Description of the Interaction of Fermions with an Electromagnetic Field Based on Cartan Mechanics*

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Abstract. In this paper, the mechanics of Eli Cartan is used, which is an alternative to the Lagrange-Hamiltonian formalism, has certain advantages in the formulation of quantum electrodynamics. To demonstrate this fact, it was described as the interaction of fermions with an electromagnetic field.

We demonstrated the possibility of using the mechanics of E. Cartan in quantum field theory. Based on the use of these mechanics, additional conditions can be introduced directly into the Cartan equations. Such conditions include, for example, switching conditions between pulses and coordinates, as well as Lorentz calibration conditions

Keywords: Cartan mechanics; quantum electrodynamics, fermions, electromagnetic interaction.

1 Introduction

The mechanics of Eli Cartan, which is an alternative to the Lagrange-Hamiltonian formalism, has certain advantages in the formulation of quantum electrodynamics. To demonstrate this fact, we describe the interaction of fermions with an electromagnetic field.

All particles that make up the Universe fall into two groups: fermions and bosons. Graduate students of Leiden University (Holland) Samuel Gaudsmith and George Uhlenbeck introduced this distinction. Gaudsmith, who was more engaged in research, noticed an additional splitting of the emission spectrum of helium atoms. Uhlenbeck, who knew better classical physics, saw the reason for this splitting in some internal property of the electron. Together they concluded that the electron initially has a certain angular momentum - spin [1-4].

The foundations of quantum mechanics were only then laid, so this idea led to the addition of a fourth quantum number (in addition to the main, orbital, and magnetic), called the spin quantum. The electron is depicted as a tiny, rapidly spinning top, but such a description does not need to be taken literally. In 1928, the development by the British physicist P. Dirac of relativistic quantum mechanics created a theoretical basis for the spin of an electron; the guess of Gaudsmith and Uhlenbeck turned out to be very successful [5-7].

2 Main content. Interaction of fermions with an electromagnetic field based on Cartan mechanics

In 1925, the Austrian physicist Wolfgang Pauli concluded that two electrons couldn’t be in the same quantum state in the same place. This principle of Pauli's prohibition lies at the heart of the Periodic Table of Chemical Elements.

In studying the statistical behavior of electrons, the Italian-American physicist Enrico Fermi and Dirac developed the Fermi-Dirac statistics theory. Its provisions were subsequently extended to other particles with a half-integer spin. These particles, called fermions, encompass all leptons and quarks. Thus, the mass of the universe is made up of fermions [8-10].

The study of particles with zero or integer spin in 1924 was carried out by the Indian physicist Chatyatranat Bose. While working at the University of Dhaka (Bangladesh), Bose sent the results of his research for review to Einstein. He translated his work into German and strongly advised him to publish it. The following year, Einstein expanded the Bose results to include all particles that are not fermions. The statistical behavior of such particles came to be called Bose-Einstein statistics. Particles obeying these statistics, Dirac called bosons [11-13]. The carriers of all interactions — the photon in the electromagnetic, the gluons in the strong, and the W and Z particles in the weak — are bosons.

If two fermions cannot be in the same quantum state, then there is no such restriction for bosons. Indeed, the more bosons are in a certain energy state, the greater the likelihood that all other bosons will be in this state. This phenomenon underlies stimulated emission in lasers when photons are brought into the same energy state. This kind of "herd" helps to explain the superfluidity of helium and even superconductivity when the electrons collide in pairs and act like bosons. In 1995, it was possible to reduce the temperature of gaseous rubidium in such a way that all atoms found the same quantum state. Such a cluster is called the Bose-Einstein condensate [9, 10].

The tendency to “loneliness” in fermions and the “sociability” of bosons make them so dissimilar. However, this difference turns out to be decisive for the nature of the universe. For example, if fermions united like bosons, all the electrons in the atom would collect at the lowest energy level, and then there could be no talk of chemical reactions, and therefore, of life.

The electromagnetic interaction is one of four fundamental interactions. It exists between particles with an electric charge [14-17]. According to the generally accepted view, such an interaction between charged particles does not occur directly, but only using an electromagnetic field.

In the framework of quantum field theory [11, 16], such an interaction is carried by a massless boson — a photon.

Fermions are among the fundamental particles that have an electric charge and participate in electromagnetic interaction.

Along with electromagnetic, there are also weak [3, 7, 9] and strong interactions. The electromagnetic interaction is distinguished by its long-range nature. According to Coulomb's law, the force of interaction between charges decreases only as of the second power of the distance. Gravitational interaction also complies with this law, but it is much weaker than electromagnetic [17-19].

According to the classical (non-quantum) approach, electromagnetic interaction is described by classical electrodynamics [17-20].

First, we quantize the electromagnetic field.

Consider a 2-form Ω of the form:

$$\Omega = \int dV' \left\{ d\left(\frac{1}{2}\frac{\partial A^{\nu}}{\partial x'^{\alpha}}\frac{\partial A_{\nu}}{\partial x'^{\beta}} g^{\alpha\beta}\right) \wedge dt \right\} =$$

$$\int dV' \frac{\partial^2 A^{\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} d\, A_{\nu} g^{\alpha\beta} \wedge dt,\ \ \nu,\alpha,\beta = \overline{0,3} \tag{1}$$

The equation of E. Cartan for her has the form:

$$0 = \frac{\delta\Omega}{\delta dA^{\nu}} = \int dV' \frac{\partial^2 A^{\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} \delta(\vec{x}' - \vec{x})\delta_{\nu}^{\gamma}\, g^{\alpha\beta} dt = \frac{\partial^2 A^{\gamma}(x)}{\partial x^{\alpha}\partial x^{\beta}} g^{\alpha\beta} dt = 0$$

The equation: $\frac{\partial^2 A^{\gamma}(x)}{\partial x^{\alpha}\partial x^{\beta}} = 0$ – is the equation of the dynamics of the electromagnetic field vector potential.

We introduce the vectors $\vec{E}$ and $\vec{H}$, which also describe the electromagnetic field:

$$\vec{E} = -\vec{\nabla} A^0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{2}$$

$$\vec{H} = \vec{\nabla} \times \vec{A}. \tag{3}$$

Then for $\vec{H}$ and $\vec{E}$ we obtain the equations:

$$\vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$$

$$\vec{\nabla} \cdot \vec{E} = -\Delta A^0 - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} = -\Delta A^0 + \frac{1}{c^2}\frac{\partial^2 A^0}{\partial t^2} = g^{\alpha\beta}\frac{\partial^2 A^0}{\partial x^{\alpha}\partial x^{\beta}} = 0 \tag{4}$$

Lorentz calibration used here:

$$\frac{\partial A^0}{c\partial t} + \vec{\nabla} \cdot \vec{A} = 0 = \frac{\partial A^{\alpha}}{\partial x^{\alpha}} \tag{5}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \tag{6}$$

$$\vec{\nabla} \cdot \vec{H} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\,\vec{\nabla} \cdot \vec{A} - \Delta\vec{A} = -\vec{\nabla}\frac{1}{c}\frac{\partial A_0}{\partial t} - \Delta\vec{A} =$$

$$= -\vec{\nabla}\frac{1}{c}\frac{\partial A_0}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}A^0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \qquad (7)$$

This is the continuity equation for the energy of an electromagnetic field [21-23].

To record the energy of the electromagnetic field in the secondary quantization representation, we express $\vec{E}$ and $\vec{H}$ through the generalized coordinates and momenta of the electromagnetic field:

The energy of the electromagnetic field is:

$$\mathcal{H} = \frac{1}{8\pi}\int dV\left(\vec{E}^2 + \vec{H}^2\right). \qquad (8)$$

Really:

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{1}{4\pi}\int dV\left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{H}}{\partial t}\right) = \frac{1}{4\pi}\left(c\vec{\mathrm{E}}\cdot\left(\vec{\nabla}\times\vec{H}\right) - c\vec{\mathrm{H}}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right) =$$

$$-\vec{\nabla}\cdot\frac{c}{4\pi}\vec{E}\times\vec{H}$$

$$\frac{\partial}{\partial t}\int_V dV\,\mathcal{H} = \int_{dV} -\frac{c}{4\pi}V\left(\vec{\mathrm{E}}\times\vec{H},\cdots,\cdots\right)$$

This is the continuity equation for the energy of an electromagnetic field. To record the energy of the electromagnetic field in the secondary quantization representation, we express $\vec{E}$ and $\vec{H}$ through the generalized coordinates and momenta of the electromagnetic field:

$$\vec{H} = \int d\vec{K}\sum_\alpha P_{\vec{K}\alpha}(t)\vec{h}_{\vec{K}\alpha}(\vec{r})\sqrt{4\pi};\ \vec{E} = \int d\vec{K}\sum_\alpha \omega_{\vec{K}}\vec{q}_{\vec{K}\alpha}(t)\vec{e}_{\vec{K}\alpha}(\vec{r})\sqrt{4\pi}$$

And $P^{\cdot}\_\,(K^{\to}\alpha) = -$ ⟦ω ^ 2⟧ $\_K^{\to}\, q\_\,(K^{\to}\alpha)$

Using Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{H} = 0 \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \quad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

we get $\vec{\nabla}\cdot\vec{h}_{\vec{K}\alpha}(\vec{r}) = \vec{\nabla}\cdot\vec{e}_{\vec{K}\alpha}(\vec{r}) = 0.$ (9)

$$\vec{\nabla}\times\vec{E} = \int d\vec{K}\sum_\alpha \omega_{\vec{K}} q_{\vec{K}\alpha}(t)\vec{\nabla}\times\vec{e}_{\vec{K}\alpha}(\vec{r}) = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = \int d\vec{K}\sum_\alpha -\frac{1}{c}\dot{P}_{\vec{K}\alpha}(t)\vec{h}_{\vec{K}\alpha}(\vec{r})$$

Hence: $\vec{\nabla}\times\vec{e}_{\vec{K}\alpha}(\vec{r}) = \frac{\omega_{\vec{K}}}{c}\vec{h}_{\vec{K}\alpha}(\vec{r})$ (10)

and $\dot{P}_{\vec{K}\alpha} = -\omega^2{}_{\vec{K}} q_{\vec{K}\alpha}$ (11)

$$\vec{\nabla}\times\vec{H} = \int d\vec{K}\sum_\alpha P_{\vec{K}\alpha}(t)\vec{\nabla}\times\vec{h}_{\vec{K}\alpha}(\vec{r}) = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \int d\vec{K}\sum_\alpha \frac{\omega_{\vec{K}}}{c}\dot{q}_{\vec{K}\alpha}(t)\vec{e}_{\vec{K}\alpha}(\vec{r}).$$

Hence: $\vec{\nabla}\times\vec{h}_{\vec{K}\alpha}(\vec{r}) = \frac{\omega_{\vec{K}}}{c}\vec{e}_{\vec{K}\alpha}$ (12)

and $P_{\vec{K}\alpha}(t) = \dot{q}_{\vec{K}\alpha}(t).$ (13)

Thus, $P_{\vec{K}\alpha}(t)$ и $q_{\vec{K}\alpha}(t)$ are canonical Hamilton variables, their equation:

$$\begin{cases} \dot{q}_{\vec{K}\alpha} = P_{\vec{K}\alpha} \\ \dot{P}_{\vec{K}\alpha} = -\omega^2{}_{\vec{K}} q_{\vec{K}\alpha} \end{cases} \tag{14}$$

It describes a mathematical pendulum for each momentum $\hbar\vec{K}$ and polarization α.

Let us prove that $\vec{h}_{\vec{K}\alpha}$ are orthogonal to $\vec{h}_{\vec{K}'\alpha'}$ for $\vec{K} \neq \vec{K}'$, α≠α', as well as $\vec{e}_{\vec{K}\alpha}$: Using (10) we obtain [22-25]:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{e}_{\vec{K}\alpha}(\vec{r})\right) = -\Delta \vec{e}_{\vec{K}\alpha}(\vec{r}) = \frac{\omega_{\vec{K}}}{c} \vec{\nabla} \times \vec{h}_{\vec{K}\alpha} = (\frac{\omega_{\vec{K}}}{c})^2 \vec{e}_{\vec{K}\alpha}(\vec{r}). \tag{15}$$

Using (13) we obtain:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{h}_{\vec{K}\alpha}(\vec{r})\right) = -\Delta \vec{h}_{\vec{K}\alpha}(\vec{r}) = \frac{\omega_{\vec{K}}}{c} \vec{\nabla} \times \vec{e}_{\vec{K}\alpha}(\vec{r}) = (\frac{\omega_{\vec{K}}}{c})^2 \vec{h}_{\vec{K}\alpha}(\vec{r}). \tag{16}$$

Equations (15) and (16) show that $\vec{e}_{\vec{K}\alpha}(\vec{r})$ and $\vec{h}_{\vec{K}\alpha}(\vec{r})$ are eigenfunctions of the operator - Δ (self-adjoint) with eigenvalues$(\frac{\omega_{\vec{K}}}{c})^2$.

Hence: $\int dV\, \vec{e}_{\vec{K}\alpha} \cdot \vec{e}_{\vec{K}'\alpha'} = \int dV\, \vec{h}_{\vec{K}\alpha} \cdot \vec{h}_{\vec{K}'\alpha'} = \delta_{\vec{K}\vec{K}'} \delta_{\alpha\alpha'}$.

Therefore:

$$\mathcal{H} = \frac{1}{2} \int dV \left(\int d\vec{K}\, d\vec{K}' \sum_{\alpha\alpha'} \omega_{\vec{K}}\, \omega_{\vec{K}'} q_{\vec{K}\alpha}(t) q_{\vec{K}'\alpha'}(t) \times \vec{e}_{\vec{K}\alpha}(\vec{r}) \cdot \vec{e}_{\vec{K}'\alpha'}(\vec{r}) + \int d\vec{K}\, d\vec{K}' \sum_{\alpha\alpha'} P_{\vec{K}\alpha}(t) P_{\vec{K}'\alpha'}(t) \times \vec{h}_{\vec{K}\alpha}(\vec{r}) \cdot \vec{h}_{\vec{K}'\alpha'}(\vec{r}) \right) =$$

$$= \frac{1}{2} \left(\int d\vec{K}\, d\vec{K}' \sum_{\alpha\alpha'} \omega_{\vec{K}}\, \omega_{\vec{K}'} q_{\vec{K}\alpha}(t) q_{\vec{K}'\alpha'}(t) \int dV\, \vec{e}_{\vec{K}\alpha}(\vec{r}) \cdot \vec{e}_{\vec{K}'\alpha'}(\vec{r}) + \right.$$
$$\left. + \int d\vec{K}\, d\vec{K}' \sum_{\alpha\alpha'} P_{\vec{K}\alpha}(t) P_{\vec{K}'\alpha'}(t) \int dV\, \vec{h}_{\vec{K}\alpha}(\vec{r}) \cdot \vec{h}_{\vec{K}'\alpha'}(\vec{r}) \right) = \frac{1}{2} \int d\vec{K} \sum_{\alpha} \left(P^2{}_{\vec{K}\alpha}(t) + \right.$$
$$\left. + \omega^2{}_{\vec{K}} q^2{}_{\vec{K}\alpha}(t) \right). \tag{17}$$

The energy of the electromagnetic field $\mathcal{H}$ is the Hamiltonian of the Hamilton equation for the electromagnetic field (14). And can be represented in a second quantized form [26]:

$$H = \frac{1}{2} \int d\vec{K} \sum_{\alpha} \left(\omega^2{}_{\vec{K}} q^2{}_{\vec{K}\alpha} + P^2{}_{\vec{K}\alpha} \right) = \int d\vec{K} \sum_{\alpha} \hbar\omega_{\vec{K}} \left(\frac{\omega_{\vec{K}}}{2\hbar} q^2{}_{\vec{K}\alpha} + \frac{1}{2\hbar\omega_{\vec{K}}} P^2{}_{\vec{K}\alpha} \right) =$$

$$= \sum_{\alpha} \int d\vec{K} \hbar\omega_{\vec{K}} \left[\left(\sqrt{\frac{1}{2\hbar\omega_{\vec{K}}}} P_{\vec{K}\alpha} + i \sqrt{\frac{\omega_{\vec{K}}}{2\hbar}} q_{\vec{K}\alpha} \right) \times \left(\sqrt{\frac{1}{2\hbar\omega_{\vec{K}}}} P_{\vec{K}\alpha} - i \sqrt{\frac{\omega_{\vec{K}}}{2\hbar}} q_{\vec{K}\alpha} \right) + \frac{1}{2} \right] =$$

$$= \sum_{\alpha} \int d\vec{K} \hbar\omega_{\vec{K}} \left(a^+{}_{\vec{K}\alpha} a_{\vec{K}\alpha} + \frac{1}{2} \right) \tag{18}$$

The following notation is introduced here:

$$a^+{}_{\vec{K}\alpha} = \sqrt{\frac{1}{2\hbar\omega_{\vec{K}}}} P_{\vec{K}\alpha} + i \sqrt{\frac{\omega_{\vec{K}}}{2\hbar}} q_{\vec{K}\alpha} \tag{19}$$

$$a_{\vec{K}\alpha} = \sqrt{\frac{1}{2\hbar\omega_{\vec{K}}}} P_{\vec{K}\alpha} - i\sqrt{\frac{\omega_{\vec{K}}}{2\hbar}} q_{\vec{K}\alpha} \tag{20}$$

As is well known, fermions obey the Dirac equation.

To obtain it, it is enough to extract the square root from the Klein-Gordon equation.

Klein-Gordon equation:

$$g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0 \tag{21}$$

Using equality [24-26]: $\frac{1}{2}\left(\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha\right) = g^{\alpha\beta}$, we obtain from (21):

$$\frac{1}{2}\left(\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha\right) \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0 = \left(\gamma^\alpha \gamma^\beta \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \varphi + \frac{m^2 c^2}{\hbar^2} \varphi\right) =$$

$$= \left(i\gamma^\alpha \frac{\partial}{\partial x^\alpha} + \frac{mc}{\hbar}\right)\left(i\gamma^\beta \frac{\partial}{\partial x^\beta} - \frac{mc}{\hbar}\right) \varphi, \text{ from here: } \left(i\gamma^\alpha \frac{\partial}{\partial x^\alpha} - \frac{mc}{\hbar}\right) \Psi = 0. \tag{22}$$

This is the Dirac equation of the γ^α- matrix 4x4, and Ψ- is a 4-component column vector. This equation can also be obtained as the Cartan equation by taking the 2-form Ω in the form:

$$\Omega = \int d\vec{x} \{d\overline{\Psi} \wedge d\Psi i\hbar - dH \wedge dt\} \tag{23}$$

taking H in the form:

$$H = \int \overline{\Psi} \{c\vec{\alpha} \cdot \vec{p} + mc^2 \beta\} \Psi dV \tag{24}$$

Then the equations of E. Cartan give:

$$0 = \frac{\delta\Omega}{\delta d\overline{\Psi}} = \frac{\delta\Omega}{\delta d\Psi} = i\hbar d\Psi - \{c\vec{\alpha} \cdot \vec{p} + mc^2 \beta\} \Psi dt =$$

$$= -i\hbar d\overline{\Psi} - \{c\vec{\alpha} \cdot \vec{p} + mc^2 \beta\}^+ \overline{\Psi} dt \tag{25}$$

Hence:

$$i\hbar \frac{\partial \Psi}{\partial t} = \{c\vec{\alpha} \cdot \vec{p} + mc^2 \beta\} \Psi \tag{26}$$

Using $\vec{\alpha}^+ = \vec{\alpha}, \quad \beta^+ = \beta$, we obtain:

$$i\hbar \frac{\partial \overline{\Psi}}{\partial t} = -\{c\vec{\alpha} \cdot \vec{p} + mc^2 \beta\} \overline{\Psi} \tag{27}$$

We introduce new matrices to write these equations in covariant form:

$$\beta = \gamma^0 \quad \beta^2 = (\alpha^k)^2 = 1 \quad \beta\vec{\alpha} = \vec{\gamma} \quad x^\mu = (ct, \vec{r})$$

Multiplying equation (26) by$\gamma^0 = \beta$ and dividing by $\hbar c$, we obtain:

$$\left(i\gamma^\alpha \frac{\partial}{\partial x^\alpha} - \frac{mc}{\hbar}\right) \Psi = 0 \tag{28}$$

Thus, we again obtained the Dirac equation, and, therefore, proved the effectiveness of the 2-form and Cartan equations, that is, the Mechanics of E. Cartan [27-29].

To quantize the Dirac field a second time, we consider the eigenvectors and eigenvalues of the operator H of equation (26):

$$\{c\vec{\alpha}\cdot\vec{p}+mc^2\beta\}\mathcal{V}(\vec{K}',\vec{r},i')=E_{\vec{K}'i'}\mathcal{V}(\vec{K}',\vec{r},i') \tag{29}$$

We expand the solutions of equation (26) in these vectors:

$$\Psi(\vec{r},t)=\sum_{\vec{K},i}\mathcal{V}(\vec{K},\vec{r},i)a_{\vec{K}i}(\mathrm{t}) \quad i=\overline{1,4} \tag{30}$$

$$\Psi^{+}(\vec{r},t)=\sum_{\vec{K},i}\tilde{\mathcal{V}}^{*}(\vec{K},\vec{r},i)a^{+}{}_{\vec{K}i}(\mathrm{t}) \quad i=\overline{1,4} \tag{31}$$

The energy of the fermion field is equal to the sum of the energies of its quanta.

The momentum of the fermion field P $^{\rightarrow}$ is equal to:

$\mathcal{V}$ – 4 component spinor, $\tilde{\mathcal{V}}^{*}$- complex conjugated transposed spinor.

The operator of the number of fermions N has the form:

$$N=\int dV\,\Psi^{+}\Psi=\int dV\sum_{\overline{K},\overline{K}',i,i'}a^{+}{}_{\vec{K}i}\,a_{\vec{K}i}\tilde{\mathcal{V}}^{*}(\vec{K},i,\vec{r})\mathcal{V}(\vec{K}',i',\vec{r})=$$

$$=\sum_{\vec{K},\vec{K}',i,i'}a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\delta_{\vec{K}\vec{K}'}\delta_{ii'}=\sum_{\vec{K}i}a^{+}{}_{\vec{K}i}a_{\vec{K}i} \tag{32}$$

The Hamiltonian of fermions H in the second quantization representation has the form:

$$H=\int dV\Psi^{+}\{c\vec{\alpha}\cdot\vec{p}+mc^2\}\Psi=\sum_{\vec{K},\vec{K}',i,i'}\int dV\,a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\tilde{\mathcal{V}}^{*}(\vec{K},i,\vec{r})\{c\vec{\alpha}\cdot\vec{p}+$$
$$+mc^2\beta\}\mathcal{V}(\vec{K}',i',\vec{r})=\sum_{\vec{K},\vec{K}',i,i'}\int dV\,a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\tilde{\mathcal{V}}^{*}(\vec{K},i,\vec{r})\mathcal{V}(\vec{K}',i',\vec{r})E_{\vec{K}'i'}=$$

$$=\sum_{\vec{K},\vec{K}',i,i'}a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}E_{\vec{K}'i'}\delta_{\vec{K}\vec{K}'}\delta_{ii'}=\sum_{\vec{K}i}E_{\vec{K}i}a^{+}{}_{\vec{K}i}a_{\vec{K}i}. \quad i=\overline{1,4} \tag{33}$$

The energy of the fermion field is equal to the sum of the energies of its quanta.

The momentum of the fermion field $\vec{P}$ is equal to:

$$\vec{P}=\int dV\Psi^{+}-i\hbar\vec{\nabla}\Psi=\sum_{\vec{K},\vec{K}',i,i'}\int dV\,a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\tilde{\mathcal{V}}^{*}(\vec{K},\vec{r},i)-i\hbar\vec{\nabla}\mathcal{V}(\vec{K}',\vec{r},i')=$$

$$=\sum_{\vec{K},\vec{K}',i,i'}a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\hbar\vec{K}'\int dV\,\tilde{\mathcal{V}}^{*}(\vec{K},\vec{r},i)\mathcal{V}(\vec{K}',\vec{r},i')=$$

$$=\sum_{\vec{K},\vec{K}',i,i'}a^{+}{}_{\vec{K}i}\,a_{\vec{K}'i'}\hbar\vec{K}'\delta_{\vec{K}\vec{K}'}\delta_{ii'}=\sum_{\vec{K}i}\hbar\vec{K}'a^{+}{}_{\vec{K}i}a_{\vec{K}i}$$

here we have used the explicit form$\mathcal{V}(\vec{K}',\vec{r},i')$:

$$\mathcal{V}(\vec{K}',\vec{r},i')=L^{-\frac{3}{2}}U(\vec{K},i)e^{-i\vec{K}'\cdot\vec{r}}. \tag{34}$$

The momentum of the Dirac field is equal to the sum of the momenta of its quanta.

To determine the type of current of charged fermions, we use the Hermitian conjugation of the Dirac equation:

$$\left(i\gamma^{\mu}\frac{\partial}{\partial x^{\alpha}}-\frac{mc}{\hbar}\right)\Psi=0=i\gamma^{0}\frac{\partial\Psi}{\partial ct}+i\gamma^{K}\frac{\partial\Psi}{\partial x^{K}}-\frac{mc}{\hbar}\Psi$$

Hermitian mating:

$$-\frac{i}{c}\frac{\partial\Psi^{+}}{\partial t}\gamma^{0}-i\frac{\partial\Psi^{+}}{\partial x^{K}}(-\gamma^{K})-\frac{mc}{\hbar}\Psi^{+}=0 \tag{35}$$

Multiplying (35) on the right by γ^0 and using: $\gamma^K\gamma^0=-\gamma^0\gamma^K$, we obtain:

$$i\frac{\partial\Psi^{+}}{\partial x^{0}}\gamma^{0}\gamma^{0}+i\frac{\partial\Psi^{+}}{\partial x^{K}}\gamma^{0}\gamma^{K}-\frac{mc}{\hbar}\Psi^{+}\gamma^{0}=0=i\frac{\partial\overline{\Psi}}{\partial x^{0}}\gamma^{0}+i\frac{\partial\overline{\Psi}}{\partial x^{K}}\gamma^{K}+\frac{mc}{\hbar}\overline{\Psi}=0=$$

$$=i\frac{\partial\overline{\Psi}}{\partial x^{\mu}}\gamma^{\mu}+\frac{mc}{\hbar}\overline{\Psi}=0 \tag{36}$$

Here $\overline{\Psi}\equiv\Psi^{+}\gamma^{0}$.

Replacing the ordinary derivative with the covariant derivative in equation (28), we obtain a fermion interacting with the electromagnetic field:

$$\left[i\gamma^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}}-ie\mathcal{A}_{\alpha}\right)-\frac{mc}{\hbar}\right]\Psi=0 \tag{37}$$

Taking the density of the Lagrange function (Bethe G., 1964) in the form:

$$\mathcal{L}=\overline{\Psi}\left[i\gamma^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}}-ie\mathcal{A}_{\alpha}\right)-\frac{mc}{\hbar}\right]\Psi=\overline{\Psi}\left[i\gamma^{\alpha}\frac{\partial}{\partial x^{\alpha}}-\frac{mc}{\hbar}\right]\Psi+i-ie\overline{\Psi}\gamma^{\alpha}\Psi\mathcal{A}_{\alpha}=$$

$$=L_{0}+e\overline{\Psi}\gamma^{\alpha}\Psi \tag{38}$$

L_0- is the standard Lagrangian of the Dirac equation and $e\overline{\Psi}\gamma^{\alpha}\Psi\mathcal{A}_{\alpha}$ interaction potential with an electromagnetic field: $U=-e\overline{\Psi}\gamma^{\alpha}\Psi\mathcal{A}_{\alpha}$.

The Hamiltonian of the interaction of a fermion with an electromagnetic field is:

$$H_{1}=\int dV\left[-e\overline{\Psi}\gamma^{\mu}\Psi\mathcal{A}_{\mu}\right]=$$

$$=\sum_{\vec{P},\vec{K},n_i}\left(g_{\vec{P}\vec{K}i}a^{+}_{\vec{P}-\hbar\vec{K},i}a_{\vec{P}i}b^{+}_{n\vec{K}}+g^{*}{}_{\vec{P}Ki}a^{+}_{\vec{P}i}a_{\vec{P}-\hbar\vec{K},i}b_{n\vec{K}}\right) \tag{39}$$

The obtained results of the second quantization of fermions can be combined by writing in form 2 the forms of E. Cartan for the Cartan equation:

$$\Omega=d\xi_{1}^{i}\wedge d\eta_{1}^{j}\left[a_{i},a_{j}\right]_{+}+d\xi_{2}^{i}\wedge d\eta_{2}^{j}\left[a_{i}^{+},a_{j}^{+}\right]_{+}+d\xi_{3}^{i}\wedge d\eta_{3}^{j}\left(\left[a_{i},a_{j}^{+}\right]_{+}-\delta_{ij}\right)+$$

$$+d\zeta_{1}^{i}\wedge d\eta\theta_{1}^{j}\left[b_{i},b_{j}\right]_{-}+d\zeta_{2}^{i}\wedge d\theta_{2}^{j}\left[b_{i}^{+},b_{j}^{+}\right]_{-}+d\zeta_{3}^{i}\wedge d\theta_{3}^{j}\left(\left[b_{i},b_{j}^{+}\right]_{-}-\delta_{ij}\right)+$$

$$+d\mu\wedge\Big\{i\hbar d|\,t1\rangle-\Big(\sum_{\vec{K},i}E_{\vec{K}i}a^{+}{}_{\vec{K}i}a_{\vec{K}i}+\sum_{\alpha}\int d\vec{K}\hbar\omega_{\vec{K}}\left(b^{+}{}_{\vec{K}\alpha}b_{\vec{K}\alpha}+\frac{1}{2}\right)+$$

$$+\sum_{\vec{P},\vec{K},\alpha,i}g_{\vec{P}\vec{K}i}a^{+}_{\vec{P}-\hbar\vec{K},i}a_{\vec{P}i}b^{+}_{\vec{K}\alpha}+g^{*}{}_{\vec{P}Ki}a^{+}_{\vec{P}i}a_{\vec{P}-\hbar\vec{K},i}b_{\vec{K}\alpha}\Big)\times|\,t1\rangle dt\Big\} \tag{40}$$

The equations of E. Cartan for 2 forms (40) have the form:

$$0=\frac{\partial^{2}\Omega}{\partial d\xi_{1}^{i}\partial d\eta^{1j}}=\frac{\partial^{2}\Omega}{\partial d\xi_{2}^{i}\partial d\eta^{2j}}=\frac{\partial^{2}\Omega}{\partial d\xi_{3}^{i}\partial d\eta_{3}^{j}}=\frac{\partial^{2}\Omega}{\partial d\xi_{3}^{i}\partial d\eta_{3}^{j}}=\frac{\partial^{2}\Omega}{\partial d\zeta_{1}^{i}\partial d\theta_{1}^{j}}=\frac{\partial^{2}\Omega}{\partial d\zeta_{2}^{i}\partial d\theta_{2}^{j}}=$$

$$= \frac{\partial^2 \Omega}{\partial d\zeta_3^i \partial d\Theta_3^j} = \frac{\partial \Omega}{\partial d\mu} = \left[a_i, a_j\right]_+ = \left[a_i^+, a_j^+\right]_+ = \left[a_i, a_j^+\right]_+ - \delta_{ij} = \left[b_i, b_j\right]_- =$$

$$= \left[b_i^+, b_j^+\right]_- = \left[b_i, b_j^+\right]_- - \delta_{ij} = i\hbar d|\, t1\rangle - \left(\sum_{\vec{K},i} E_{\vec{K}i} a^+{}_{\vec{K}i} a_{\vec{K}i} + \right.$$
$$+ \sum_\alpha \int d\vec{K} \hbar\omega_{\vec{K}} \left(b^+{}_{\vec{K}\alpha} b_{\vec{K}\alpha} + \frac{1}{2}\right) + \sum_{\vec{P},\vec{K},\alpha,i} g_{\vec{P}\vec{K}i} a^+_{\vec{P}-\hbar\vec{K},i} a_{\vec{P}i} b^+_{\vec{K}\alpha} +$$
$$\left. + g^*{}_{\vec{P}Ki} a^+_{\vec{P}i} a_{\vec{P}-\hbar\vec{K},i} b_{\vec{K}\alpha}\right) |\, t1\rangle dt \qquad (41)$$

All equations (41), except [25, 26] the last, are permutation relations between the operators, indicating that the operators a_i, a_j^+ - are fermionic, and b_i, b_j^+ - are bosonic operators. The last equation is the Schrödinger equation for interacting fields: fermionic with electromagnetic (bosonic) fields.

We rewrite it in the form:

$$i\hbar \frac{d|t1\rangle}{dt} = (H_0 + H_1)|\, t1\rangle, \qquad (42)$$

where:

$$H_0 = \sum_{\vec{K},i} E_{\vec{K}i} a^+{}_{\vec{K}i} a_{\vec{K}i} + \sum_\alpha \int d\vec{K} \hbar\omega_{\vec{K}} \left(b^+{}_{\vec{K}\alpha} b_{\vec{K}\alpha} + \frac{1}{2}\right)$$
$$H_1 = \sum_{\vec{P},\vec{K},\alpha,i} g_{\vec{P}\vec{K}i} a^+_{\vec{P}-\hbar\vec{K},i} a_{\vec{P}i} b^+_{\vec{K}\alpha} + g^*{}_{\vec{P}Ki} a^+_{\vec{P}i} a_{\vec{P}-\hbar\vec{K},i} b_{\vec{K}\alpha}.$$

Representing $|\, t1\rangle = e^{-\frac{iH_0 t}{\hbar}} |\, t\rangle$, we obtain:

$$i\hbar - \frac{iH_0}{\hbar} e^{-\frac{iH_0 t}{\hbar}} |\, t\rangle + e^{-\frac{iH_0 t}{\hbar}} i\hbar \frac{d|t\rangle}{dt} = H_0 e^{-\frac{iH_0 t}{\hbar}} |\, t\rangle + H_1 e^{-\frac{iH_0 t}{\hbar}} |\, t\rangle$$

$$\text{Or } i\hbar \frac{d|t\rangle}{dt} = e^{\frac{iH_0 t}{\hbar}} H_1 e^{-\frac{iH_0 t}{\hbar}} |\, t\rangle \qquad (43)$$

$$e^{\frac{iH_0 t}{\hbar}} H_1 e^{-\frac{iH_0 t}{\hbar}} = \sum_{\vec{P},\vec{K},\alpha,i} g_{\vec{P}\vec{K}i} e^{\frac{i}{\hbar}\left(E_{\vec{P}-\hbar\vec{K},i} - E_{\vec{P},i} + \hbar\omega_{\vec{K}}\right)t} a^+_{\vec{P}-\hbar\vec{K},i} a_{\vec{P}i} b^+_{\vec{K}\alpha} +$$
$$+ g^*{}_{\vec{P}Ki} e^{\frac{i}{\hbar}\left(E_{\vec{P},i} - E_{\vec{P}-\hbar\vec{K},i} - \hbar\omega_{\vec{K}}\right)t} a^+_{\vec{P}i} a_{\vec{P}-\hbar\vec{K},i} b_{\vec{K}\alpha} \qquad (44)$$

We represent the differential equation:

$$i\hbar \frac{d|t\rangle}{dt} = H_1 \,|\, t\rangle, \qquad (45)$$

in the form of an integral equation:

$$|\, t\rangle = |\, 0\rangle + \frac{1}{i\hbar} \int_0^t H_1(\theta) d\theta \,|\, \theta\rangle \qquad (46)$$

In the first order of perturbation theory, its solution has the form:

$$|\, t\rangle \approx |\, 0\rangle + \frac{1}{i\hbar} \int_0^t H_1(\theta) d\theta \,|\, 0\rangle \qquad (47)$$

In the first order of the perturbation theory, we calculate the probability of spontaneous emission of a photon by an electron. To do this, take: $|\,0\rangle = a^{+}_{\vec{P}_0 i}|\,vacuum\rangle$. Then:

$$|\,t\rangle \approx a^{+}_{\vec{P}_0 i}|\,vacuum\rangle +$$

$$+\frac{1}{i\hbar}\int_0^t d\theta \sum_{\vec{P},\vec{K},\alpha,i} g_{\vec{P}\vec{K}i\alpha} e^{\frac{i}{\hbar}\left(E_{\vec{P}-\hbar\vec{K},i}-E_{\vec{P},i}+\hbar\omega_{\vec{K}}\right)\theta} a^{+}_{\vec{P}-\hbar\vec{K},i} a_{\vec{P}i} b^{+}_{\vec{K}\alpha}\, a^{+}_{\vec{P}_0 i_0}|\,vacuum\rangle =$$

$$= a^{+}_{\vec{P}_0 i_0}|\,vacuum\rangle -$$

$$-\sum_{\vec{P},\vec{K},\alpha,i} \frac{g_{\vec{P}\vec{K}i\alpha} e^{\frac{i}{\hbar}\left(E_{\vec{P}-\hbar\vec{K},i}-E_{\vec{P},i}+\hbar\omega_{\vec{K}}\right)t}-1}{E_{\vec{P}-\vec{K},i}-E_{\vec{P},i}+\hbar\omega_{\vec{K}}} a^{+}_{\vec{P}-\hbar\vec{K},i} b^{+}_{\vec{K}\alpha} \delta_{\vec{P}\vec{P}_0}\, \delta_{ii_0}|\,vacuum\rangle =$$

$$= a^{+}_{\vec{P}_0 i}|\,vacuum\rangle - \sum_{\vec{K},\alpha} \frac{g_{\vec{P}_0\vec{K}i} e^{\frac{i}{\hbar}\left(E_{\vec{P}_0-\hbar\vec{K},i}-E_{\vec{P}_0,i}+\hbar\omega_{\vec{K}}\right)t}-1}{E_{\vec{P}_0-\hbar\vec{K},i_0}-E_{\vec{P}_0,i}+\hbar\omega_{\vec{K}}} a^{+}_{\vec{P}_0-\hbar\vec{K},i_0} b^{+}_{\vec{K}\alpha}\,|\,vacuum\rangle \quad (48)$$

Thus, the probability for an electron to emit a photon with momentum $\hbar\vec{K}$ is equal to:

$$\frac{\left|g_{\vec{P}_0,\hbar\vec{K},i_0}\right|^2 \left|\sin^2\frac{\left(E_{\vec{P}_0-\hbar\vec{K},i_0}-E_{\vec{P}_0,i}+\hbar\omega_{\vec{K}}\right)}{2}t\right|^2}{\left|\frac{E_{\vec{P}_0-\hbar\vec{K},i_0}-E_{\vec{P}_0,i}+\hbar\omega_{\vec{K}}}{2}\right|^2} \quad (49)$$

At $t \gg 1$, this value is equal to:

$$\left|g_{\vec{P}_0,\hbar\vec{K},i_0}\right|^2 \pi^2 \delta^2\left(E_{\vec{P}_0-\hbar\vec{K},i} - E_{\vec{P}_0,i} + \hbar\omega_{\vec{K}}\right) =$$

$$= \left|g_{\vec{P}_0,\hbar\vec{K},i_0}\right|^2 \pi^2 \delta\left(E_{\vec{P}_0-\hbar\vec{K},i} - E_{\vec{P}_0,i} + \hbar\omega_{\vec{K}}\right) \frac{1}{2\pi}\int_{-T}^{T} e^{i\Delta K} dK =$$

$$= \left|g_{\vec{P}_0,\hbar\vec{K},i_0}\right|^2 \pi \delta\left(E_{\vec{P}_0-\hbar\vec{K},i} - E_{\vec{P}_0,i} + \hbar\omega_{\vec{K}}\right) T \quad (50)$$

The probability of radiation per unit time W is equal to:

$$W = \left|g_{\vec{P}_0,\hbar\vec{K},i_0}\right|^2 \pi \delta\left(E_{\vec{P}_0-\hbar\vec{K},i} - E_{\vec{P}_0,i} + \hbar\omega_{\vec{K}}\right)$$

3 Conclusions

We demonstrated the possibility of using the mechanics of E. Cartan in quantum field theory. Based on the use of these mechanics, additional conditions can be introduced directly into the Cartan equations. Such conditions include, for example, switching conditions between pulses and coordinates, as well as Lorentz calibration conditions.

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