# Mathematical Models of Pseudorandom Processes Behavior for Nonlinear Dynamical Systems

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### Abstract

This paper considers the processes in maps, which are examples of nonlinear dynamical systems. Analysing dynamical systems, it is necessary to take into account and analyze properties of iterative functions that determine the length of nonrepetitive iterative process. It is shown that not only properties of functions, but also properties of numbers from the considered functions domain influence the nonlinear maps behavior. Also this work consider nonlinear maps as a background in the analysis of financial data and complex dynamical systems, such as stock exchange and economics.

### Keywords

Chaos, randomness, nonlinear maps, financial analysis.
[1]

## 1. Introduction

Modeling of random and pseudorandom processes in nonlinear dynamical systems based on randomized algorithms of cyclic trajectories of fixed points requires formation of information technologies that would include a wide range of mathematical methods for predicting occurrences of such trajectories and pattern recognition for their behavior that depends on previous states. It is necessary to take into account the random nature of the depth of transition from one state or sequence of states to another state. An effective solution to this problem is provided that there are semantic databases that allow to take into account the experience of preliminary analysis of nonlinear dynamic systems behavior under different conditions. Estimation methods are required to define true randomness or pseudorandomness if so-called chaotic behavior observed during number sequence generations. Algorithms for analyzing such behavior increase the range of approaches that can be used to effectively solve various problems in areas such as modeling of dynamical systems, functional analysis, function theory, cryptography and others. The obtained sequences are widely used in the Monte-Carlo method where sequences of pseudorandom points are used to calculate multidimensional integrals [1], in the theory of machine learning to obtain training and test sequences [2] and others. Various methods of number sequence generation are based on chaotic processes and the next question arises: “What processes can be considered as chaotic?” There is no precise and constructive axiomatic definition of the concept of randomness at this moment. In Kolmogorov axioms, random sequences were left outside the theory and only general approaches to the definition of randomness were proposed, such as von Mises's approach. In this case, even if a method of sequence generation is used, there is a problem of uncertainty of the degree of approximation to the real randomness, because a truly random sequence is infinite and cannot be generated for real-world problems. Numerical sequence generators provide pseudorandomness with a certain degree of approximation to a given distribution law. Thus, the construction of random number generators is a way to define the formal concept of randomness, which is important and necessary in modern probability theory, stochastic process theory and others. Since a truly random sequence is a mathematical model that is a

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completely unpredictable and therefore non-periodic infinite sequence, the problem of generating a pseudorandom sequence (PRS) is formulated [3]. Separate branch of science, which is called deterministic dynamical systems also refers to the problem of chaos. The basis of this branch is the analysis of the dynamics of iterative fixed points using recursive functions. In this case, iterative cycles or orbits are considered as deterministic chaos [7], because they depend on the initial conditions, and also show irregularity [4]. Deterministic dynamical systems arise as a result of approximation of complex processes from the physical and mental world and can be used as generators of pseudorandom sequences. Another advantage of dynamical systems is that the required level of approximation to randomness can be achieved by a combination of several algebraic equations. In the case of PRS generators and dynamical systems, it is necessary to take into account and analyze properties of iterative functions that determine the length of nonrepetitive iterative process, which is one of the main properties of generators. In this case, the properties of the set of numbers on which these iterative functions are defined draw much less attention. There is a direct connection with number theory and above-stated problem. Prime numbers are of considerable interest because factorization can not be applied to them while usage of large compound numbers does not provide required cycle length. In many practical problems, the initial value is determined by random selection of a prime number of large dimension, but some prime numbers also do not allow obtaining of desired cycle length and approaching truly random sequence. The numbers of this class include of Fermat, Mersenne, Wagstaff numbers and their generalizations [1]. In this case, neighboring prime numbers provide a cycle length corresponding to the dimensionality of the number and meet the requirements to the resulting sequence. Thus, insufficient analysis of the selected numbers can lead to significant errors in the obtained sequences and incorrect conclusions.

## 2. Analysis of iteration processes in nonlinear maps

To analyze above-formulated problem, this paper considers the processes in maps, which are examples of nonlinear dynamical systems. By process it is meant a function $F$, that present the final sequences (words) in words such that if for the word $x$ $F(x)$ is defined and $y \subset x$, then $F(y)$ also defined and $F(y) \subset F(x)$. Let $\omega$ - some sequence. The process $F$ will be reapplied as long as possible. As a result, it will be obtained fragments of some new sequence $P$ - the result of application $F$ to $\omega$, ie $P = F(\omega)$.Nonlinear maps of the following classes are considered: “Tent”, “Asymmetric tent”, “Sawtooth” and multiplicative order map. It is important to note that maps (1,2,3) are continuous functions, while maps (4) are defined only on the set of integers. The choice of these maps allows us to consider and analyze the chaotic processes that are also observed in complex dynamical systems.

$$t_1(x_n) = x_{n+1} = \begin{cases} 2x_n, & x_n < 1/4 \\ 1-2x_n, & x_n \geq 1/4 \end{cases} \quad (1) \qquad t_2(x_n) = x_{n+1} = \begin{cases} 2x_n, & x_n < 1/2 \\ 1-x_n, & x_n \geq 1/2 \end{cases} \quad (2)$$

$$t_3(x_n) = x_{n+1} = \begin{cases} 2x_n, & x_n < 1/2 \\ 2x_n - 1, & x_n \geq 1/2 \end{cases} \quad (3)$$

It should be noted that computer systems in calculations operate with numbers in binary form and of limited length, while mathematics operates with numbers of infinite length and these circumstances lead to errors during operations with fractional numbers, which ultimately creates a problem of reliability and gives false conclusions about the processes in dynamic systems due to the fact that one of the properties of any dynamic system is the sensitivity to initial conditions [4]. In order to minimize this circumstance, the transition to integer maps is performed, which are presented as follows:

$$x_{n+1} = \begin{cases} 2x_n, & 4x_n < p \\ p-2x_n, & 4x_n \geq p \end{cases} \quad (1) \qquad x_{n+1} = \begin{cases} 2x_n, & 2x_n < p \\ p-x_n, & 2x_n \geq p \end{cases} \quad (2)$$

$$x_{n+1} = \begin{cases} 2x_n, & 2x_n < p \\ 2x_n - p, & 2x_n \geq p \end{cases} \quad (3) \qquad\qquad x_{n+1} = 4x_n (\text{mod } p), \quad (4)$$

where $p$ – prime number. The graphs representing maps (1,2,3) in the interval [0,1] are shown in Figures 1, 2, 3, respectively, and the graph 4 gives the definition of the map (4) on the set of integers. In the presented figures, the graphs of the maps intersect the diagonal $y = x$ at some points, which are fixed periodic points. It should also be noted that the map (1) is algebraically congruent to the map (4) on the set of integers, i.e. the cycle lengths for all prime numbers coincide. And map (2) does not satisfy in the general case to Fermat's little theorem.

Despite the simplicity of the above maps, their iterative cycles have properties that confirm the above statements. According to them, the structure of iterative cycles is determined not only by the properties of the maps themselves, but also by the properties of the numbers on which these maps are based and which have a significant impact on the structure and can significantly change it. The presented nonlinear maps allow to divide the set of primes $p$ into a system of classes based on the length of the iterative process for given prime numbers [5]. Since such an internal structure is an exception, the question of introducing a certain degree of similarity of internal structures arises. For example, for the prime number 649657, Figures 5 and 6 show the internal structure of iterative processes for maps, where the dotted line shows the resulting sequences and the solid line shows the internal parts within the sequences that give the maximum value of the correlation coefficient.

It is worth noting that the choice of a prime number of large dimension does not allow to avoid sequences with a simple internal structure and internal exponential and periodic components, which are indicated by the green region on the Figure 7. At the same time, the previous and next prime numbers generate sequences for which the length of the period corresponds to the dimension of the number and may indicate a better approximation to the conditions for pseudorandom sequences, but also have periodic components observed in places of compaction on the Figure 7.

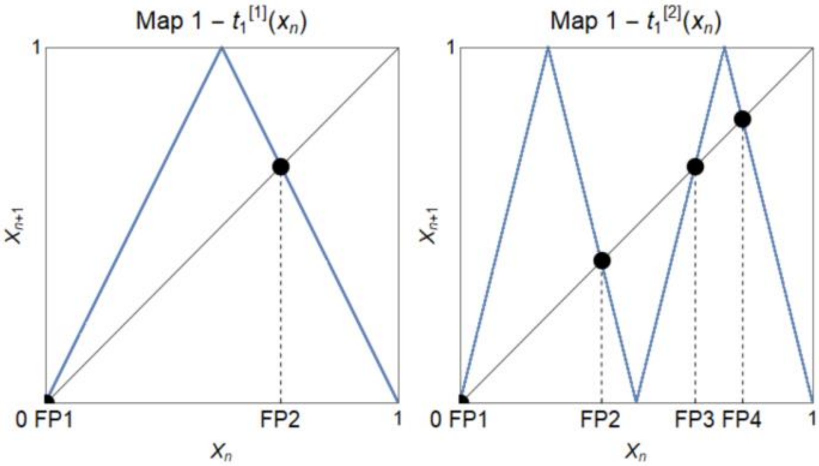


**Figure 1.** 1st and 2nd iteration for the map 1

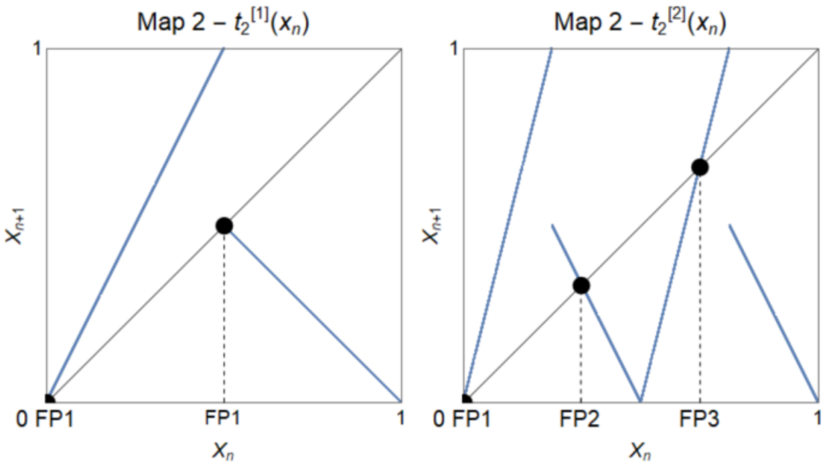


**Figure 2.** 1st and 2nd iteration for the map 2

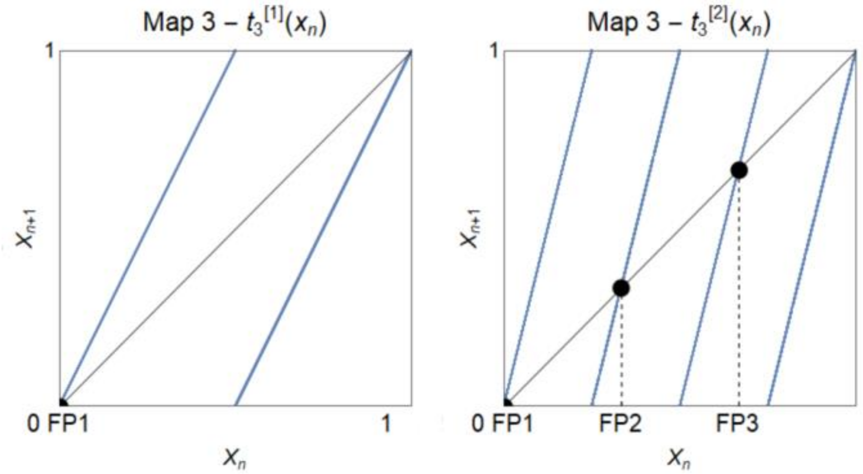


**Figure 3.** – 1st and 2nd iteration for the map 3

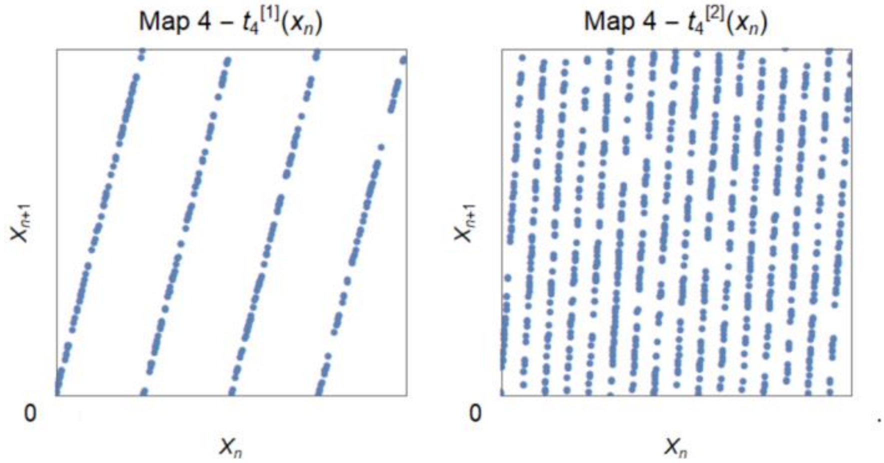


**Figure 4.** – 1st and 2nd iteration for the map 4

Considering prime numbers, special attention is drawn to prime numbers of a special form, such as generalized Gaussian Mersenne numbers - prime numbers $p^* = 2^p - 1$ for which the number $((1+i)^p - 1)((1-i)^p - 1)$ and $p$ are also prime numbers. It is possible to identify sequences that have a simple structure with exponential components that are repeated with a certain frequency and slightly different in amplitude. These numbers completely violate the conditions of randomness, which are essential for PRS. It is proved that such an internal regular structure is characteristic to Mersenne, Wagstaff numbers and their various generalizations. Thus, Figure 8 shows the structure of PRS for the generalized Gaussian Mersenne number.

## 3. Synergetic behavior of nonlinear dynamical systems of stock exchanges

### a. Wave fluctuations in nonlinear processes of stock exchanges

The processes of investing financial resources in the world economy for the purposes of specific countries or their regions are multidimensional random processes $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ that are a dynamical system. Each component $\xi_i(t)$ in $\xi(t)$ is a multidimensional random process $\xi_i(t) = g_i(\xi_{i1}(t), \ldots, \xi_{im}(t), \ldots)$ of different dimensions.

In all cases of examination of a certain random process of formation of security quotations it is not possible to build a mathematical stochastic model of a random vector $\xi(t)$ and its individual components. In real life, stock market processes are registered using modern computer information technologies, which allow to calculate various parameters, such as prices of stocks, options and other securities at the opening and closing of the stock exchange and the number of their sales.

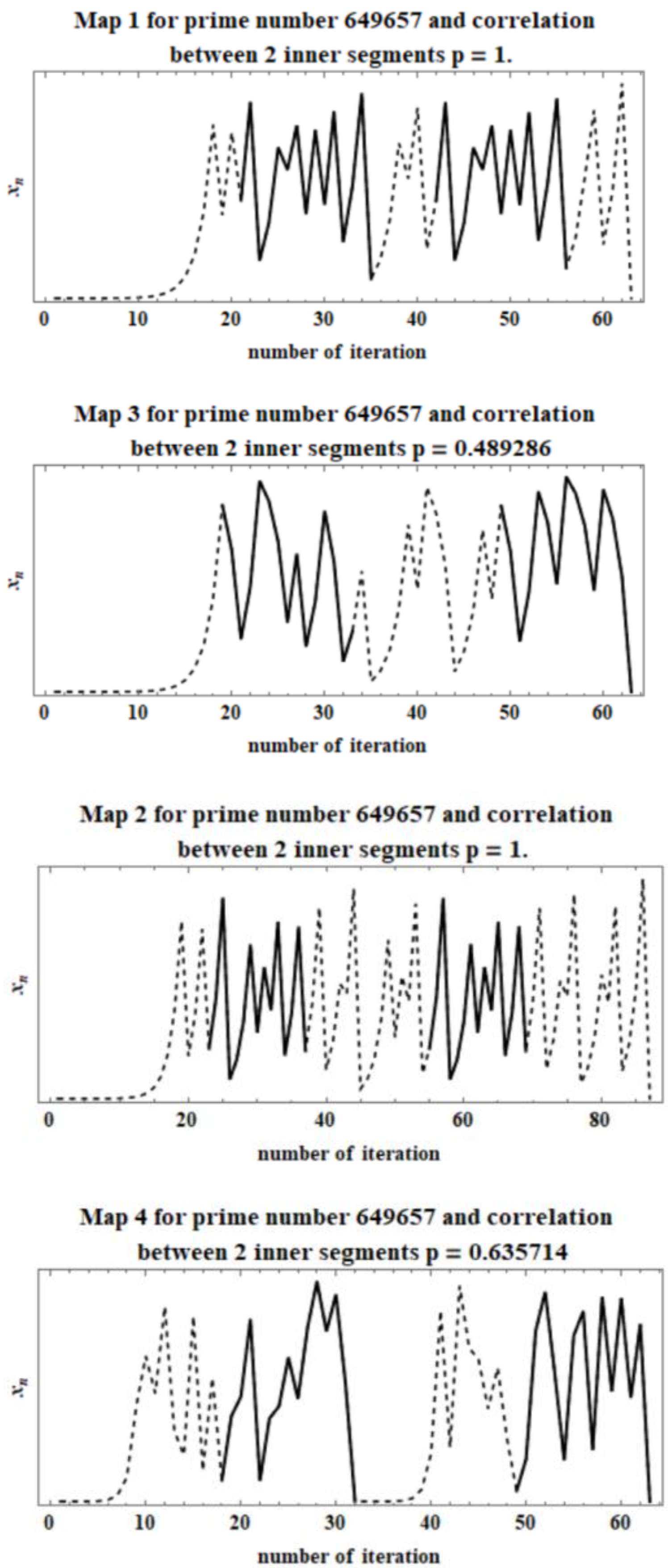


**Figure 5.** Sequence structure generated by prime number 649657

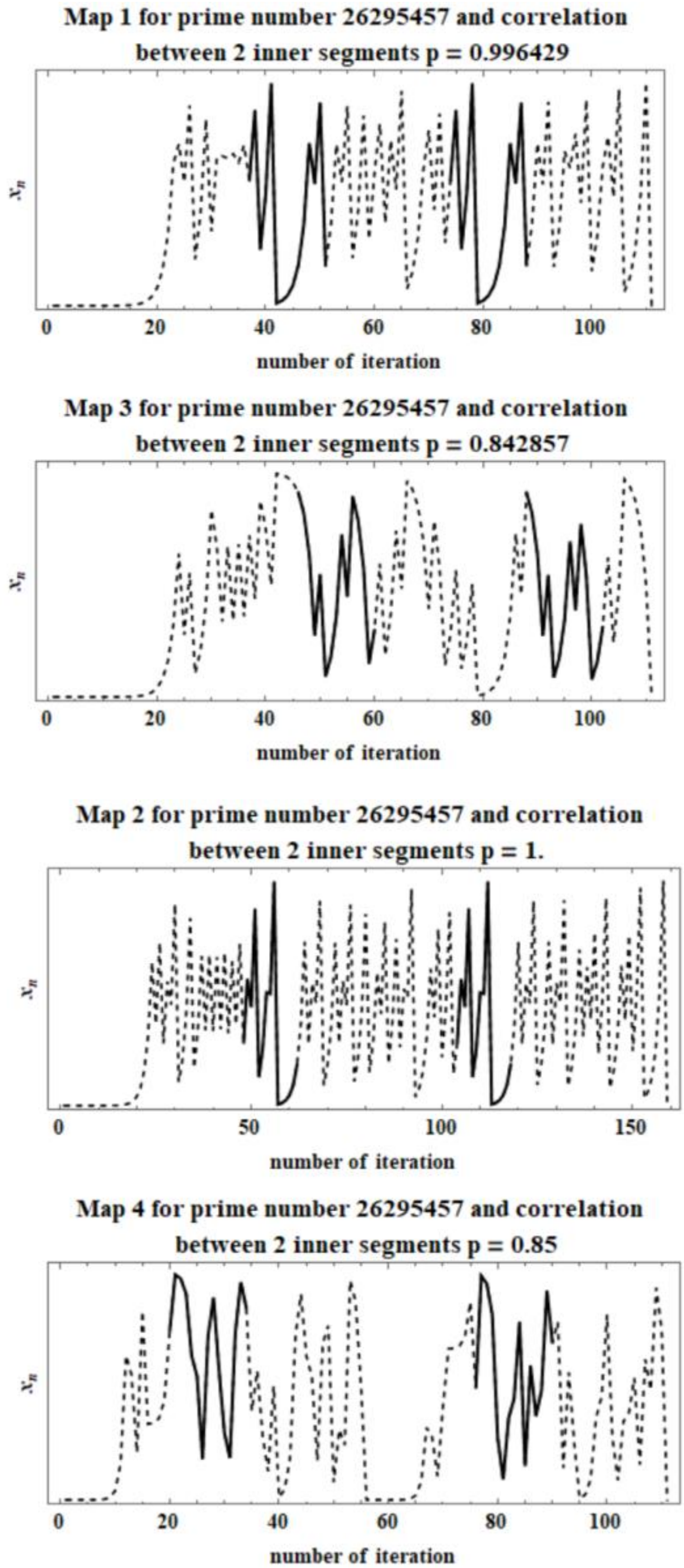


**Figure 6.** Sequence structure generated by prime number 26295457

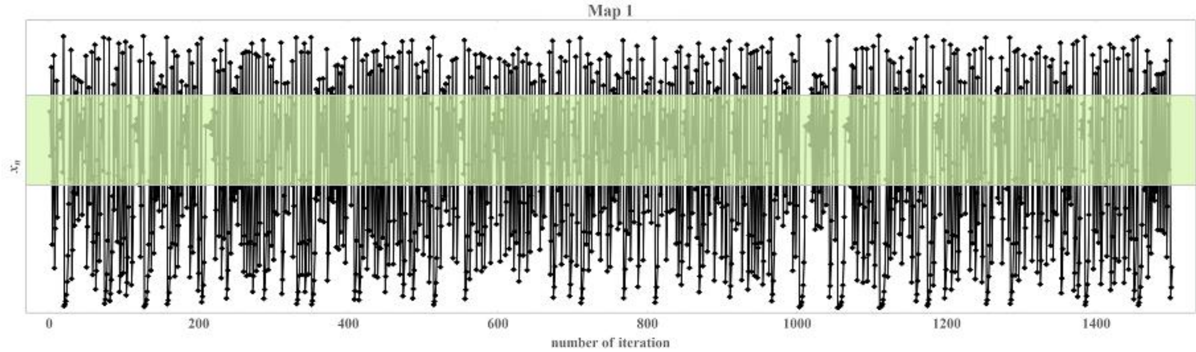


**Figure 7.** Nonlinear dynamics by prime 160465519 for map 1

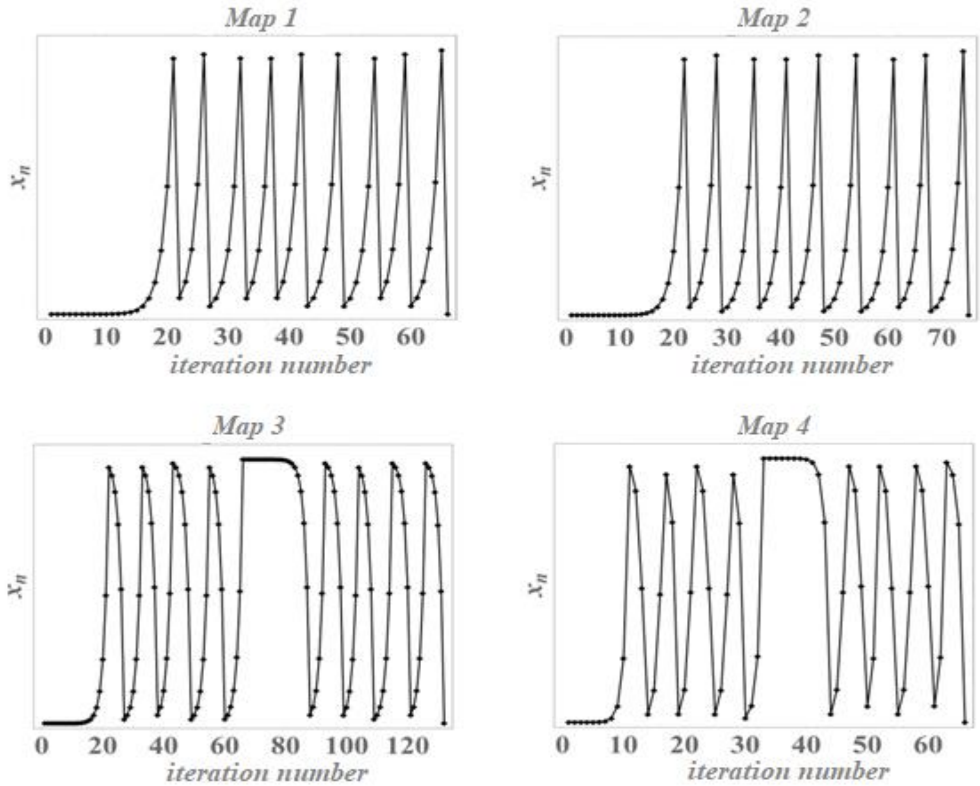


**Figure 8.** PRS based on Gaussian Mersenne prime

The dynamics of price fluctuations is represented by time series in which time can be recorded with high accuracy (up to a millionth of a second). Time series are registered for all bidders in the markets. The time series $X(t_1),\ldots,X(t_n),\ldots$ for each company is essentially a statistical model of a random process $\xi(t)$.

The main mathematical tools for studying self-organized nonlinear dynamic processes of stock markets are the analysis, processing and forecasting of dynamic processes for all market participants on the basis of time series. Next, it needs to explore some mathematical features of each one-dimensional time series. It is proved that the dynamics of price fluctuations is cyclical [1]. The study of the structure of cycles of price fluctuations is an important and very complex problem of statistical analysis of time series [3].

It is established that the one-dimensional time series of share quotations and options on the stock markets *X(t)* is a mixture of at least three random processes. The term "mixture" emphasizes that the components of the mixture are not independent random variables at all, but the relationship between them is nonlinear, and they can be uncorrelated in individual time intervals, where nonlinear dependence has a significant meaning of behavior with a particular strategy. The three underlined components $X(t)$ are related to three random processes with different evolutional mechanisms. Computer analysis of time series of stock prices allows to establish that the dynamics of time series is largely determined by the following factors:

1) Economic fluctuations of chaotic deterministic nature are due to regular fixed points, which can asymptotically approach the general limit [6];
2) Fluctuations of $X(t)$ are in the nature of random processes with independent increments due to the large number of market participants' interests, which are constantly changing when buying and selling securities [2];
3) The structure of wave fluctuations is due to the ongoing Elliott wave generation process in which fluctuation trend always consists of five growth fluctuations associated with price increases and three decreasing fluctuations due to opposite corrective actions.

In the first case, at any time $t_i$, the time series $X(t_i)$ have a form that corresponds to the logistic equation $X(t_{n+1}) = aX(t_n)(1 - X(t_n))$, which has the following graphical representation.

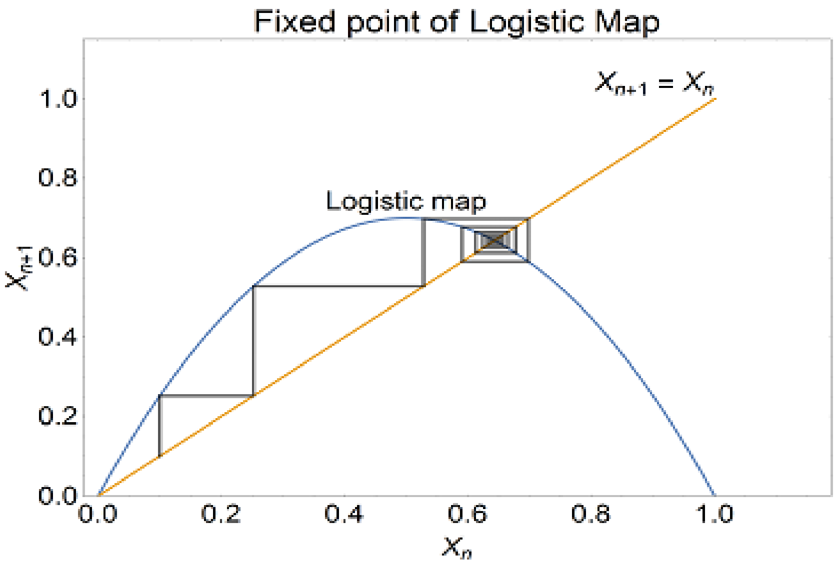


**Figure 9.** Graphical representation of Logistic map

This process describes the acquisition of stocks in the interval $(t_n, t_{n+1})$, which leads to the exhaustion to some extent of this resource at the time $t_{n+k}$ (this may be the end of the trading day). As proposed in [6] for the chaotic nature of logistic map it is proved that it connects with the map "Tent" which allows to consider in detail the dynamic nature of processes and to investigate the iterative process of fixed point formation [7,8].

In the second case, based on the fact that the continuous flow of orders on the stock exchange contains unrelated orders, the model "Geometric Brownian motion" can be used to simplify the description of the buying and selling process. This approach to stock markets was first proposed by M. F. M.Osborne [5]. This model, in contrast to the Brownian motion, is everywhere positive and has a log-normal law of price distribution. Figure 10 shows a representation of the Brownian motion trajectory for a particle on the 2D plane and a simulation of the geometric Brownian motion process. Processes of this kind lead to constant fluctuations during one trading day.

However, it follows by randomness that it is impossible to predict the rise or fall of any particular stock, but due to the law of large numbers it is possible to obtain information about the behavior of markets in the long run.

In the third case, it is possible to observe the basic law of nature [4], according to which the development of the price of any stock (option) consists of five fluctuations with increasing and decreasing its values, but close to a certain value and with the next three stages of price adjustment of the financial instrument. A graphical representation of the process is shown in Figure 11:

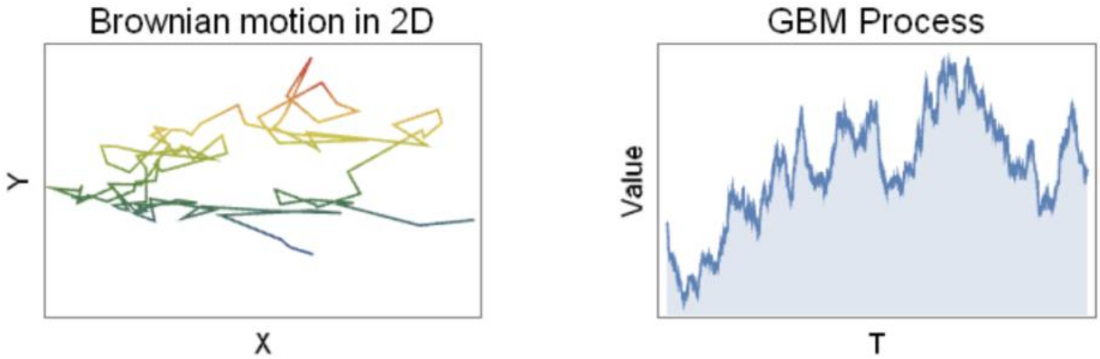


**Figure 10.** Brownian motion simulation

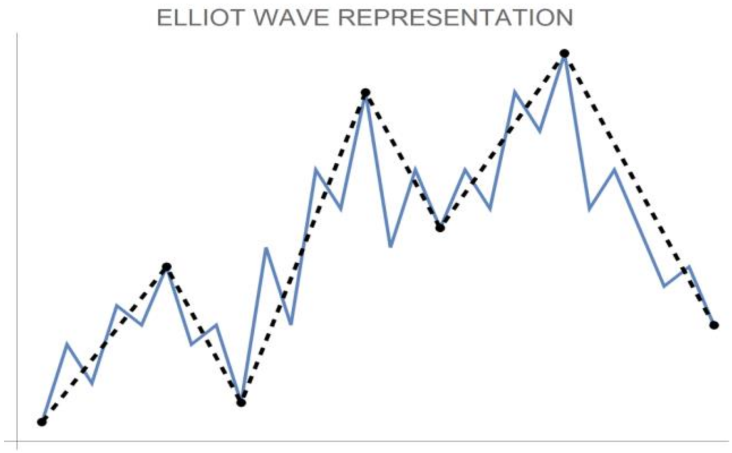


**Figure 11.** Elliot wave structure

The number of development waves can be more than five, but the number of waves of correction is always smaller. The construction of the mathematical theory of Elliott waves is associated with the development of mathematical methods for modeling human decision-making processes in a socio-dynamic environment. An analysis of the decision-making processes of people (brokers) in the stock markets confirms that the desire to sell a financial asset at a higher price always requires greater fluctuations in decisions than decisions related to adjusting prices at some acceptable level. The structure of Elliott waves agrees with the wave lengths of individual waves at the stage of development and at the stage of correction. It is established that the number of waves at the development stage always exceeds the number of waves at the correction stage. This is due to the fact that the adjustment phase is always associated with minimization of financial losses and maximization of the possible rate of return.

Analysis of the observed Elliott waves confirms the view that these waves are the result of the collective intellectual activity of a large number of people with agreed strategies for success, and also confirms the Keynesian business cycle model [2]. It has been proven that the use of Elliott waves, the theory of fixed iteration points and random walk processes create a basis for stabilization and improvement of the efficiency of stock markets as a regulator of financial assets investing in the developing world economy.

## b. Nonlinear dynamical processes in synergetic economics

Analysis of developed and emerging economies of the world suggests that they all function as synergetic systems. There is still no exact mathematical definition of the synergetic nature of such processes. The model of synergetic dynamics of interaction of various components of nonlinear maps describing processes of development and realization of innovative projects is considered.

The processes of innovative projects consist of two components. The first stage is the creation of a new technology for studying the management of economic systems and the introduction of an innovative development process based on investment in the business plan. This process is systematically repeated in high-tech companies:

1) new development process;
2) investment in a business plan;
3) implementation of development in the markets

And again, this sequence of stages of innovation is repeated at a new level.

The stability of these processes must be accurately explained from a mathematical point of view. Research papers [1, 9, 10] contain the results of research aimed at explaining the synergistic nature of economic development associated with the development of high technology (Microsoft, Google, IBM, Tesla, etc.). The considerations presented in these works do not always give a full justification of their synergetic nature.

The authors of [2, 11, 12] first drew attention to the Levy factor, when investment in a project is sharply reduced, and very quickly allocated to the implementation of a new investment (promising) project in the same field and the same companies.

This is especially evident in projects related to electric vehicles, solar power plants, system software, smartphones, information technology and more.

Analysis of nonlinear dynamic processes in such areas allows us to formulate and prove that such synergetic processes in these companies and the economy as a whole always contain several nonlinear components of such processes, close to the logistic model of resource consumption, with moments of restructuring in the form of Levy flight [12 ], ie periodic changes in investment strategy. Levy's flight is one of the forms of random wandering, the graphic image of which is shown in Figure 12.

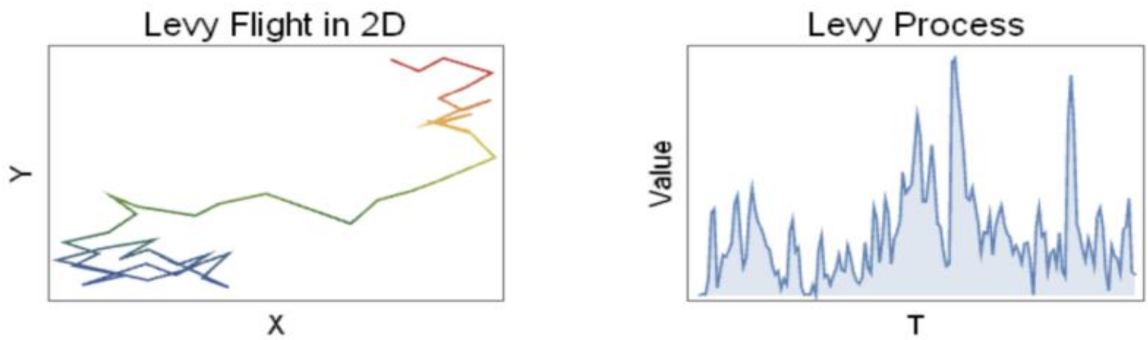


**Figure 12.** Representation of Levy Flight process

Each component performs (realizes) its contribution to the formation of stability. Functions of type $S_1(t)$ are the process of transferring funding from one project to a completely new one. The process $S_2(t)$ is implemented when at some point the implemented innovation project receives corrective funding, which leads to a decrease in the level of innovation and its gradual decline. Maps $S_3(t)$ and $S_4(t)$ describes the single processes of investing financial resources and their depletion. A graphical representation of these processes is shown in Figure 13.

Undoubtedly, the number of components of these types can be much larger. Analysis of the $S$ function using its computer modeling and mathematical analysis with the right choice of weight functions always provides the necessary level of stability for these types of investment management processes in all areas of developed economies and much less in developing countries. A more complete mathematical analysis of this approach and the results of computer modeling will be the subject of a separate publication.

## c. Fourier analysis in finance: price formation and Fourier transform

A wide range of real systems demonstrates the properties of randomness, including finance and stock markets in particular. The question arises: "Does a system like the stock market develop by accident?". The hypothesis of random wandering as the basis of the market was formalized and popularized in [3]. In order to test the correctness of this hypothesis, it is necessary to determine whether the results formed in the market are stochastic or deterministic.

To solve this problem, a statistical approach was used, using a set of NIST tests and other cryptographic tests for randomness. It has been shown that markets are not random because financial data do not satisfy the properties of randomness, such as chaos, frequency stability and unpredictability.

Accordingly, it is possible to build methods for forecasting financial data on historical changes in stock prices and a wide range of financial derivatives. Among the various mathematical tools that can be applied to finance, the Fourier transform is of particular interest because it makes it possible to identify patterns or loops in time series data and also provides a real-time pricing method for a financial instrument such as the option that was considered in the work of Carr and Madan [19].

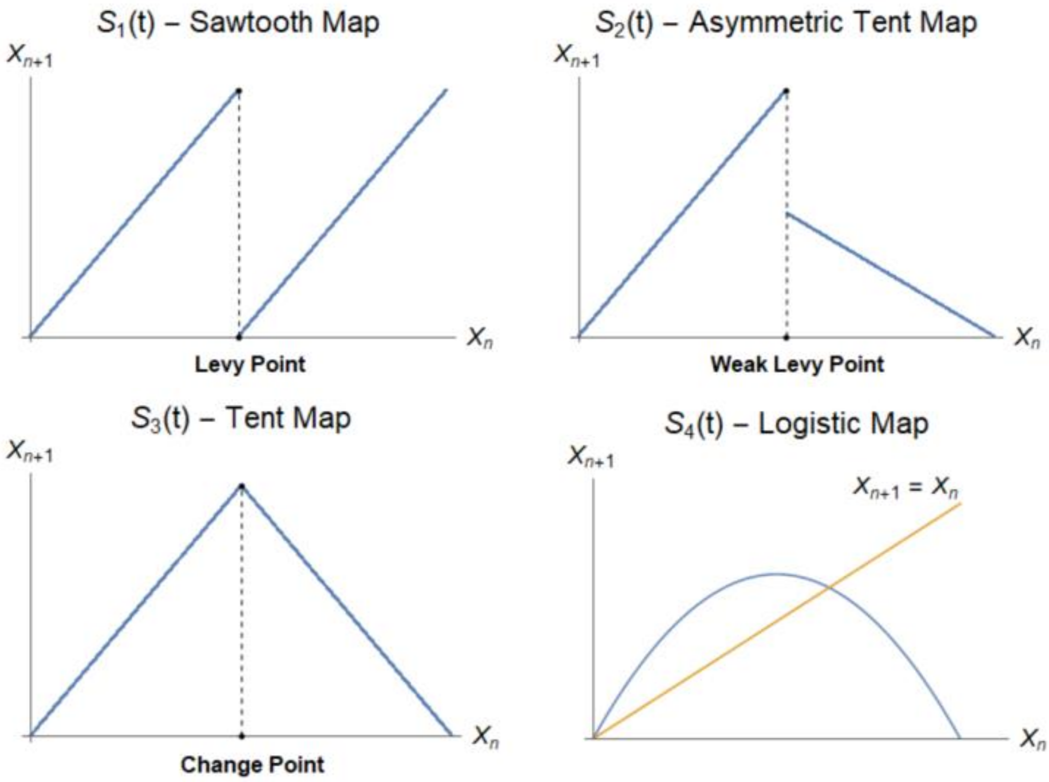


**Figure 14.** Representation of investment processes

The very first option pricing model used a geometric Brownian process to model the basic pricing process. But due to some known shortcomings of this approach, it was proposed to move to the options of pricing models based on Levy processes.

Therefore, the Fourier transform technique is an effective approach to estimating the option according to the Levy model, because the Levy process $X_t$ can be fully described by its characteristic function $\phi_X(u)$, which is defined as the Fourier transform of the density function $X_t$.

To describe the application of the Fourier transform (FT) in the option pricing model, it is important to establish a definition of the transformation. If we analyze the piecewise-smooth function $f(x)$ on $(-\infty,+\infty)$, then the FT is defined as:

$$F_f(u) = \int_{-\infty}^{\infty} e^{iuy} f(y)dy \tag{1}$$

If we consider a sequence of numerical values $\{x_k\}$ as a result of a discrete Fourier transform (DFT) for another sequence $\{y_j\}$ and is defined as follows:

$$y_j = \sum_{k=0}^{N-1} x_k e^{i\frac{2\pi jk}{n}} \tag{2}$$

Here, the Fourier transform can be considered as a projection of the sequence $\{x_k\}$ on the vector space, because each product is a projection on a single basis vector. Next it is necessary to describe the Levy process.

An actual process is called a Levy process if it exhibits certain properties: independent increments, stationary increments, and almost certainly a continuous right map with a limit on the left. The Levy process can be represented as a combination of linear drift, Brownian process and jump process.

Turning to the application of PF for option pricing, we can describe an alternative formulation of option pricing, which uses an analytical expression of the characteristic function of the basic process of asset price using fast Fourier transform (FFT).

To obtain a quadratic-integral function, it was proposed to consider the Fourier transform of the declining price of the transaction $c(k)$, where $c(k) = e^{ak}C(k)$ for $a > 0$. Positive values $a$ improve the integration of the modified transaction value over the negative axis k.

Consider $\psi_T(u)$ as a Fourier transform $c(k)$, $p_T(s)$ as a function of the density of the underlying process of the asset price, where $c(k)s = \log S_T$ and $\phi_T(u)$ as a characteristic function $p_T(s)$. This gives:

$$\psi_T(u) = \int_{-\infty}^{\infty} e^{iuk} c(k) dk = \frac{e^{-rT}\phi_T(u-(a+1)i)}{a^2 + a - u^2 + i(2a+1)u} \tag{3}$$

The transaction price $C(k)$ can be restored by the inverse Fourier transform, where

$$C(k) = \frac{e^{-ak}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du = \frac{e^{-ak}}{\pi}\int_{0}^{\infty} e^{-iuk}\psi_T(u)du \tag{4}$$

The above integral can be calculated using FFT. And it is important to note that $a$ should be such that the denominator has only imaginary roots in $u$, because the integration is performed on the real value $u$.

Next, it is necessary to describe the evaluation process for the integral (4). Let's start with choosing the number of intervals $N$ and bandwidth $\Delta u$. Therefore, the numerical approximation for $C(k)$ is given by the formula:

$$C(k) \approx \frac{e^{-ak}}{\pi}\sum_{j=1}^{N} e^{-iu_j k}\psi_T(u_j)\Delta u, \tag{5}$$

where $u_j = (j-1)\Delta u, j = 1,\ldots,N$. The semi-infinite integral domain $[0,\infty)$ in the integral in (4) is approximated by a finite integral domain, where the upper limit for in numerical integration is represented as $N\Delta u$.

Also, the Fourier variable $u$ is selected at discrete points instead of a continuous representation. From one set of FFT calculations it is possible to get the price of the transaction (call) for the option for a set of strike prices.

This gives market analysts the opportunity to record the price sensitivity of the transaction with different execution prices.

The transaction price is multiplied by the corresponding exponential attenuation coefficient, converted into a square-integral function, and the Fourier transform of the modified transaction price is converted into an analytical function of the characteristic function of the registration price (log price). However, the short maturity causes significant numerical errors, so the time component of the option should be considered to obtain smaller pricing errors for all execution price levels.

## 4. Conclusion

Summarizing the above information, it should be noted that Fourier transform methods provide an approach to pricing for financial instruments based on Levy processes, because the analytical representation of the characteristic function is more accessible and can be obtained using the previously described approach with high accuracy and efficiency. It is possible to get option prices for the full range of exercise prices in one set of FFT calculations. And this approach can be further developed for other derivatives on the Levy model, where the payout function depends on the volatility of the underlying process.

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