A Panorama of Fuzzy Sets: Dedicated to the 100th Anniversary of the Birth of Lotfi Zadeh

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Abstract

The paper overviews a variety of fuzzy set representations that appeared after the publication of the seminal work «Fuzzy Sets» by Lotfi A. Zadeh in 1965. Primarily, a brief reminder of its contents is given. Before considering various approaches to specifying fuzzy sets, basic postulates of classical set theory are discussed. Lesniewski's Mereology and Vopenka's Alternative Set Theory are shown as early examples of nonclassical set models. Two types of nonstandard sets with uncertainty region depending on observer's awareness are envisaged – both Overdefinite Sets and Subdefinite Sets (the latter were extensively studied by Narin'yani). Besides, some natural versions of vagueness representation, called Flou Sets and Nebular Sets respectively, are mentioned. Three main interpretations of fuzzy sets are clarified. Basic ways of extending the canonical approach by Zadeh are considered. Among them the definition and use of Goguen's L-Fuzzy Set is of special concern. The formalisms of both Homogeneous Vector-Valued and Heterogeneous Fuzzy Sets are presented. Two specific approaches of coping with uncertainty – Orlov's Random Sets and Hirota's Probabilistic Sets are highlighted. The representation theorem for fuzzy concepts is stated that establishes isomorphism between Zadeh's class of Fuzzy Sets and family of Nested Sets.
Nonstandard fuzzy sets providing unified models to deal with various sides of imperfect information are analyzed. In particular, Interval-Valued Fuzzy Sets touch both fuzziness and imprecision of membership specification, Atanassov's Intuitionistic Fuzzy Sets equipped with both membership and non-membership functions envisage incomplete information, Dubois and Prade's Two-Fold Fuzzy Sets also take into account incomplete information by specifying membership possibility and membership necessity degrees, and Zhang's Bipolar Fuzzy Sets express some overlapping of opposite properties. Some original author's results, such as two bases of operations over Interval-Valued Fuzzy Sets, representation of Level Fuzzy Sets as Parameterized Fuzzy Sets, introduction of Level Intuitionistic Fuzzy Sets are presented. The frontier between Nonstandard Fuzzy Sets and Hybrid Fuzzy Sets is rather vague, but usually hybrid fuzzy sets integrate two or more various basic types of sets, e.g. Fuzzy Sets and Rough Sets, Fuzzy Sets and Soft Sets, Fuzzy Sets and Multisets. A rather new concept of Hesitant Fuzzy Set is explained. Among hybrid fuzzy sets the main attention is paid to Rough Fuzzy Sets and Fuzzy Rough Sets, Fuzzy Multisets and Multi-Fuzzy Sets, Fuzzy Soft Sets. Some more sophisticated hybrids like Rough Fuzzy Soft sets, Interval-Valued Fuzzy Soft Sets are already formed. This overview can be useful for selecting an adequate formalism to cope with various sides of imperfect information.

Keywords 1

Fuzzy set, L-Fuzzy Set, Nested Set, Nonclassical Set, Overdefinite Set, Subdefinite Set, Flou Set, Representation Theorem, Totally Fuzzy Set, Fuzzy Level Set, Interval-Valued Fuzzy Set, Intuitionistic Fuzzy Set, Rough Fuzzy Set, Fuzzy Rough Set, Fuzzy Multiset

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1. Introduction

The significance of scientific theory strongly depends on its impact to social and everyday life, as well as possibilities of its further evolution and embedding into a changing world. Over 55 years of fuzzy set theory it has gone from difficult initial recognition to widespread use in almost all social areas (economics, industries, military field, etc.) and household items (fuzzy fridges, fuzzy washing machines, fuzzy vacuum cleaners, and so on). The seminal paper by L. Zadeh [1] has stimulated a lot of various approaches for representing information fuzziness, uncertainty, imprecision, ambiguity. Some of them extend canonical Zadeh's fuzzy set theory and others use quite different ideas.

In [2, p. 9] the arrival of fuzzy set theory has been related to "generalization and rethinking of many-valued logics, extensions of probability theory and mathematical statistics, development of discrete mathematics (in particular, weighted graphs)". The objective of this paper is both the analysis of some historical prerequisites for fuzzy set theory such as nonclassical set theory and review of existing representations of fuzzy sets and close formal models, including nonstandard fuzzy sets and hybrid fuzzy sets together with the links between them.

In 1965 Lotfi Zadeh published his pioneering paper "Fuzzy Sets" [1] in Information and Control. Primarily, he noticed that the theory of fuzzy set could be viewed as an attempt to develop a body of concepts for dealing in a systematic way with such important type of imprecision as a lack of sharp boundaries in a class of objects (common examples "young women", "small cars", "narrow streets", "short sentences", "funny jokes" were cited). Membership in such classes is a matter of degree, Thus, informally, a fuzzy set may be considered as a class in which there is a graduality of progression from membership to nonmembership. Instead of classical characteristic function $f \in \{0, 1\}$ we have a membership function $\mu \in [0, 1]$, and in this perspective, a set in the conventional mathematical sense can be viewed as a special case of fuzzy set. However, a fuzzy set is assumed to be embedded in a nonfuzzy universe of discourse: it is a subset of a universal set.

In [1] *Fuzzy Set* is defined as a family of ordered pairs

$$A = \{(x, \mu_A(x))\} \text{ or simply by a membership function } \mu: X \to [0, 1]. \quad (1)$$

The membership function associates with each element $x \in X$ its membership value $\mu_A(x)$ (the grade of membership of x in A). According to L. Zadeh, a basic grade of membership $\mu_A(x)$ interpretation is the degree of compatibility between x and the concept represented by A.

Key attributes of fuzzy sets such as its support, height, crossover point were introduced. Besides, an α-level set (α-cut) of fuzzy set is a nonfuzzy set denoted A_α that comprises all elements of X whose grade of membership in A is greater or equal to the level α, $A_\alpha=\{(x \in X \mid \mu_A(x) \geq \alpha\}$, $\alpha \in [0, 1]$. So a variant of transition from fuzzy set to ordinary crisp set by using α-level as a sort of decision threshold was outlined. Moreover, any fuzzy set can be decomposed into its level sets through the resolution identity.

The level set being a union of at most countable number of intervals is an extension of an interval concept $I = \{x \in X \mid a \leq x \leq b\}$, where a and b are consequently left and right boundaries of interval. Therefore, interval algebra can be viewed as a special case of α-cuts algebra.

Main definitions of complement, union, intersection operations were specified: already in [1] L. Zadeh considered three various versions of intersection (min, product, bounded difference) and union (max, probabilistic sum, bounded sum). New unary operations of concentration and dilation of membership function were introduced by taking square and square root respectively. The containment of fuzzy set was proposed too. The Cartesian product of fuzzy subsets was built, both fuzzy relations and their projections have been defined. The concept of a cylindrical extension was suggested to provide an intuitively appealing interpretation of the composition of fuzzy relations.

An extension principle for fuzzy sets was introduced to allow the extension of a mapping or relation from points in X to fuzzy subsets of X. Fuzzy sets of type 2, i.e. fuzzy sets with fuzzy membership functions were proposed. A possibility/probability consistency principle was formulated. The concepts of fuzzy proposition and fuzzy restriction were considered, translation rules for fuzzy propositions were constructed.

Below we shall start with a short description of two main postulates of classical set theory and reasons for their breaking.

2. Alternative Set Theories and Nonclassical Sets Depending on Observer’s Awareness

Classical sets satisfy the following basic postulates – membership postulate and distinguishability (or discernibility) postulate. According to membership postulate, any element either belongs or does not belong to the set. The discernibility postulate means that any set is considered as a collection of different, clearly discernible elements that can be enumerated, presented in the form of a list. Moreover, in classical sets repeating elements are not allowed.

The rejection of these two rigid postulated leads to the emergence of nonclassical set theories. Early examples of such theories are *Mereology* by S. Lesniewski [3] and *Alternative Set Theory* (AST) by P. Vopenka [4]. In the latter classes are viewed as more general categories with respect to sets and concepts of semisets are used. A class is defined as any collection of sets that can be unambiguously determined by a property that all its members share. To differ from sets the concept of membership is not employed here. A semiset is a proper class contained by a set.

So the main Vopenka’s principle may be formulated as follows: the set-theoretic universe is so large that in every proper class some members are similar to others, with this similarity formalized through elementary embeddings.

In AST an interpretation of infinity as vagueness inherent in «boundless finite» is adopted. In the framework of relativism the concept of horizon is suggested (near which phenomena of indistinguishability and fuzziness are encountered).

In some sense, the AST rises to mereology. Mereology (partonomy) is the theory of parthood relations, where both relations of part to whole and relations of part to part within a whole are considered. It falls outside the scope of classical set theory. In Cantor’s set theory the distinguishability postulate and empty set are widely used. To differ from Cantor’s approach mereology: a) makes emphasis to the wholistic nature of set as a collective class; b) is based only on “part of” relation; 3) does not employ empty set. It is founded on the following axioms: 1) everything is part of itself (reflexivity axiom); 2) two distinct things cannot be part of each other: if *A* is a part of *B*, then *B* is not a part of *A* (anti-symmetry axiom); 3) any part of any part of the thing is itself part of this thing: if *A* is a part of *B* and *B* is a part *C*, then *A* is a part of *C* (transitivity axiom).

Thus, the parthood relation is a partial ordering. Due to the absence of singular expressions such a mereological approach gives the opportunity of useful information granulation by nonclassical sets.

Let us consider a nested set as a basis for constructing non-classical sets with uncertainty area and flexible boundaries depending on observer’s awareness. It can be given by a triple

$$X=(X^+, X^-, X^0), \qquad (2)$$

where $X^+=\{x|\ x \in X\}$, $X^-=\{x|\, x \notin X\}$, $X^0=\{x|\ x?X\}$ or by three-valued characteristic function $f \in \{1, 0.5, 0\}$. The way of interpreting $x?X$ or 0.5 leads to specifying such uncertainty sets as either *Overdefinite* or *Subdefinite* sets. In case of overdefinite set we have redundant, ambiguous information about membership $x\ (\in \wedge \notin)\ X^0$ and subdefinite set corresponds to a lacking or incomplete information about membership $x\ (\rceil{\in} \wedge \rceil{\notin})\ X^0$. Subdefinite sets were firstly introduced by A.S. Narinyani in 1980 (see [5,6]) in the form $A = (X^+, X^-, P^l, P^u)$, where P^l, P^u are lower and upper estimates of the power of uncertainty region.

The transition from two nested sets to *n* nested sets means the specification of *n-Flou Set* (in the sense of Y. Gentilhomme [7]) and rather close concept of *Nebular Set* by V.T. Kulik [8] based on membership relation. By the way, the term “Nebular” seems to be quite suitable for a general expression of both vagueness and fuzziness, because the word “nebula” makes an association with a cloud having ill-defined boundaries and its sound reminds of non-boolean nature of fuzzy algebra.

In this section we have focused on nonclassical sets and, first of all, nested sets as natural steps towards fuzzy sets. In [9] W.Pedrycz followed an opposite way by defining a *Shadowed Set* in the form A: $X \to [0, [0,1], 1]$, i.e. giving a three-valued characterization of Zadeh’s fuzzy set. Inversely, numeric membership values are classified into three categories: full membership, complete exclusion and unknown. The codomain of *A* consists of three components: “1” viewed as a core of the shadowed set, "0" standing for excluded elements and [0, 1] interpreted as an ignorance. Here the above denotations X^+, X^-, X^0 in (2) may be also used.

3. Fuzzy Sets – Interpretations, Representation Forms and Basic Theorem

The following basic ways of developing the canonical approach by L.Zadeh can be mentioned:

- modification of membership function codomain;
- modification of membership function domain;
- transformation of mapping from the domain to codomain;
- transition from homogeneous to heterogeneous membership values.

Depending on applications, there exist three main interpretations of membership functions [10]: similarity, preference and uncertainty. According to D.Dubois and H.Prade, these three semantics can be related to such measurement issues as distance, cost and possibility. Here a fuzzy distance is often associated with fuzzy clustering. The preference interpretation means that an ordered structure is given in the universe X as follows: $x >_A y \Leftrightarrow x$ is preferred to y with respect to the property A satisfaction [11]. This preference is performed with using some membership scale: $\boldsymbol{x} >_A \boldsymbol{y} \Leftrightarrow \mu_A(x) \geq \mu_A(y)$. It is tightly connected with value (or utility) functions. Fuzzy set interpretation as uncertainty concerns both Gentilhomme's «peripheral zone» of flou set [7] and the representation of membership by possibility distribution [12].

Apart of [0,1] such membership codomains as [−1,+1] and [0, ∞) were taken too. To differ from standard extension of characteristic function of a set, in *Toll Sets* [13] the membership is related to the idea of a cost to pay. Here a function $\mu: X \to [0, \infty)$ can be viewed as a sort of cost or penalty function assigning to any element $x \in X$ an infinite cost when x is outside, no cost at all in case of full membership (for instance, birth affiliation), and taking values from (0, ∞) for intermediary costs of membership (purchased membership).

The first generalization of Zadeh's fuzzy set (1) was Goguen's *L-Fuzzy Set* that appeared in 1967 [14]

$$A: X \to L, \tag{3}$$

where the membership function took its values from distributive lattice L. Later on the formula (3) was modified to use such codomain structures as lattice ordered semigroup, semiring, category [15].

If a family of fuzzy sets $A_i \subset X, i = 1, ..., n$ expresses n properties of studied object, then each element $x \in X$ is characterized by a vector of membership values $(\mu_1(x), ..., \mu_n(x))$, describing degrees of satisfaction for these properties. So a homogeneous *Vector-Valued Fuzzy Set* [16] is given by a function

$$\mu: X \to [0, 1]^n. \tag{4}$$

More generally, a definition of homogeneous fuzzy set (4) means that for any $x \in X$ the same structure of membership function co-domain is taken. Otherwise, if for different $x \in X$ membership function can take values from different suitable mathematical structures, we obtain a *Heterogeneous Fuzzy Set* [17]. It was introduced by A. Kaufmann in 1972 in the form

$$\mu \in L_1^{\{x_1\}} \times L_2^{\{x_2\}} \times \ldots \times L_n^{\{x_n\}}, \tag{5}$$

where $L_1, \ldots, L_n$ are different lattices. The definition (5) shows the extreme case "own structure for each $x \in X$".

An important extension of Zadeh's approach was *Totally Fuzzy Set* by D.Ponasse [18]. It is given by a triple

$$(X, \mu, \sigma), \tag{6}$$

where $\mu: X \to [0,1]$ is a membership function, and $\sigma: X \times X \to [0,1]$ is an indistinguishability function.

A natural way of modifying basic fuzzy set representation by L.Zadeh consists in taking other membership domains. Let us take another universal set instead of conventional $U=X$ that implies Zadeh's definition $\mu: X \to [0,1]$. Some representative examples are $U=2^X$, $U=[0,1]^X$, $U=\Re$, where $\Re$ is a set of real numbers, etc. Hence we obtain suitable patterns to define: fuzzy numbers $\Re\rightarrow[0,1]$, fuzzy measures $2^X \rightarrow [0,1]$, fuzzy sets of type two $[0,1]^X\rightarrow[0,1]$, and so on.

A fundamental challenge consists in specifying representation theorems establishing links between various fuzzy concepts and conventional set theoretic approaches, including either links between fuzzy sets and families of crisp sets or links between different fuzzy sets. A basic representation theorem for fuzzy concepts was formulated in terms of *L*-fuzzy sets.

Let us take a *n*-flou set as family of crisp nested sets

$$M = \langle M_1, \ldots, M_n\rangle, \tag{7}$$

where $M_i \subseteq X$, $i = 1, \ldots ,n$, and $M_1 \subseteq \ldots \subseteq M_n$. A generalized *n*-flou set (7) can be written in the form [19]

$$M: L \rightarrow 2^X, \tag{8}$$

with: a) $M_0 = X$ (boundary condition); b) $\forall\ \alpha_1, \alpha_2 \in L$, $\alpha_1 \leq \alpha_2 \Rightarrow M_{\alpha 1} \supseteq M_{\alpha 2}$ (anti-monotony condition),

Now let

$$L^X = \{A \mid A: X \rightarrow L\} \tag{9}$$

be a set of all *L*-fuzzy subsets, where *L* is a complete lattice, and let take a family of ordinary subsets *X* (lattice of flou sets)*:*

$$\Phi(L) = \{(M_\alpha), \alpha \in L\}, \tag{10}$$

where $M_\alpha \subseteq X$, $\forall \alpha \in L$ и $M \vee \alpha_i = \cup M_\alpha$. The main representation theorem by C.V. Negoita and D. Ralescu [19] establishes the isomorphism between complete lattices L^X given by (9) and $\Phi(L)$ (formula (10)).

It is obvious that for $L = [0, 1]$ we have a set of fuzzy subsets denoted by $\Im[0,1]^X = \{\mu_A \mid \mu_A: X \rightarrow [0, 1]\}$ and a family of mappings $\Phi([0, 1]) = \{M_\alpha \mid M_\alpha: [0, 1] \rightarrow 2^X\}$. Any element M_α put in correspondence to each $\alpha \in [0, 1]$ some subset of the universe *X*. If in addition the boundary and anti-monotony conditions are satisfies, we obtain the representation theorem [11]: the classes $[0, 1]^X$ и $\Phi([0, 1])$ are isomorphic under intersection and union operations. Therefore, fuzzy information processing can be adequately performed in the class of crisp nested sets.

The role of representation theorem goes beyond purely set-theoretic formalisms. For instance, it can serve as a justification of equivalent use of metagraphs instead of fuzzy graphs [20].

A further development of fuzziness representation by ordinary sets is related to the concept of *Soft Set* by D. Molodtsov [21]. Its softness consists in the flexibility of the boundary depending on the parameters. Let *X* be a universal set and *P* stands for a set of parameters, $Q \subseteq P$. A soft set is defined by a pair (φ, Q), where φ: $Q \rightarrow 2^X$. In other words, the soft set specifies a parameterized family of subsets of the universe *X*.

An earlier representation of fuzzy set by A. Orlov [22] transforms it into a *Random Set* based on Kolmogorov's probability space (Ω, B, Pr), where Ω is a non-empty space of elementary events, B stands for a field of Borel sets (σ-algebra of subsets) and Pr is a probability measure. A random set is given by a mapping: A: $\Omega\rightarrow 2^Y$, where the inverse image of any subset $X \subset Y$ is measurable, i.e. $A^{-1}(X) \in$ B.

4. Nonstandard Fuzzy Sets and Hybrid Fuzzy Sets

Among nonstandard and hybrid fuzzy sets level fuzzy sets, interval-valued and intuitionistic fuzzy sets, two-fold fuzzy sets are of special concern.

For many practical applications it is enough considering fuzzy sets defined on subsets of the universe *X*. This idea can be implemented by specifying *Level Fuzzy Sets* [23].

Let $\alpha \in [0, 1]$. We can define two types of ordinary crisp level sets A_α and $A_{\underline{\alpha}}$ of fuzzy set *A* in the following way $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$, $A_{\underline{\alpha}} = \{x \in X \mid \mu_A(x) \leq \alpha\}$.

We suggested the specification of level fuzzy sets as fuzzy sets depending on the parameter α [24].

***Definition* 1.** Level fuzzy set is a parameterized function $\mu_A(x, \alpha)$ defined as $\mu_A: X \times [0, 1] \to [0, 1]$ or in Kaufmann's denotations as a family of ordered pairs

$$A(\alpha) = \{x \in A_\alpha, \mu_{A(\alpha)}(x) = \mu_A(x)\}, \forall x \in X. \tag{11}$$

An Interval-Valued Fuzzy Set (*IVFS*) [25-28] is a natural instance of fuzzy set of type two with granular membership, when we are not able to exactly specify each membership value by a unique number, but give it approximately by some interval. In this case we try to take into account both fuzziness and imprecision. A membership order and imprecision order are considered; in this sense, any IVFS is a hybrid model

An IVFS is defined by

$$\mu_A: X \to 2^{[0,1]} \text{ or } A = \{(x, [\mu_A^l(x), \mu_A^u(x)])\}, \forall x \in X, \tag{12}$$

where $\mu_A^l(x)$ and $\mu_A^u(x)$ denote respectively lower and upper membership values for each x from X.

Let us take $\mu_A(x) = [\mu_A^l(x), \mu_A^u(x)]$; $\mu_B(x) = [\mu_B^l(x), \mu_B^u(x)]$. Two bases of operations corresponding to membership order and imprecision order were introduced in the following way (see [29]):

- *the membership order 1*:

intersection $A \cap_1 B \Leftrightarrow \mu_{A\cap_1 B}(x) = [\min(\mu_A^l(x), \mu_B^l(x)), \min(\mu_A^u(x), \mu_B^u(x))], \forall x \in X$;
union $A \cup_1 B \Leftrightarrow \mu_{A\cup_1 B}(x) = [\max(\mu_A^l(x), \mu_B^l(x), \max(\mu_A^u(x), \mu_B^u(x))], \forall x \in X$;
complement: $A'_1 \Leftrightarrow \mu_{A'_1}(x)\ [1 - \mu_A^u(x), 1 - \mu_A^l(x)], \forall x \in X$;

- *the imprecision order 2*:

intersection $A \cap_2 B \Leftrightarrow \mu_{A\cap B}(x) = [\max(\mu_A^l(x), \mu_B^l(x)), \min(\mu_A^u(x), \mu_B^u(x))], \forall x \in X$;
union $A \cup_2 B \Leftrightarrow \mu_{A\cup_2 B}(x) = [\min(\mu_A^l(x), \mu_B^l(x), \max(\mu_A^u(x), \mu_B^u(x))], \forall x \in X$;
complement: $A'_2 \Leftrightarrow \mu_{A'_2}(x) = [0, 1] \setminus [\mu_A^l(x), \mu_A^u(x)], \forall x \in X$.

A useful counterpart of interval-valued fuzzy set is Hirota's *Probabilistic Set* [30] with randomized membership function, where membership is characterized by an expected value and variance appears due to random noise.

Let us point out that Zadeh's membership function is a direct extension of characteristic function for classical set and meets conventional membership and distinguishability postulates. The idea of membership completeness brings about the compensation principle of membership and non-membership: more is membership μ, less is non-membership ν. Moreover, a strict negation condition $\nu_A(x) = 1 - \mu_A(x), \forall x \in X$ is supposed. In other words, the membership function is usually viewed as defined everywhere: membership gaps and gluts are forbidden.

A rather independent consideration of membership and non-membership values was proposed in 1986 by K.Atanassov, the founder of *Intuitionistic Fuzzy Set* (IFS) [31,32]. An IFS is defined by an ordered triple

$$A = \{(x, \mu_A(x), \nu_A(x)\}, \forall x \in X, \tag{13}$$

where $\mu_A(x)$ specifies the degree of membership, and $\nu_A(x)$ – the degree of non-membership of the element x to the set A, $A \subset X$. Here $\mu_A: X \to [0, 1]$, $\nu_A: X \to [0,1]$, and for every element $x \in X$ $0 \le \mu_A(x) + \nu_A(x) \le 1$. Besides, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation margin of x in A.

Let us take two IFS $A=\{(x,\mu_A(x),\nu_A(x)\}$ and $B=\{(x,\mu_B(x),\nu_B(x)\}$. Main set-theoretic operations for IFS are the following:

union $A \cap B = \{(x, (\max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))\}, \forall x \in X$;
intersection $A \cap B = \{(x, (\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))\}, \forall x \in X$;
complement $A' = \{(x, \nu_A(x), \mu_A(x)\}, \forall x \in X$.

An inclusion $A \subseteq B$ is defined by the conditions $\mu_A(x) \le \mu_B(x))$ and $\nu_A(x) \ge \nu_B(x), \forall x \in X$.

In [33] E. Kerre and G. Deschriver proved the existence of tight relationship between IVFS (12) and IFS (13) by using the concepts of L-fuzzy sets and bilattices (more specifically, bilattice-based squares and triangles).

A further extension of IFS is an Interval-Valued Intuitionistic Fuzzy Set [34] that integrates the ideas of approximate membership setting with partial independence between membership and non-membership.

The α, β-cut of IFS can be defined as a crisp set $A_{\alpha\beta} = \{x \in X \mid \mu_A(x) \ge \alpha, \nu_A(x) \le \beta\}, \alpha, \beta \in [0, 1]$.

***Definition* 2.** A level IFS is given by a pair of parameterized membership functions $\mu_A(x,\alpha)$ and $\nu_A(x)$ or in the form $A(\alpha, \beta) = \{(x \in A_{\alpha\beta}, \mu_{A(\alpha)}(x), \nu_{A(\beta)}(x)\}, \forall x \in X$.

A concept of bipolar fuzzy set (BPFS) rather close to IFS was generated by W. Zhang in 1994 [35] on the basis of classical bipolar scales. A natural way of specifying BPFS consists in considering pairs of opposite properties such as "strong-weak", "large-small", "good-bad" by using the symmetric membership codomain [−1, +1]. Here the membership degree 0 means that the element *x* is irrelevant to a considered dyad. The membership value (0, +1] shows that the element *x* satisfies with some degree a positive property, and the membership degree [−1, 0) indicates that *x* meets with some grade a negative property (counter-property). The status of property (positive or negative) is often conventional and depends on the problem domain. For example, large is a positive property for sumo wrestler, but a negative property for short track skater.

Let *X* be a non-empty set. A *Bipolar Fuzzy Set A* in *X* is given by a triple

$$A=\{(x, \mu_A^P(x), \mu_A^N(x)\}, \forall x \in X, \tag{14}$$

where $\mu_A^P: X \to [0, +1]$, $\mu_A^N: X \to [-1, 0]$. Generally, we can consider 3 cases:

1) $\mu_A^P(x) \neq 0$, but $\mu_A^N(x) = 0$ (in this case *x* has only positive satisfaction);
2) inversely, $\mu_A^P(x) = 0$, but $\mu_A^N(x) \neq 0$ (it means that *x* satisfies only counter-property);
3) both $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ (it is a common case in (14), when the membership function for positive property overlaps the membership function for negative property; this situation is quite realistic in case of loose relationships between scale poles.

D. Dubois and H. Prade [36] introduced *Two-Fold Fuzzy Sets* to deal with incomplete information about the membership status. A basic two-fold fuzzy set is composed of a nested pair of fuzzy sets expressing membership possibility and membership necessity degrees. In his turn, P. Ren [37] proposed generalized fuzzy sets to deal with incomplete information.

Two-fold fuzzy sets are closely related to the idea of rough sets. The concept of Rough Set was proposed in 1982 by Z. Pawlak [38]. Rough set is defined by its lower and upper approximations. It is based on an indiscernibility (equivalence) relation $R \subseteq X \times X$ defined on a set *X* that represents a lack of knowledge about elements *x* of *X*. A pair $APR = (X,R)$ is called an approximation space. Equivalence classes with respect to *R* are called elementary sets in *APR*, and any family of elementary sets forms a composite set. Then we can specify two approximations: *the lower approximation* of *X* denoted by $\underline{R}X = \{x \| x|_R \subseteq X\}$ (the greatest composite set contained in *X*) and *the upper approximation* of X denoted by $RX = \{x \| x|_R \cap X\}$(the least composite set containing *X*). The lower approximation of a set *X* with respect to *R* is the set of all objects, which are certainly *X* with respect to *R*, and the upper approximation of a set *X* with respect to *R* is the set of all objects, which are possibly *X* in view of *R*. So $\underline{R}X \subseteq X \subseteq RX$.

It is obvious that the definition of fuzzy set concerns the membership postulate of set theory, whereas the specification of rough set means the refusal from the discernibility postulate. Here fuzzy sets are usually based on continuous generalization of set-characteristic functions, but rough sets embody the idea of indiscernibility between objects in a set through partitions and quotient sets.

Generally, these two approaches are referred to uncertainty modeling: fuzzy set is a way to cope with vagueness, and rough set deals with information imprecision or coarseness. A natural way of building hybrid fuzzy sets consists in integrating fuzzy set and rough set theories.

The first researchers who decided to combine these two models in order to obtain a more realistic representation of imperfect information were D. Dubois and H. Prade [39]. They introduced both rough fuzzy sets and fuzzy rough sets. At first, *Rough Fuzzy Set* is defined as a pair of approximations of a fuzzy set in a nonfuzzy approximation space. In other words, the universe of discourse is coarsened by means of an equivalence relation to construct the upper and lower approximations of a fuzzy set. At second, *Fuzzy Rough Set* is obtained by approximating a crisp set in a fuzzy approximation space $APR = (X, \mu_R)$, where fuzzy indiscernibility relation μ_R induces fuzzy equivalence classes. New results on the representation of similarity relations by means of a fuzzy partition of fuzzy clusters are of special concern. Indeed, fuzzy rough sets provide a good background for modal logics with fuzzy modalities. At third, the approximation of fuzzy set in a fuzzy approximation space is considered.

Fuzzy rough sets were extensively studied in [40-42]. A more sophisticated hybrid – an *Intuitionistic Fuzzy Rough Set* – was suggested in [43].

Another non-standard fuzzy set allowing both gradual membership and multiple instances for each of elements is a fuzzy multiset (see [24]). In a multiset (or a bag) the repetitive elements are significant. An ordinary *Multiset* is given by two basic functions: a characteristic function f: $X \to \{0, 1\}$ and a multiplicity function m: $X \to N_0$, where $N_0 = \{0, 1, 2, \ldots\}$ is a set of non-negative integers. In practice, the positive integer number of instances for any element called its multiplicity or a count of the multiset is crucial.

Let us associate x with the type of element from the set of types X. On the one hand, a standard multiset interpretation is a set with n instances viewed as exact copies of each type. On the other hand, if we interpret the multiplicity of an element as its weight n=w, then multisets can be seen as sets with weighted elements. It can be written in the form $A=\{w_1*x_1, w_2*x_2, \ldots, w_n*x_n\}$, where $w_i = w_A(x)$ and $*$ stands for algebraic product.

R. Yager [44] was the first to consider *Fuzzy Multisets* under the name of fuzzy bags. According to him, an element $x \in X$ may occur more than once with possibly the same or different membership values $\mu(x)$. For example, $\boldsymbol{A} = \{(x_1, \{0.2, 0.3\}), (x_2, \{0.5, 0,5, 0.6\}), (x_3, 0,9, 0,9, 0,9)\}$is a fuzzy bag.

So Yager's *Fuzzy Bag* ***A*** is written in the form

$$\mu_A: X \times [0,1] \to N_0 \text{ or } \boldsymbol{A} = \{(x_i \mid \mu_{A^1}(x_i), \ldots, \mu_{A^n}(x_i), I = 1, \ldots, n\}, \mu_i \in [0, 1], \quad (15)$$

i.e. the element x appears n times with a membership value $\mu(x_i)$ in a fuzzy bag ***A***. Here we deal with possibly inexact copies of x having degrees of similarity $\mu_i(x)$ with respect to some standard. So a fuzzy bag can be also viewed as a kind of parameterized fuzzy set $\mu_A(x, \alpha)$, where the parameter $\alpha \in [0, 1]$ corresponds to the level of similarity above.

In [45] a different approach with respect to Yager's definition (15) was used. Fuzzy bags are characterized through their α-cuts and ω-cuts (level fuzzy sets). The α-cut of a fuzzy bag ***A*** is defined as the crisp bag A_α which includes all the occurrences of the elements of a universe X whose grade of membership in ***A*** is greater than or equal to the degree α, $\alpha \in (0, 1]$. The number of occurrences x_i of the element x in A_α is denoted: $\omega_{A\alpha}(x)$.

Let $\omega_A(x.g)$ be the number of occurrences of the element x in ***A*** associated with the grade of membership g. Each fuzzy bag can be represented by its α-cuts via the formula: $\omega_{A\alpha}(x) = \sum \omega_A(x, g)$, (for $g \geq \alpha$). The α-cuts of a fuzzy bag are nested crisp bags and a fuzzy bag can be represented by the family of all its α-cuts.

Furthermore, the ω-cut of a fuzzy bag A is the fuzzy set A^ω such that the grade of membership of the element x in A^ω denoted by $\mu_{A\omega}(x)$ defines the extent to which A contains at least ω occurrences of x, $\omega \in N$ (the set of positive integers). Here $\mu_{A\omega}(x)$ is seen as the best α, such that A_α contains at least ω occurrences of x. So a fuzzy bag can be represented by a family of the nested fuzzy sets A^ω.

Thus, the ω-cuts define a homomorphism between fuzzy sets and fuzzy bags. The operations on fuzzy sets can be easily extended to fuzzy bags.

In [45] the cardinality of a fuzzy bag and some operations on fuzzy bags using fuzzy numbers were also introduced. Fuzzy multisets and their operations were extensively studied by S. Miyamoto [46]. The concept of *Multi-Fuzzy Set* in some sense dual to Yager's fuzzy bag was proposed in [47] in the form $X \to [0, 1] \times N_0$.

Further extension and merging of the ideas of fuzzy sets of type 2, IVFS, IFS and fuzzy multisets led to the concept of *Hesitant Fuzzy Sets* (HFS)[48]. A hesitant fuzzy set is a family of ordered pairs $\mathbb{A} = \langle (x, h(x)) \rangle$, where $h_{\mathbb{A}}(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set $\mathbb{A}$. For convenience, $h_{\mathbb{A}}(x)$ is called a hesitant fuzzy element (HFE) and H stands for the set of all HFE.

A hybridization of fuzzy set and soft set concepts is also useful. Let X be a universal set, P a set of parameters, and φ is a mapping from P to a set of fuzzy subsets in X. *A Fuzzy Soft Set* [49] is defined by a pair (φ, P), where φ: $P \to [0, 1]^X$. In other words, the fuzzy soft set is a parameterized family of fuzzy subsets of the universe X.

Other hybrid sets such as rough fuzzy soft sets, interval-valued fuzzy soft sets (IVFSS) were proposed in [50-52]. In case of IVFSS we take a mapping φ^I from P to the set of interval-valued fuzzy sets. The hybridization of various nonstandard fuzzy sets and their various counterparts seems to become a main trend in modern fuzzy set theory.

5. Conclusion

This paper is not only the retrospective of various fuzzy sets, dedicated to the 100th anniversary of the birth of the "Father of Fuzzy Logic", that tends to show a great fruitfulness of the basic concepts proposed by L.A. Zadeh and revive the discussion about important and useful results of their development obtained during 55 years. It is also a "memory on the future", a kind of anticipation of new branches of fuzzy mathematics, starting from parameterized fuzzy sets and fuzzy graphs of new types such as two-fold fuzzy graphs, fuzzy rough graphs, bipolar fuzzy soft graphs, fuzzy multigraphs and so on. Moreover, it stimulates the development of new hybrid models related to Soft Computing, which are based on nonstandard and heterogeneous fuzzy sets, together with new generation neural networks and various bionic algorithms. It is a necessary prerequisite to conceiving a sort of Mendeleev's table for vagueness and uncertainty factors. The blossom era of fuzzy mathematics and its applications still continues.

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7. References

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