# Constructive Geometric Models with Imaginary Objects

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**Abstract.** The article is devoted to the consideration of a number of theoretical questions of projective geometry related to specifying and displaying imaginary objects, especially, conics. The lack of development of appropriate constructive schemes is a significant obstacle to the study of quadratic images in three-dimensional space and spaces of higher order. The relationship between the two circles, established by the inversion operation with respect to the other two circles, in particular, one of which is imaginary, allows obtain a simple and effective method for indirect setting of imaginary circles in a planar drawing. The application of the collinear transformation to circles with an imaginary radius also makes it possible to obtain unified algorithms for specifying and controlling imaginary conics along with usual real second-order curves. As a result, it allows eliminate exceptional situations that arise while solving problems with quadratic images in spaces of second and higher order.

**Keywords:** Scientific Visualization, Constructive Geometric Modeling, Geometric Experiment, Projective Geometry, Quadric, Imaginary Geometric Objects.

## 1 Introduction

Nowadays the attention of many researchers working in the field of constructive geometry is directed to solving problems related to modeling quadrics. Until recently, the well-known, but almost forgotten constructive schemes for constructing quadrics given with nine points located in three-dimensional space could not be implemented practically due to the instrumental complexity of such schemes in the planar drawing [1]. The development of automation tools for geometric constructions based on the theoretical principles of projective geometry, the beginning of introducing imaginary objects into constructive geometry make it possible to remove from the agenda the issue of instrumental limitations of the geometric method [2]. However, the elimination of this problem revealed a number of other problems, primarily related to the insufficient development of constructive geometric algorithms when working with imaginary objects that appear in abundance, for example, when solving problems with curves or surfaces of the second order. One of the simple but illustrative examples of such problem is the projective algorithm for determination the intersection of a quadric surface with a plane according to the initial data, represented by three conics located on its surface. Since the choice of the plane position in the general case can be arbitrary, the probability of the absence of an explicit (real) intersection of a straight line with the projections of conical sections is very high. This situation, however, should not led to the algorithm failure in general problem solving, because it may have real result. For obvious reasons, the “manual” solution of such problems is very difficult, but in the case of computer simulation, solution can be obtained. Of course, corresponding effective project algorithms, developed on effective structural geometric schemes, should present. These considerations explain the relevance of the study, some of obtained results will be discussed below.

## 2 The imaginary circle construction

Diagnostic methods for assessment of impairment of conscious level are still widely used in the treatment in an Intensive Care Unit. It includes tests of orientation, attention, memory, language and visual-spatial skills. At first, J. Bennett invented Glasgow Coma Let two visibly intersecting circles $a$ and $b$ be given on the plane: $U,V = a \times b$. Let’s find out the central points $A$ and $B$ of the circles $a$ and $b$ and draw a straight line $l = A \circ B$ passing them. Our goal is to construct an inversion circle that transmits the circle $a$ into a circle $b$. It is easy to understand that this problem has two solutions.

We define an involution $\xi_1 \mid \begin{matrix} l; A_1; A_2 \\ l; B_2; B_1 \end{matrix}$ and find its dual points $P_1$ and $P_2$, where $A_1, A_2 = a \times l$, $B_1, B_2 = b \times l$. Taking them as diametrical ones, we construct a circle $p = P_1 \circ P_2$ (See Fig. 1). Then, $b = \mathrm{I}_p(a)$ where $\mathrm{I}_p(a)$ is the inversion transformation induced in the plane by a circle $p$. Due to the involution properties of inversion it follows that $a = \mathrm{I}_p(b)$.

Now we define an involution $\xi_2 \mid \begin{matrix} l; A_1; B_2 \\ l; B_1; A_2 \end{matrix}$ and determine its dual points $Q_1$ and $Q_2$. Taking them as diametrical ones, we construct a circle $q = Q_1 \circ Q_2$ (See Fig. 2). Then $b = \mathrm{I}_q(a)$, where $\mathrm{I}_q(a)$ is the inversion transformation induced in the plane by a circle $q$. Due to the involution properties of inversion it follows that $a = \mathrm{I}_q(b)$.

Thus, two circles are detected that can equally map the original circles onto each other. This, however, does not mean the equality of the point series transformation, i.e. at $T \sim a$, $T_p = \mathrm{I}_p(T)$, $T_q = \mathrm{I}_q(T)$ – $T_p \neq T_q$. Moreover, the circle $d = T \circ T_p \circ T_q$ drawn through the inverse image and its images on the circle $b$ will be orthogonal to the circles of the bundle $(U,V)$ and have a center lying on its radical axis. In other words, we can say that the points $T$, $T_p$ and $T_q$ model each other and the circle $d$ performs a communication line in the internal apparatus of modeling the transformation under consideration.

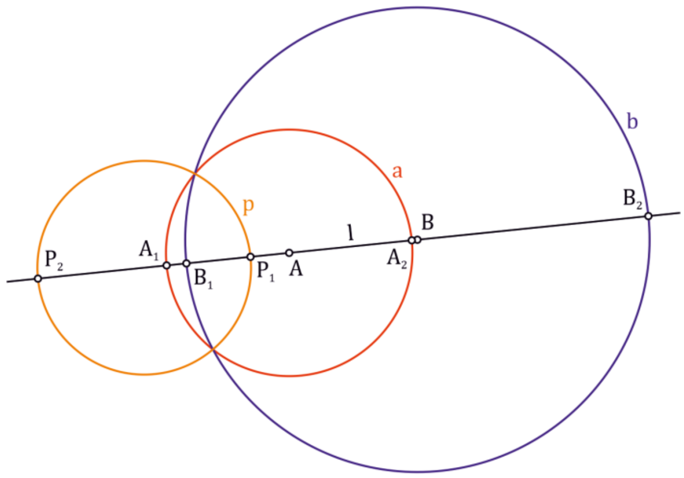


**Fig. 1.** Demonstration of first circle of inversion definition

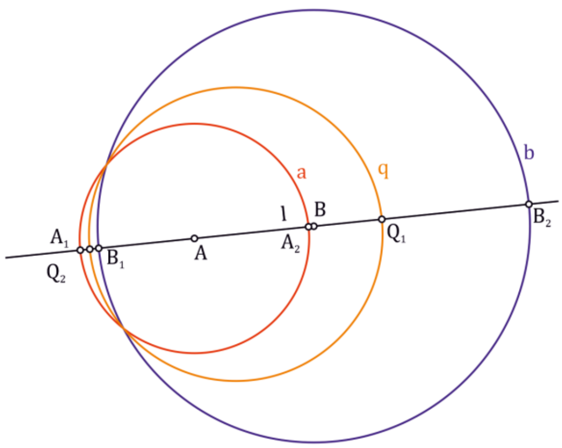


**Fig. 2.** Demonstration of first circle of inversion definition

The resulting circles $p$ and $q$ are orthogonal: $p \perp q$. They form an orthogonal pair in a bundle of circles defined by the centers $U, V = a \times b$ and induced by a pair of circles $(a,b)$. The following statements are true for this pair: $p=\mathrm{I}_q(p)$ and $q=\mathrm{I}_p(q)$ (See Fig. 3).

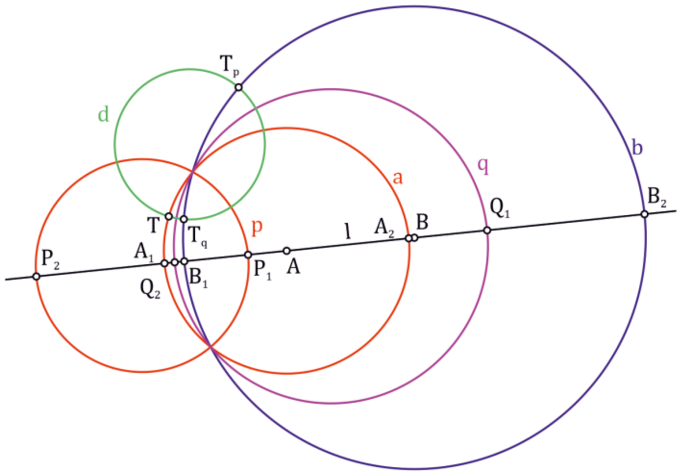


**Fig. 3.** Demonstration of non-equality of point sequences

Carrying out further arguments in the context of the considered problem, we can formulate the task of constructing inversion circles for the resulting pair of circles $p$ and $q$. Acting by analogy with the previous case, we obtain two circles $r$ and $s$: $q=\mathrm{I}_r(p)$, $p=\mathrm{I}_r(q)$, $q=\mathrm{I}_s(p)$, $p=\mathrm{I}_s(q)$. The circles $r$ and $s$ form the second orthogonal pair in the bundle defined by the centers $U,V=a\times b$ and induced by the pair of circles $(a,b)$. Continuing the construction, we find that $r=\mathrm{I}_p(s)$, $s=\mathrm{I}_p(r)$, $r=\mathrm{I}_q(s)$, $s=\mathrm{I}_q(r)$ i.e. the components of the pair $(p,q)$ are inversion circles for the circles of the opposite pair $(r,s)$ and vice versa (See Fig. 4).

Let’s shift one of the circles $a$ or $b$ so that their intersection points $U,V=a\times b$ become implicit, i.e. imaginary. This action will lead to the fact that the points of one of the pairs of dual points of involutions $\xi_1$ and $\xi_2$ either take imaginary values, and the explicit construction of one of the circles $p$ or $q$ becomes instrumentally impossible. However, from a mathematical point of view, nothing fatal happens. If we consider the analytical equivalent of the corresponding state of the obtained geometric system, we can find that in this case we are dealing with intersections of circles, one of which has a real center but an imaginary radius.

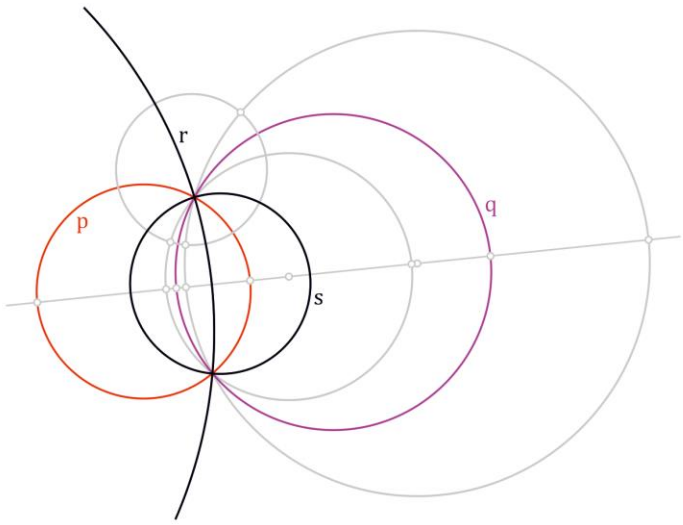


**Fig. 4.** The second pair of orthogonal circles in a common bundle

Now we have to answer the next question: since the problem of finding inversion circles can lead to the appearance of circles with imaginary radii, can such circles play the role of source data in geometrical problems, and what result will be obtained? It should be answered in the affirmative. So, for example, constructing the inversion circles for one real circle and the other – the imaginary circle with a real center and imaginary radius, in the general case we get two inversion circles having not only an imaginary radius, but also an imaginary center.

Using only traditional tools for performing geometric models in the form of drawings, the imaginary circle cannot be imagined explicitly. However, relying on the dependences just obtained, it is possible to interact indirectly with such an object and control it by influencing existing real objects, which led to the formation of an imaginary object. It should be recognized that despite the purely formal and typical nature of the actions performed for this, such a technique entails an increase in the complexity of the practical application of the model. Therefore, when developing systems for automating geometric constructions, it is necessary to provide not only methods for the intrasystem representation of imaginary objects, but also to create algorithmic tools for interactive-graphic control of these data to provide comprehensive functionality of constructive geometry algorithms with images of a real and imaginary mathematical nature.

In relation to the problem under consideration, we can assume that the imaginary circle that is the result of an operation on real circles can be fully replaced by them, while the operations performed with this circle and the images generated as a result of these actions can be represented in real and complex-valued values, matched with model circles in a constructive relationship.

Let some imaginary circle $a$ be given on the plane. We set ourselves the goal of indicating a point incident with this circle. If such a problem were formulated for a real circle, then its solution did not cause any difficulties. However, under the condition of the imaginary circle $a$, it becomes not quite clear how to perform such an appointment.

To solve the problem we draw an additional real circle $b$ (See Fig. 5). Through the centers of both circles, draw a straight line $l$ and find the points of intersection of the circles $a$ and $b$ with it: $A_1, A_2 = l \times a$, $B_1, B_2 = l \times b$. We define an involution $\xi_1 \left| \begin{matrix} l; A_1; A_2 \\ l; B_1; B_2 \end{matrix} \right.$, find its double points $P_1$ and $P_2$, through which, as diametrical, we draw a circle $p = P_1 \circ P_2$. A circle $p$ is one of the inversion circles that translate the circle $a$ into $b$ and vice versa: $a = \mathrm{I}_p(b)$, $b = \mathrm{I}_p(a)$. The second inversion circle is constructed similarly. We should pay attention to the fact that the circles of inversion $p$ and $q$ are essentially imaginary. Choosing any of them, for example $p$, and transferring an arbitrary point $T$ of a circle $b$ relative to it in inversion, we obtain an image of a point $a \sim T_p = \mathrm{I}_p(T)$, thereby providing the ability to indirectly specify a point incident with an imaginary circle $a$.

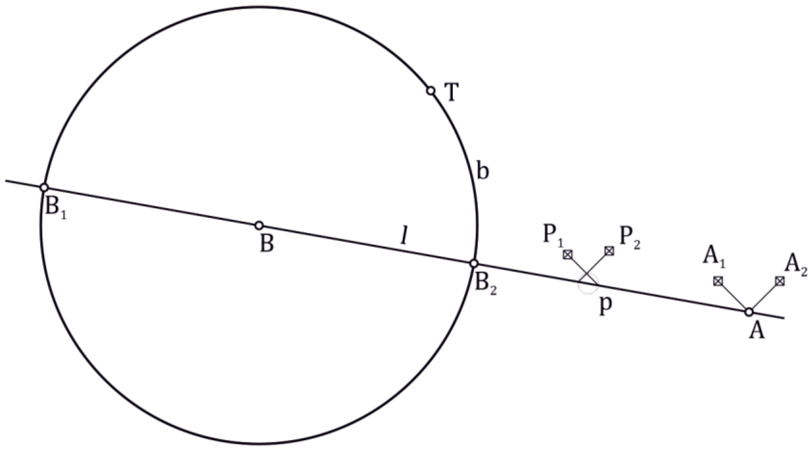


**Fig. 5.** Demonstration of algorithm on incidental point to image circle assignment

## 3 Imaginary conic

Let's consider one of the possible methods of indirect synthetic determination of imaginary conics. Without conducting global generalizations, in the framework of this article, by the term imaginary conics we mean conics having a real center and imaginary half-diameters.

It is well known that in projective geometry the conics do not differ, and any non-degenerate conic can be transformed by a collinear transformation into another, including case of a specific conic. Since the circle is essentially a conic, we use a combination of inversion and collineation transformations to determine imaginary conics and control them, but before that we should formulate and solve one auxiliary problem on a set of objects that have real values to ensure the visibility of the drawing that explains it.

Let two real conics $a$ and $b$ be given on the plane. We find such a collinear transformation $\chi$ by which an arbitrary real point $T \sim a$ is mapped to a real point $T' = \chi(T)$; $T' \sim b$.

In order to solve this problem, we have to construct the major directions of both conics. Consider a method of constructing these directions for one of them (See Fig. 5).

First of all we have to determine a point $C$ – the image of an infinitely distant line in polar transformation relative to the conic $a$. We draw an arbitrary line $m$, passing through point $C$, thereby defining one of the diameters of this conic. Taking the diameter as the polar, we find the pole $M$ corresponding to this polar, and draw a line passing it and the point $C$ that corresponds to the diameter conjugated to the diameter $m$, producing a pair of conjugated diameters $(m, n = S \circ M)$. Let's repeat the same action, changing the position of the line $m$ and marking it as $m^* \neq m$. Now we get a pair of conjugate diameters $(m^*, n^* = S \circ M^*)$ as a result. We draw an arbitrary line $t$ on the plane and find the points of intersection of pairs of conjugate diameters $P = m \times t$ , $Q = n \times t$, $P^* = m^* \times t$, $Q^* = n^* \times t$ on it. The formed pairs of points define an involution $\xi \,|\, \begin{matrix} t;P;P^* \\ t;Q;Q^* \end{matrix}$ on the line $t$ with dual points $R$ and $S$.

Then we draw a circle $z$ of zero radius centered at a point $S$ and define the involution $\varsigma$ induced by this circle on a straight line $t$. This involution distinguishes two dual points $U$ and $V$ on a line $t$. Now we define an involution $\zeta \,|\, \begin{matrix} t;V;R \\ t;U;S \end{matrix}$ on the line $t$. Its dual points $X$ and $Y$, together with the point $S$, indicate the directions of the major diameters of the conic $a$, and they, in turn, highlight the diametric points on it $X_1^a, X_2^a = (S \circ X) \times a$, $Y_1^a, Y_2^a = (S \circ Y) \times a$ (See Fig. 5).

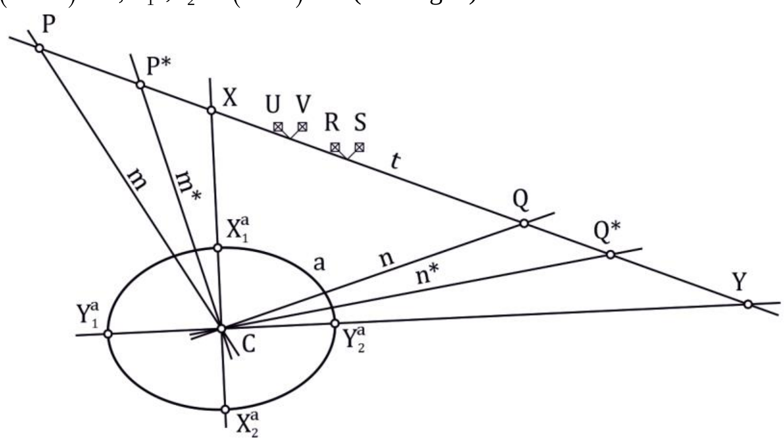


**Fig. 6.** Demonstration of new synthetic algorithm on major diameters of conic definition

Having completed similar constructions with respect to the conic $b$, we find four points of the same nature: $X_1^b$, $X_2^b, Y_1^b$, $Y_2^b$. Defining collineation $\chi\left|\begin{matrix} X_1^a, Y_1^a, X_2^a, Y_2^a \\ X_1^b, Y_1^b, X_2^b, Y_2^b \end{matrix}\right.$ at the obtained points, we can get a solution of the original problem, i.e. provided $T \sim a$ we are able to set the point $T' \sim b$ by performing found collinear transformation $\chi$: $T' = \chi(T)$ (See Fig. 6).

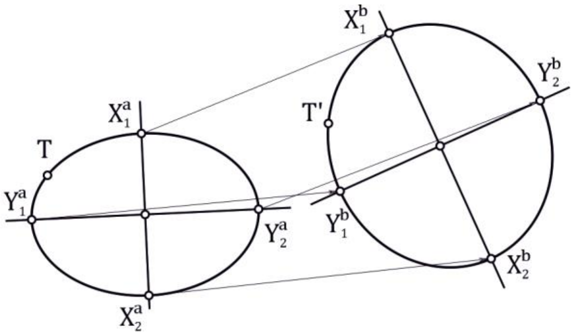


**Fig. 7.** Demonstration of unified collinear transformation of one conic to another

Despite the fact that the solution of the problem was carried out using real images of the plane, the presented algorithm is applicable to imaginary objects. Now let the conic $a$ be real and the conic $b$ be imaginary. Having set a point $T$ on the conic $a$, as a result of performing the collinear transformation, we obtain an imaginary point $(T' \sim b) = \chi(T \sim a)$ that is clearly incident to the conic $b$; therefore, the method of indirectly indicating the point incident to the imaginary conic is defined.

We now consider some issues of mapping a pair of imaginary conics onto a pair of real conics.

Let two imaginary conics $a$ and $b$ be given on the plane. Find the points of their intersection $A, B, C, D = a \times b$. We also indicate on this plane four arbitrary real points $A'$, $B'$, $C'$ and $D'$, no triple of points of which lie on one straight line. We establish a collinear transformation $\chi\left|\begin{matrix} A, B, C, D \\ A', B', C', D' \end{matrix}\right.$. Applying this transformation with respect to the initial conics, we obtain two real explicitly intersecting conics $a' = \chi(a)$ and $b' = \chi(b)$. The resulting conics are elements of a conic bundle given by the centers $A'$, $B'$, $C'$ and $D'$. Thus, a correspondence between two bundles of conics through collineation $\chi$ is established and objects of one bundle are unambiguously interpreted by the real images of the other bundle (See Fig. 7).

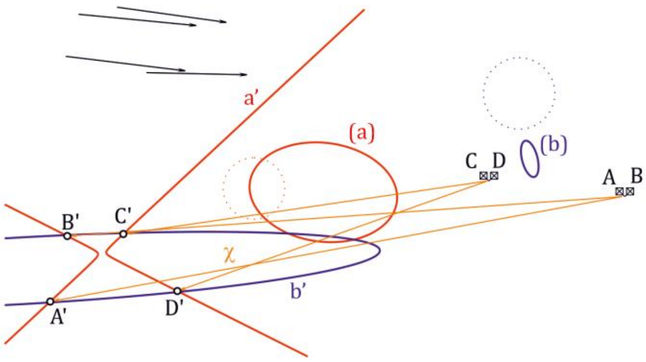


**Fig. 8.** Demonstration of unified collinear transformation of one conic to another

Having the opportunity to build the center and the main directions of the imaginary conic, it is no longer difficult to carry out a constructive solution to problems that are practically inaccessible with the traditional method of implementing geometric algorithms. Among them are the problems of finding focal points of imaginary second-order curves based on the general constructive method [3], determining imaginary quadratic involutions, and many others that have well-known real analogues.

## 4 Conclusion

As a result of the study, the following tasks were solved:

- A constructive algorithm for constructing an imaginary circle based on the inversion transformation has been developed;
- A method of indirect constructive control of imaginary quadrics through real images is presented;
- A method for organizing a visual-graphic interface of an information environment that executes geometric algorithms is proposed;
- A constructive complex of algorithms has been developed that is equally suitable for operations with real and imaginary conics, which allowed expanding the capabilities of geometric methods in solving problems associated with modeling quadrics and their multidimensional analogues using planar drawing methods.

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