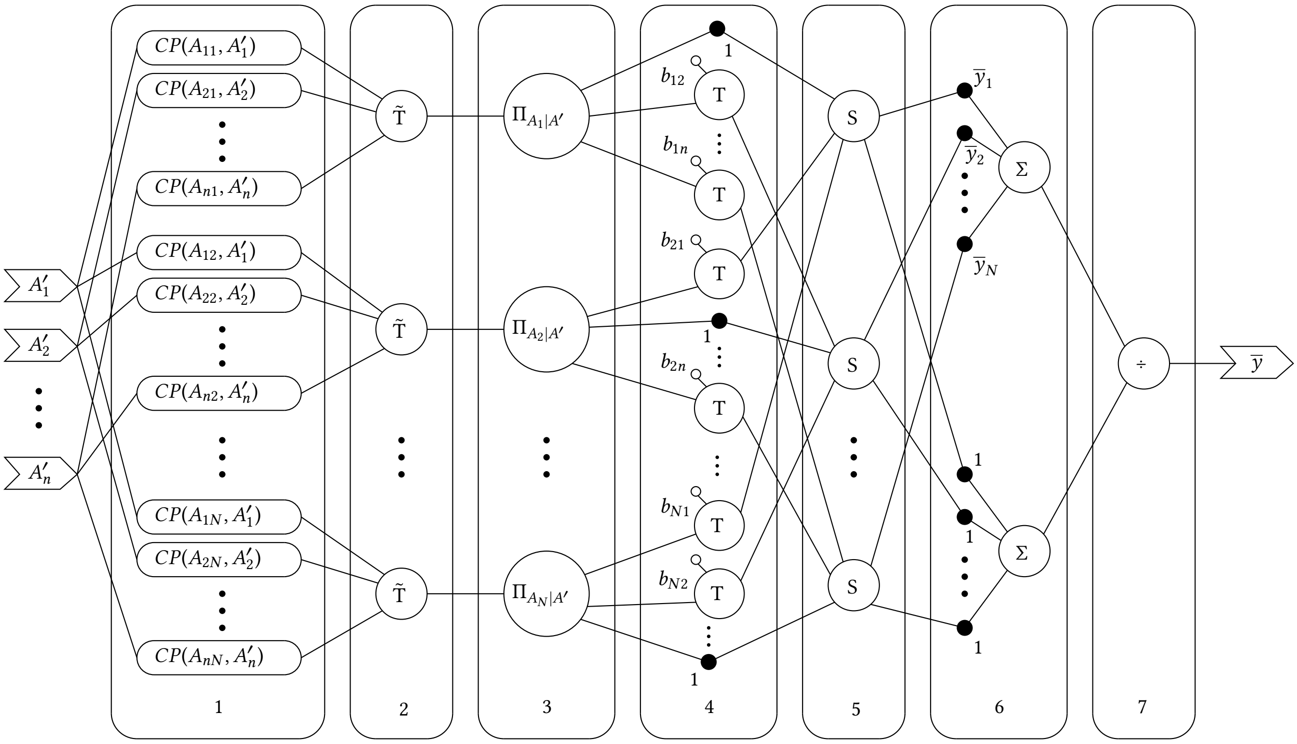


**Figure 4:** Network structure of inference process based on (22)

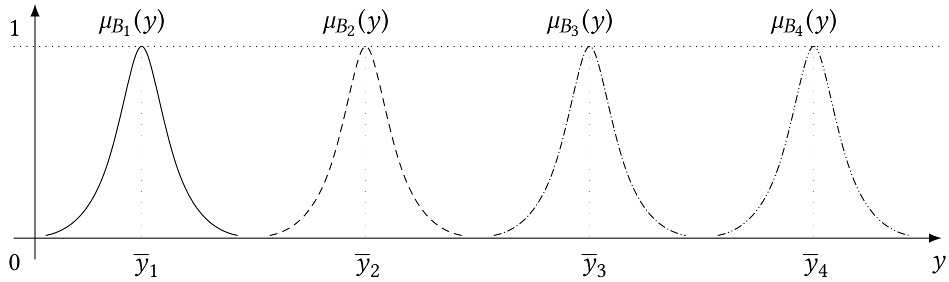


**Figure 5:** An example of set of terms meeting condition (23)


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