Opinion Dynamics Models with Noise

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Abstract. The paper analyses noise effect on opinion dynamics behaviour in connection with different noise distribution parameters. The coefficient of variation, which determines the degree of noise homogeneity, was chosen as the principal characteristic of noise. In this study, a dependence was detected between the degree of noise homogeneity and the consensus-reaching time in Friedkin-Johnsen and Hegselmann-Krause opinion dynamics models.

Keywords: Opinion dynamics · Consensus · Noise · Modelling · Hegselmann-Krause Dynamics · Friedkin-Johnsen Dynamics · Coefficient of Variation · Stochastic Process

1 Introduction

Individual opinion dynamics in social systems is a complex process, which shapes the relationships among individuals inside the system depending on similarities and differences between individuals, their inner conflicts, presence or absence of leaders in groups. Opinion dynamics is construed as the process of opinion formation in a group through a fusion of individual opinions in which interacting agents within a group continuously update and fuse their opinions on the same issue based on the established fusion rules and reach a consensus, polarization, or fragmentation in the final stage [10]. Opinion formation in a group is a stochastic process, since agents' opinions are random variables with some distribution.

In a majority of mathematical models for opinion dynamics in social systems [2,3,8,19,26,27], the final state of the dynamics is either a perfect consensus or a division of the population into groups whose members share a certain opinion [22]. In reality however, public opinion in social groups does not reach such ideal states of complete consensus [30]. To obtain more realistic results from the modelling of opinion dynamics in social groups, an additional random element is introduced, termed noise impact or, simply, noise. The authors of [28] view noise as "free will" owing to which individuals can change their opinion randomly, while in [39] noise is interpreted as a replacement of individuals with new ones in groups of non-fixed size. Noise can also be interpreted as partial loss of information or its distortion in the course of interactions for various reasons. Research of the noisy opinion dynamics models is a way to analyse the noise resistance of opinion dynamics and to expose new, previously not investigated, complex phenomena in social systems generated by various noise interventions. This is particularly important when studying how opinions are formed in various societies in the situation where many activities are migrating to the digital space, where there are multiple factors that influence, directly or indirectly, the process of decision making by individuals or groups.

This article explores the stochastic process of opinion formation in the presence of noise using the Friedkin-Johnsen and Hegselmann-Krause opinion dynamics models as examples. The two models were chosen owing to their continuous nature, availability of an individual parameter for agents in groups, which represents the agent's susceptibility to influence from the rest of the group in the former dynamics, and the confidence threshold in the latter. The coefficient of variation, which defines the degree of noise homogeneity, was chosen as the principal characteristic of noise. The study aims to reveal the relationship between the degree of noise homogeneity and the consensus-reaching time in the cases of uniform and normal noise distribution.

The article is structured as follows. The second section briefly reviews the literature on the topic. The third section examines the effect of noise on the convergence of Friedkin-Johnsen and Hegselmann-Krause opinion dynamics for the cases of uniform and normal noise distribution taking into account changes in the coefficient of variation. The conclusions summarise the key findings of the study.

2 Literature Review on the Subject

The papers [10,22,34] give a classification of the existing models for opinion dynamics based on the format of opinion expression by agents, according to which the pool of opinion dynamics models can be divided into continuous or discrete models. The authors of [10,34] additionally distinguish a group of hybrid models. The group of continuous models includes the classical DeGroot model [1,8], the Deffuant et al. model [9], the Hegselmann-Krause model [19], as well as the Friedkin-Johnsen model [15,16], which is one of the few to have been experimentally validated in small and medium-size groups [17]. A distinctive feature of continuous models for opinion dynamics is that the opinion of any agent is expressed as a continuous variable, so that situations can be simulated where the opinions of agents in a group regarding an issue vary continuously over time, taking any values ranging from "fully agree" to "completely against" [28]. Discrete opinion dynamics models, studied in the papers [6,7,14,20,31,35,36,38], are used to model situations with a limited choice of solutions for an issue. The group of hybrid models encompasses the continuous opinions and discrete actions (CODA) model [23,24,25], the DeGroot-Friedkin model [21], and the model for public opinion dynamics in an online-offline social network context [11,13].

Exposure to noise in the process of opinion formation in social systems plays a key role, wherefore one of research alleys is the design and analysis of opinion dynamics models with noise [4,5,12,28,41]. The effects of noise on opinion formation are the most commonly studied by the continuous Deffuant [4,28,29] and Hegselmann-Krause [30,37,41] models. Both models employ the bounded confidence concept, which implies that agents can influence one another only if the distance between their opinions is below a certain threshold, and the mechanism for common opinion formation based on the averaging rule. The fundamental difference between these models is the mode of interactions between agents in groups. In the Hegselmann-Krause model, agents interact in a large group, whereas the Deffuant model focuses on pairwise interactions only. The original Deffuant and Hegselmann-Krause models are studied and their similarities and distinctions are described in the papers [22,40].

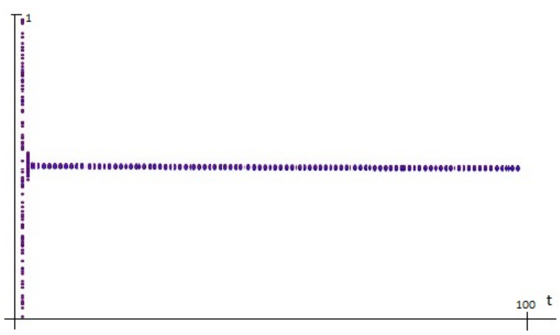

Fig. 1. The original Friedkin-Johnsen opinion dynamics.

The paper [30] explores the noisy Hegselmann-Krause model for opinion dynamics where agents are allowed, with certain probability, to spontaneously change their opinion to another one, selected randomly inside the opinion space. This paper analysed how the opinion dynamics evolved in the context of unbounded and bounded random opinion jumps inside the whole opinion space or in a limited interval centred around the current opinion. The authors of [30] have also scrutinised the similarities and differences of this noisy Hegselmann-Krause opinion dynamics and the matching Deffuant opinion dynamics, described previously in [4,29].

The convergence conditions for the noisy Hegselmann-Krause opinion dynamics are the main subject for the papers [37,41]. The paper [37] provides a rigorous theoretical analysis of the consensus behaviour of opinion dynamics in noisy environments and demonstrates that noise helps to "synchronise" opinions.

The main difference of this article from the above works on the study of the behavior of the dynamics of opinions with noise is a new approach to the study of the influence of noise, based on the dependence of the rate of convergence of the dynamics on the degree of homogeneity of the noise. We explore the effect of homogeneous and heterogeneous noise on the behaviour of known opinion dynamics provided that the noise is a random variable with uniform or normal distribution.

3 Continuous Opinion Dynamics Models with Noise

In this section we will examine two continuous opinion dynamics models which use the averaging rule in opinion formation, and investigate the effect of noise perturbations represented by independently and identically distributed random variables on the convergence of these dynamics. Firstly we take the Friedkin-Johnsen opinion dynamics for the model, and then we analyse the Hegselmann-Krause model, whose main distinction from the former is that it uses the bounded confidence concept. The convergence of both opinion dynamics will be investi-

Fig. 2. Noisy Friedkin-Johnsen dynamics with $CV = 0.2$.

gated from the point of view of the effect produced on the convergence of agents' opinion dynamics by the noise variation coefficient, which defines the degree of noise homogeneity.

3.1 Friedkin-Johnsen Model with Noise

The Friedkin-Johnsen model describes the stochastic process of opinion formation in a social group with n agents under the condition that each agent has an opinion [18] of their own on the issue, which tends to change in the process of discussions depending on the agent's susceptibility to the opinions of other members of the group.

In the original model, the opinion at current time instant is formed as the mean weighted value of the agent's initial opinion and the opinions of all agents at the preceding time instant according to the formula [15,16]:

$$y(k+1) = \Lambda W y(k) + (I - \Lambda) y(0), \tag{1}$$

where $y_i(k) \in [0,1]$ — is the value of i-th agent's opinion at k-th time instant; $y(0) = (y_1(0), y_2(0), \dots y_n(0))^T$ is the vector of the agents' initial opinions; $y(k) = (y_1(k), y_2(k), \dots y_n(k))^T$ is the vector of the agents' opinions at k-th time instant; $W = [w_{ij}]$ is the stochastic matrix of the agents' influences on one another ($0 \le w_{ij} \le 1$, $\sum_{j=1}^{n} w_{ij} = 1$, $i, j = 1, \dots, n$); $\Lambda = I - diag(W)$ is the diagonal matrix of the agents' susceptibility to the influence (opinion) of others.

In the original opinion dynamics in a group of $n = 100$ agents holding different opinions $x_i \in [0,1]$ it is already since the third stage of the discussion that all agents in the group hold the same opinions, i.e. reach a consensus [18] (see Figure 1).

Fig. 3. Noisy Friedkin-Johnsen dynamics with $CV = 0.33$.

When noise is added to the model for the dynamics (1), the values of agents' opinions may go beyond the interval $[0,1]$. To exclude such situations, a boundary condition for absorption is used, which stipulates that all opinion values beyond the interval $[0,1]$ are equated to 0 if the values are to the left of 0, or to 1 if the values are to the right of 1. Thus, agents' opinion dynamics in the noisy Friedkin-Johnsen model is given by the following formula:

$$y_i(k+1) = \begin{cases} 0, \quad y_i^*(k+1) < 0, \\ y_i^*(k+1), \quad 0 \le y_i^*(k+1) \le 1, \\ 1, \quad y^*(k+1) > 1, \end{cases} \tag{2}$$

for any $i \in \{1, 2, \ldots, n\}$ and $k \geq 0$, where

$$y_i^*(k+1) = \Lambda W y_i(k) + (I - \Lambda) y_i(0) + \xi_i(k) \tag{3}$$

for any $i \in \{1, 2, \ldots, n\}$ and $k > 0$. The noises $\{\xi_i(k)\}$ are independently and identically distributed random variables with the distribution function F.

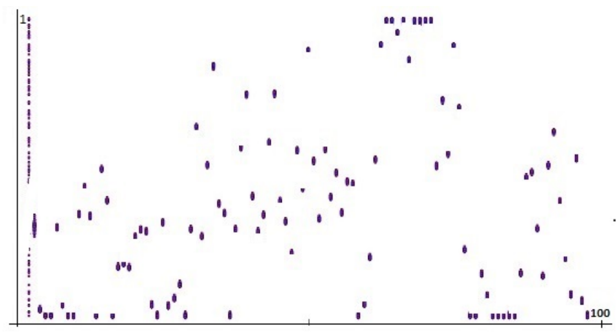

Fig. 4. Noisy Friedkin-Johnsen dynamics with $CV = 0.7$.

Consider the effect of noise homogeneity on the convergence of the dynamics (2)-(3) in the case of a uniform and normal noise distribution. Noise is said to be homogeneous if its coefficient of variation is below 0.33 [32,33]. In the case of uniform noise distribution, the opinion dynamics (2)–(3) would always converge if noises are homogeneous, but the convergence time would increase compared to the original dynamics (1). Where noises are heterogeneous, the opinion dynamics (2)–(3) converges to a consensus if the coefficient of variation takes a value in the interval $[0.33, 0.65)$, and diverges if the coefficients of variation satisfy the condition $CV \geq 0.65$. Numerical simulation results indicate that in the case of reaching a consensus it is not only the time of convergence that is enlarged by an increase in the coefficient of variation but also the scatter of opinions. The convergence behaviour of the opinion dynamics (2)–(3) in a group of $n = 100$ agents given different coefficients of variation is shown in Figures 2, 3 and 4. The scatter of opinions in Figure 4 can be interpreted as a chaos of opinions inside the group. Table 1 shows the numerical simulation results for the convergence period t_U of the Friedkin-Johnsen dynamics for different coefficients of variation in the case of uniform noise distribution. In the normal noise distribution case, numerical simulation yields similar results: the opinion dynamics (2)–(3) converges to a consensus when the coefficient of variation takes values that satisfy the inequality $CV < 0.5$, and does not reach consensus if $CV \geq 0.5$. Observe that if noise in the opinion dynamics (2)–(3) has a normal distribution, a chaos of opinions

Table 1. Friedkin-Johnsen dynamics convergence period in the case of uniform noise distribution.

CV	0.01	0.15	0.20	0.3	0.33	0.55	$[0.65, 1]$
t_U	$[6; 100]$	$[12; 100]$	$[20; 100]$	$[21; 100]$	$[30; 100]$	$[85, 100]$	—

would occur sooner than if noise is distributed uniformly. Another distinction of the uniform distribution case is the discontinuous nature of consensus, i.e. there exist several convergence regions (see Figure 5). As the coefficient of variation

Table 2. Friedkin-Johnsen dynamics convergence period in the case of normal noise distribution.

CV	t_N
0.01	$[3; 45] \cup [47; 100]$
0.15	$[5; 45] \cup [50; 100]$
0.20	$[5; 25] \cup [37; 100]$
0.30	$[5; 30] \cup [45; 85] \cup [90; 100]$
0.33	$[3; 10] \cup [13; 20] \cup [21; 25] \cup [35; 54] \cup [56; 65] \cup [68; 72] \cup [75; 80] \cup [85; 100]$
$[0.5, 1]$	—

grows in the opinion dynamics (2)–(3), the number of such convergence regions increases, as shown in Table 2.

Thus, numerical simulation of the noise variation coefficient effect on the convergence of the opinion dynamics (2)–(3) revealed that irrespective of the noise distribution pattern, the dynamics converges to a consensus when noise is homogeneous, and either converges or diverges in the presence of heterogeneous noise depending on the variation coefficient. Where noise has a normal distribution, the convergence pattern of the dynamics is discontinuous, and a chaos of opinions would occur sooner than in the case of uniform noise distribution.

3.2 Noisy Hegselmann-Krause Model

The Hegselmann-Krause opinion dynamics model is based on the bounded confidence mechanism, according to which two agents influence each other only if the distance between their opinions is below a certain threshold [30]. Keeping this concept in mind, the stochastic process of i th agent's opinion formation at any given time instant is described by the mean weighted value of i th agent's opinion and the opinions of other agents belonging to that agent's circle of trust,

$$y_i(k+1) = \left|N\left(i, y(k)\right)\right|^{-1} \sum_{j \in N(i,y(k))} y_j(k), \quad i \in \{1, 2, \ldots, n\}, \tag{4}$$

where $y_i(k) \in [0, 1]$ i th agent's opinion at k th time instant and

$$N\left(i, y(k)\right) = \{1 \le j \le n : |y_j(k) - y_i(k)| \le \varepsilon\}$$

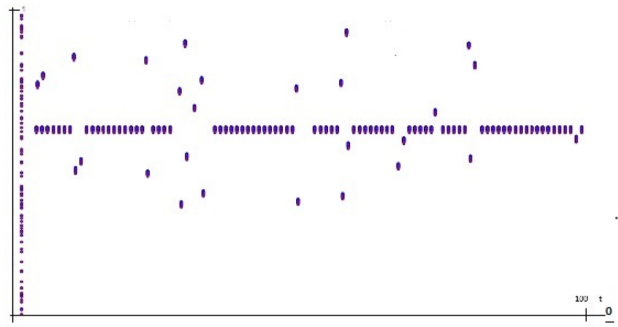

Fig. 5. Normal distribution of noise in the Friedkin-Johnsen dynamics with $CV = 0.33$.

is the set of agents belonging to i th circle of trust with the radius ε [19]. Figures 6 and 7 ïillustrate the behaviour of the opinion dynamics (4) in a group of 100 agents for different values of the radius of trust ε.

The Hegselmann-Krause dynamics convergence depends on the value of the confidence threshold ε.

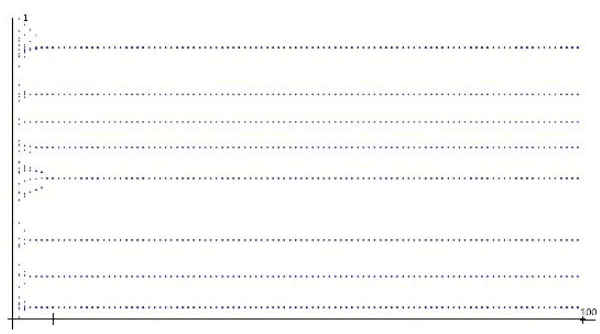

Fig. 6. The original Hegselmann-Krause dynamics with $\varepsilon = 0.05$.

A low level of trust among agents in the group leads to opinion fragmentation, i.e. the group splits into several subgroups holding similar opinions, inside which the participants reach a consensus, whereas with a high confidence threshold a consensus would be reached inside the original group. Thus, the original

Hegselmann-Krause opinion dynamics would converge to a consensus if $\varepsilon > 0.5$ and would diverge if $\varepsilon \le 0.5$. When ε values are low, the following pattern is observed: as the confidence threshold declines, the speed of opinion fragmentation in the group and the number of subgroups holding different opinions increase.

To describe the noisy stochastic process of opinion formation, noise $\xi_i(k)$, represented by a random variable with a distribution function F, is added to the original model (4) for the formation of i th agent's current opinion.

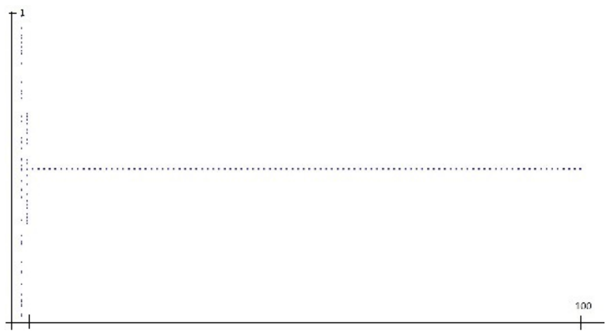

Fig. 7. The original Hegselmann-Krause dynamics with $\varepsilon = 0.6$.

In this case i th agent's opinion dynamics in the noisy Hegselmann-Krause model has the form

$$y_i(k+1) = \begin{cases} 0, & y_i^*(k+1) < 0, \\ y_i^*(k+1), & 0 \le y_i^*(k+1) \le 1, \\ 1, & y^*(k+1) > 1, \end{cases} \tag{5}$$

for any $i \in \{1, 2, \ldots, n\}$ and $k \ge 0$, where

$$y_i^*(k+1) = |N(i, y(k))|^{-1} \sum_{j \in N(i, y(k))} y_j(k) + \xi_i(k), \quad i \in \{1, 2, \ldots, n\} \tag{6}$$

with noise $\{\xi_i(k)\}$ for any $i \in \{1, 2, \ldots, n\}$ and $k > 0$ [42].

The analysis of the variation coefficient effect on the convergence rate of the opinion dynamics (5)–(6) in the case of uniform and normal noise distribution $\{\xi_i(k)\}$ showed that the effect of the confidence threshold ε on the opinion fragmentation process remains as in the original model. An increase in the coefficient of variation extends the time it takes to reach a consensus in the initial group or in the subgroups formed through fragmentation. Numerical simulation results for the convergence time of the opinion dynamics (5)–(6) in the uniform

Table 3. Hegselmann-Krause dynamics convergence period in the case of uniform noise distribution.

CV	$\varepsilon = 0.05$	$\varepsilon = 0.2$	$\varepsilon = 0.5$	$\varepsilon = 0.6$	$\varepsilon = 0.9$
0.01	$T_U = 8$	$T_U = 11$	$T_U = 9$	$T_U = 8$	$T_U = 2$
0.20	$T_U = 15$	$T_U = 18$	$T_U = 22$	$T_U = 13$	$T_U = 5$
0.33	$T_U = 60$	$T_U = 28$	$T_U = 25$	$T_U = 27$	$T_U = 20$
0.60	$T_U = \infty$	$T_U = 75$	$T_U = 88$	$T_U = 80$	$T_U = 64$
0.80	$T_U = \infty$	$T_U = \infty$	$T_U = 97$	$T_U = 84$	$T_U = 65$
0.90	$T_U = \infty$	$T_U = \infty$	$T_U = 160$	$T_U = 150$	$T_U = 145$
1	$T_U = \infty$	$T_U = \infty$	$T_U = \infty$	$T_U = \infty$	$T_U = 320$

noise distribution case are shown in Table 3. Thus, in the case of uniform noise distribution the opinion dynamics (5)–(6) converges to a consensus if noise is homogeneous: for the confidence threshold values $\varepsilon < 0.5$ convergence is possible only inside the subgroups formed through fragmentation, and for $\varepsilon \geq 0.5$ it is possible inside the whole group. Where noise is heterogeneous, the vector of the Hegselmann-Krause dynamics depends on the value of the confidence threshold ε: the dynamics diverges if ε is low and converges if its values are close to 1.

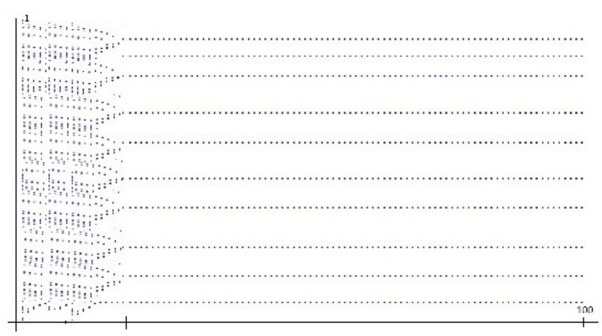

Fig. 8. Hegselmann-Krause dynamics for $\varepsilon = 0.05$ and $CV = 0.2$.

Similar results were obtained in the numerical simulation of the opinion dynamics (5)–(6) for the case of normal noise distribution. According to the results in Table 4, in the presence of homogeneous noise the opinion dynamics reaches a consensus either in some subgroups formed through fragmentation or in the whole group depending on the value of the confidence threshold ε.

With a high confidence threshold, near 1, the opinion dynamics (5)–(6) reaches a consensus in the initial group irrespective of the degree of noise homogeneity, but the convergence period is significantly extended by an increase in the coefficient of variation.

Table 4. Hegselmann-Krause dynamics convergence period in the case of normal noise distribution.

CV	$\varepsilon = 0.05$	$\varepsilon = 0.2$	$\varepsilon = 0.5$	$\varepsilon = 0.6$	$\varepsilon = 0.9$
0.01	$T_N = 10$	$T_N = 13$	$T_N = 10$	$T_N = 2$	$T_N = 2$
0.2	$T_N = 30$	$T_N = 25$	$T_N = 25$	$T_N = 15$	$T_N = 5$
0.33	$T_N = 70$	$T_N = 52$	$T_N = 50$	$T_N = 47$	$T_N = 30$
0.6	$T_N = \infty$	$T_N = \infty$	$T_N = 90$	$T_N = 85$	$T_N = 67$
0.8	$T_N = \infty$	$T_N = \infty$	$T_N = 105$	$T_N = 90$	$T_N = 70$
0.9	$T_N = \infty$	$T_N = \infty$	$T_N = 165$	$T_N = 160$	$T_N = 150$
1	$T_N = \infty$	$T_N = \infty$	$T_N = \infty$	$T_N = \infty$	$T_N = 350$

The behaviour of the Hegselmann-Krause dynamics with homogeneous and non-homogeneous noise, where the confidence threshold in a group of 100 agents is 0.05, is shown in Figures 8 and 9. They demonstrate that an increase in the coefficient of variation is accompanied by a decrease in the number of subgroups formed through fragmentation and by an increase in the scatter of opinions at the stages preceding consensus inside each of the groups. A similar situation is observed with any confidence threshold values that satisfy the condition $\varepsilon < 0.5$.

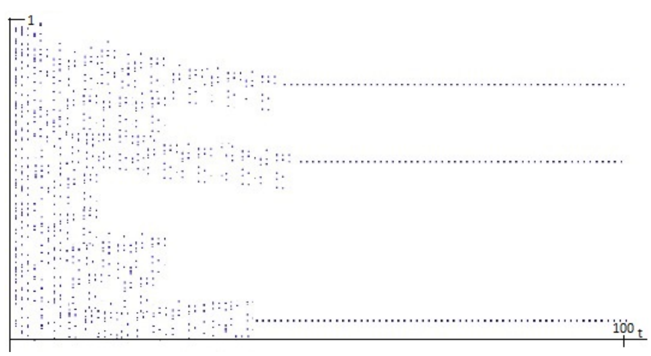

Fig. 9. Noisy Hegselmann-Krause dynamics for $\varepsilon = 0.05$ and $CV = 0.33$.

Thus, irrespective of the noise distribution pattern, an increase in the degree of noise homogeneity leads to an extension of the consensus-reaching time in the opinion dynamics (5)–(6), while the effect of the confidence threshold ε

on consensus attainment in the noisy Hegselmann-Krause dynamics remains as in the original model. Numerical simulation of the effect of the noise variation coefficient on the convergence of the dynamics (5)–(6) revealed that where the confidence threshold is high, near 1, the opinion dynamics would reach a consensus irrespective of the variation coefficient, and the latter influences only the speed of approaching consensus.

4 Conclusions

This paper explored the effect of noise on the convergence rate of the Friedkin-Johnsen and Hegselmann-Krause opinion dynamics, chosen owing to their continuous nature, availability of an individual parameter for agents in groups, which represents the agent's susceptibility to influence from the rest of the group in the former model of dynamics, and the confidence threshold in the latter. It was found through this study that irrespective of the noise distribution pattern the time it takes to reach a consensus is extended by an increase in the noise variation coefficient in both opinion dynamics. In other words, as noise becomes more heterogeneous, it gets more difficult for agents in a group to form a common opinion on an issue.

Where noise is homogeneous, the Friedkin-Johnsen opinion dynamics always reaches a consensus, while with non-homogeneous noise the dynamics would either converge or diverge depending on the coefficient of variation. Where noise has a normal distribution, this dynamics would converge in a discontinuous manner, and the convergence discontinuity periods can be interpreted as "temporary conflicts" inside the group in the process of common opinion formation.

Studying the noisy Hegselmann-Krause opinion dynamics we found that the agents' confidence threshold had the same value as in the original model. When the confidence threshold is 0.5, consensus can be reached inside the initial group, otherwise a fragmentation of opinions takes place, i.e. the group of agents splits into several opinion-sharing subgroups.

Numerical simulations with homogeneous noise showed that depending on the confidence threshold level the dynamics would converge either in the subgroups formed through fragmentation or in the whole group, but the time needed to reach a consensus would increase as noise gets less homogeneous. In the case of heterogeneous noise, opinion dynamics in a group would reach a state of "chaos" if confidence thresholds are low, whereas high confidence thresholds would lead the dynamics to a consensus, but the convergence would be slower. Thus, at any level of noise homogeneity members of consolidated groups, which have a high confidence threshold, would anyway reach a consensus in dealing with an issue, whereas a divided group would be pushed towards "chaos" even by "slight" noise.

Numerical simulation of the effect of noise on the behaviour of both dynamics revealed that the period of the dynamics' convergence to a consensus was greater in the case of normal noise distribution compared to uniform distribution. The difference in the dynamics convergence time suggests that noise with a normal distribution has a “stronger” effect on the dynamics’ behaviour than noise with a uniform distribution.

It is noteworthy that in the presence of noise consensus would be more frequently attained in the Friedkin-Johnsen dynamics model than in the Hegselmann-Krause model, due to the mode of interactions among agents inside the group. In the former mmodel of dynamics an interaction involves all agents in the group, whereas in the latter it is only among agents belonging to the circle of trust. Thus, a team split into several subgroups is more susceptible to noise effects than the whole team together. Furthermore, the more subgroups there are in a team the more susceptible it is to the influence of noise. This pattern demonstrates that the time needed to reach a consensus is directly dependent not only on noise homogeneity but also on the team’s “unity”

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