Concept Lattice and Soft Sets. Application to the Medical Image Analysis.

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Abstract. In this paper we recall the basic mathematical fundamentals of Formal Concept Analysis (FCA) and analyse the possibilities to define soft sets from a concept lattice. We propose two ways of finding soft sets in a concept lattice of FCA model. We give also an example of soft set in the framework of digital medical image. We prove the usefulness of working with soft sets for the ROI - RONI categorization of a medical image.

Keywords: Formal Concept Analysis (FCA) · Objects · Attributes · Formal Concept · Concept Lattice (Galois Lattice) · Soft Set · Digital Medical Image · Region Of Interest (ROI) · Region Of Non Interest (RONI).

1 Introduction

This paper is an investigation of the links between two great mathematical theories: Formal Concept Analysis (FCA) and Soft Sets (SS). FCA was founded in the late 90s by a team of German and French researchers in Darmstadt and Paris. Theoretically speaking, it is based on a main theorem using the notion of Galois connection. FCA has been developed a lot, especially from an application point of view, finding a good number of applications in different fields. SS is newer (end of 90' by Molodtsov [3]) and probably less known. It is a pure categorization theory considered by some as a generalization of fuzzy sets theory. At the first glance, the common point of both theories is the fact that both are categorization theories.

In the theory of categorization, there are two classes of models:

1. Models whose categorization objects belong to a space X and for which the categorization criteria are also defined within the space X;
2. Models whose categorization objects are in a space X and for which the categorization criteria are defined outside of the space X;

The soft set theory is a theory of categorization giving a categorization of a space X, following a set of parameters outside X.

Both soft set and FCA models are of the second type. It is for this reason that we decided to study the two models from the point of view of their basic

mathematical concepts. We found that the FCA model was a special case of soft set. It is claimed that in particular classes of applications soft set modeling may be more profitable.
The paper is organized as follows. Section 2 presents the mathematical FCA model. Section 3 gives the basic concepts of soft sets theory. In section 4, two soft sets models of FCA are proposed. Section 5 is dedicated to an application of soft sets to digital medical image analysis. Finally, some conclusions are given in section 6.

2 FCA model

The FCA model is presented following [1].

Definition 1. *(Formal context) A formal context,* **K,** is a triple **K** = (O, A, I) where O is a set of *objects*, A is a set of *attributes* and I is a binary relation from O to A defined by:
$\forall o \in O, a \in A$,
$I(o, a) = 1$ when the object o has the attribute a
$I(o, a) = 0$ otherwise.

Starting from binary relation I, one defines two *derivation operators* $I^{\uparrow}$ and $I^{\downarrow}$.

Definition 2. *(Derivation operators) Let $\mathcal{P}(O)$ and $\mathcal{P}(A)$ be respectively the set of all subsets of O and A.*
The operator $I^{\uparrow}$ is defined as follows:
$I^{\uparrow} : \mathcal{P}(O) \rightarrow \mathcal{P}(A)$. For $X \subset O$,

$$I^{\uparrow}(X) = \{a \in A / I(o, a) = 1,\ \forall o \in X\} \tag{1}$$

The operator $I^{\downarrow}$ is defined as follows: $I^{\downarrow} : \mathcal{P}(A) \rightarrow \mathcal{P}(O)$. For $Y \subset A$,

$$I^{\downarrow}(Y) = \{o \in O / I(o, a) = 1,\ \forall a \in Y\} \tag{2}$$

Three properties are established in [1]:

1. $X_1 \subseteq X_2 \Rightarrow I^{\uparrow}(X_1) \supseteq I^{\uparrow}(X_2)$;
2. $X \subseteq I^{\downarrow}(I^{\uparrow}(X))$ and $Y \subseteq I^{\uparrow}(I^{\downarrow}(Y))$;
3. $I^{\uparrow}(I^{\downarrow}(I^{\uparrow}(X))) = I^{\uparrow}(X)$ and $I^{\downarrow}(I^{\uparrow}(I^{\downarrow}(Y))) = I^{\downarrow}(Y)$.

Definition 3. *(Formal concept)[1] A formal concept, C of a formal context* **K** *is a pair C = (X,Y) with X$\subseteq$ O, Y $\subseteq$ A, X = $I^{\downarrow}$(Y) and Y = $I^{\uparrow}$(X). X is called the* extent*, denoted by Ext, and Y is called the* intent*, denoted by Int of the formal concept C.*

Definition 4. *(A formal concepts order)[1] Let be two concepts, C_1 and C_2. An order relation $\preceq$ is introduced by:*
$C_1 \preceq C_2 \iff Ext(C_1) \subseteq Ext(C_2) \iff Int(C_2) \subseteq Int(C_1)$

Taking into account properties 2 and 3, it is proved [1] that the set of all formal concepts of a context **K** is a *complete lattice* denoted by $\mathcal{G}(\mathbf{K})$. This lattice verifies the property of *Galois connection* [2] and it is called *Galois lattice of concepts.*

3 Basic concepts of soft sets theory

Mathematical categorization theory contains several categorization models each based on a mathematical theory, be it algebraic, geometric, probabilistic or other. In classical models of categorization, the idea is to split a space X in categories taking into account one or several criteria defined also based on the space X. So these criteria are in some way internal to the space X.

A *soft set* is a model giving a categorization on a space X taking into account a *cognitive element* external to X. This feature of externality represents another point of view of categorization.

We give some basic elements from soft set theory [3][4]. A *soft set* is a parameterized family of sets - intuitively, this is "soft" because the boundary of the set depends on the parameters. Formally, a soft set is defined by:

Definition 5. *(Soft set) Let X be an initial universe set and E a set of parameters with respect to X. Let $\mathcal{P}(X)$ denote the power set of X and* $\mathbf{A} \subset$ E. A pair (F, $\mathbf{A}$) is called a soft set over X, where F is a mapping given by F : $\mathbf{A} \to \mathcal{P}$(X). In other words, a soft set (F, $\mathbf{A}$) over X is a parameterized family of subsets of X. For e $\in \mathbf{A}$, F (e) may be considered as the set of e-elements or e-approximate elements of the soft sets (F, $\mathbf{A}$). Thus (F, $\mathbf{A}$) is defined as:

$$(F, \mathbf{A}) = \{\mathrm{F(e)} \in \mathcal{P}(\mathrm{X}) \text{ if } \mathrm{e} \in \mathbf{A}\} \text{ and } (\mathrm{F}, \mathbf{A}) = \emptyset \text{ if } \mathrm{e} \notin \mathrm{A} \tag{3}$$

Remark 1. A soft set is a categorization of a space X guided by a set of parameters $\mathbf{A}$ and a function F establishing a correspondence between a parameter and a subset of X.

4 Two soft set models of concept lattice

We explore in this section two possibilities to interpret the FCA model as a soft set. The "conceptual metaphor " generating this parallel is that relations between objects and attributes in the FCA model can be viewed as parameters. We have two options : either classify objects or classify formal concepts from the FCA model.

In the first case, one classifies FCA objects and the set $\mathbf{A}$ of parameters is the set A of FCA attributes. That is the categorization is constructed following attributes.
In the second case, one classifies FCA formal concepts and the set $\mathbf{A}$ of parameters is a subset of cartesian product O x A of FCA model.

Remark 2. The second SS model (Soft set 2) is possible because of the duality between extension and intention in the FCA model and the property of Galois connection.

4.1 First soft sets model of FCA

Definition 6. *(Soft set 1) Let be the formal context* $\mathbf{K} = (O, A, I)$ and its associated concept lattice $\mathcal{G}(\mathbf{K})$.
We define a soft set SS1 as follows:

SS1 = $\{F_1, \mathbf{A}\}$ where $\mathbf{A} \subset A$, $F_1 : A \rightarrow \mathcal{P}(O)$ defined by:

$F_1(a) = \{o \in O \ / \ I(o,a) = 1 \}$, if $a \in A$
$F_1(a) = \emptyset$, otherwise.

4.2 Second soft sets model of FCA

Definition 7. *(Soft set 2) Let be the formal context* $\mathbf{K} = (O, A, I)$ and its associated concept lattice $\mathcal{G}(\mathbf{K})$.
We define a soft set SS2 as follows:
SS2 = $\{F_2, \mathbf{A}\}$ where $\mathbf{A} \subset O \text{ x } A$, $F_2 : \mathbf{A} \rightarrow \mathcal{P}(\mathcal{G}(\mathbf{K}))$.

Remark 3. In this case, the categorization space is all the concept lattice.

1. One categorizes formal concepts, (not objects as in Soft set 1) and the clusters are related to a set of parameters chosen in the set O x A of FCA model.
2. F_2 can be defined in several ways depending on the purpose of the categorization. The advantage is that one can choose as parameters, object-attribute pairs, therefore, take as the set of parameters **A** a subset of the space (O x A) of the FCA model.

5 Image analysis application

From mathematical point of view, a grey-level digital image is a function I(x,y) defined on Z^2 with values in [0, 255]. The space X is a discrete space. In medical image case, we need to distinguish the following elements:

- the header (HD) containing patient informations,
- the anatomical object (AO)
- the background (BG) and,
- the disease area (DA) inside the anatomical object.

In the following example, we process a medical image with FCA model but taking into account the point of view of soft sets. The image (see Figure 1) is a grayscale image of size 512*512.

The image is divided into 16 non overlapping blocks B1, B1,......B16. The size of each block is 64*64 pixels. Their position is given in Table 1. Locations are dependent on pixels.

The FCA attributes are the following: Entropy, Header(H), Background (BG) and Anatomical Object(AO). The FCA objects are the blocks. One applies Soft set 2 model.

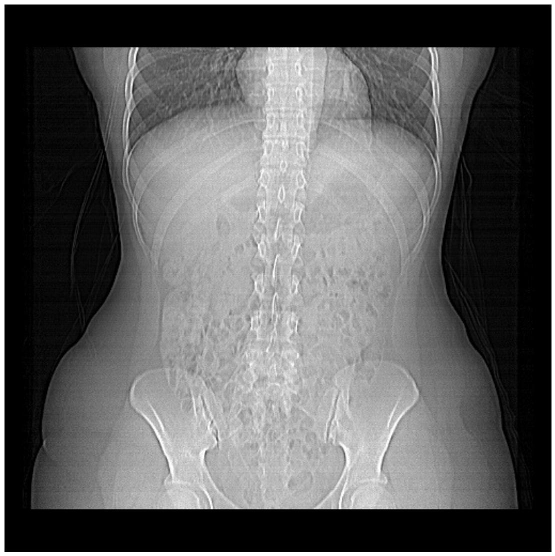

Fig. 1. A medical image

Table 1. Blocks location

B1	B2	B3	B4
B5	B6	B7	B8
B9	B10	B11	B12
B13	B14	B15	B16

In the digital medical image applications, image characteristics must be splited in two categories: the category of characteristics expressing the position versus the standard partition (background, header, anatomical object, disease area) with values 0,1 and the category of characteristics expressing the values of features related to the intensity function I(x, y).
We split entropy's values into 4 intervals: Entropy 1 = [0, 0.25[; Entropy 2= [0.25, 0.50[; Entropy 3 = [0.50, 0.75[; Entropy 4 = [0.75, 1]. Table 2 shows the blocks and corresponding entropy values.

Table 2. Blocks position and blocks entropy

Block number	Block position in columns	Block position in rows	Entropy
B1	0 - 63	0 - 63	0.5239
B2	64 -127	0 - 63	0.8223
B3	128 - 255	0 - 63	0.6223
B4	256 - 511	0 - 63	0.4123
B5	0 - 63	64 -127	0.5326
B6	64 -127	64 -127	0.8774
B7	128 - 255	64 -127	0.5820
B8	256 - 511	64 -127	0.6699
B9	0 - 63	128 - 255	0.7789
B10	64 -127	128 - 255	0.4236
B11	128 - 255	128 - 255	0.6982
B12	255 - 511	128 - 255	0.7899
B13	0 - 63	256 - 511	0.2314
B14	64 -127	256 - 511	0.2369
B15	128 - 255	256 - 511	0.4789
B16	256 - 511	256 - 511	0.5693

The FCA context is described in figure 2 and the concept lattice in figure 3.

The set of parameters is defined on O x A of FCA model with O = {B2, B6, B9, B12} and A = {Entropy 4, AO}, so A = {B2, B6, B9, B12} x {Entropy 4, AO}. Function F_2, $F_2 : \mathbf{A} \rightarrow \mathcal{P}(\mathcal{G}(\mathbf{K}))$ is defined by: for a pair p in $\mathbf{A}$, $F_2(p)$ = the subset of formal concepts in the sub-lattice corresponding to a characteristic considered as the most important in the analysis. In our case, Entropy 4 is chosen as the most important characteristic. This corresponds to concepts in the sub-lattice in blue in figure 4.

6 Conclusions

In this paper we did a bridge between FCA model and Soft Set Theory. We have analyzed the the FCA model from the point of view of Soft Sets. Because of the fact that the attributes in FCA model can be considered as parameters in

	Entropy 1	Entropy 2	Entropy 3	Entropy 4	BG	AO	DA
B1			X		X	X	
B2				X	X	X	
B3			X		X	X	X
B4		X			X	X	
B5			X		X	X	
B6				X		X	
B7			X			X	
B8			X		X	X	
B9					X	X	
B10		X				X	
B11			X			X	
B12				X	X	X	
B13	X				X	X	
B14	X				X	X	
B15		X			X	X	
B16			X		X	X	

Fig. 2. The FCA context of a medical image

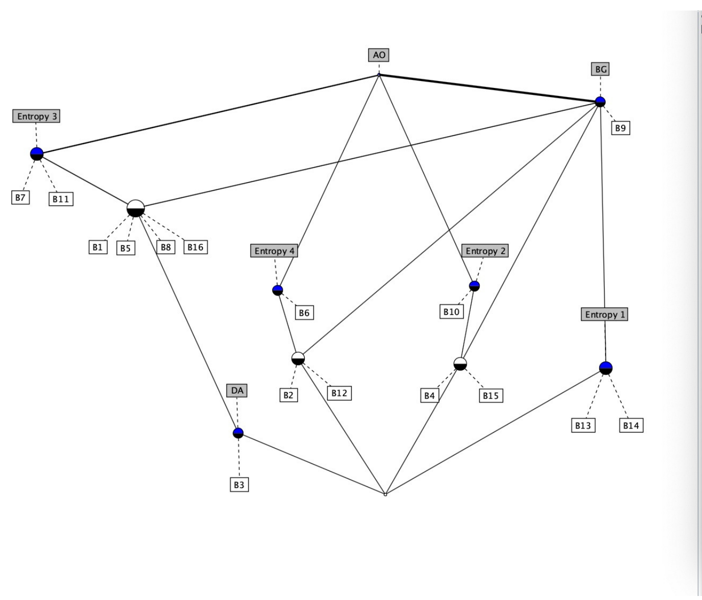

Fig. 3. The FCA lattice of a medical image

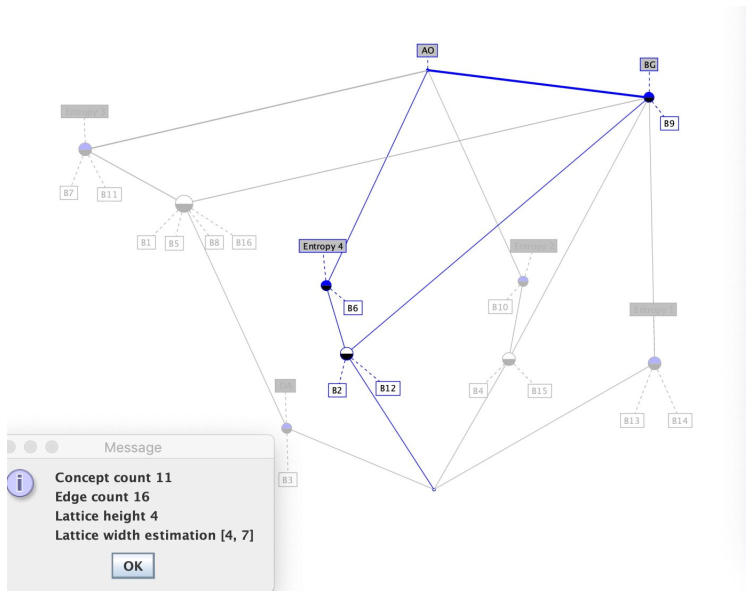

Fig. 4. The sub-lattice of a medical image

the categorization of objects, we propose two types of models of FCA as soft set: the first one that categorizes objects of the FCA model, the second one that categorizes formal concepts of FCA model. We can find out that the main theorem in FCA proved more than 20 years ago, notably that the concepts lattice is a Galois lattice make possible to view some sub lattice or path of concepts lattice as soft sets. All the examples of FCA are processed with Conexp 1.5. An application of FCA – Soft Set in the domain of medical image was presented. The point of view Soft Set may be useful in a definition of the texture for the medical image. Developing a tool dedicated in particular to process some soft sets F-type functions can be useful in the library of digital image analysis.

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