Collective Risk Estimating Method for Comparing Poly-Interval Objects in Intelligent Systems

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Abstract
Problems of comparing poly-interval alternatives under risk in the framework of intelligent computer systems are studied. The problems are common in many areas of human activities. Collective risk estimating method was chosen to compare. Another method, "mean-risk" one, which focuses on estimating of a different kind of risk, was discussed earlier. Both methods complement each other in the problems of choosing the preferred poly-interval alternative under risk and it is advisable to use the methods together. Approaches are proposed to obtain analytical expressions for indicators of preference and risk of the collective risk estimating method. The expressions are obtained for indicators of the method with using different defuzzification procedures for different configurations of poly-interval alternatives in their compared pairs. The reasons are discussed for diversity of the results for different defuzzification procedures. The results may be used in intellectual decision support systems.

Keywords1
Pairwise comparing of poly-interval alternatives, collective risk estimating approach, defuzzification procedures, analytical relations for calculation of preference and risk indicators

1. Introduction

A significant part of artificial intelligence research is connected with development of intelligent computer systems theory and practice. Problems of comparing different genesis alternatives by the effectiveness in turn play an important role in the framework of this problematic. Due to the varying degrees of uncertainty of the problems being solved, the quality indicators of alternatives compared in terms of effectiveness can be endowed with estimates of a wide spectrum, - from point to mono-interval and poly-interval estimates. In the latter case, knowledge about the parameters of the problem is expressed by a set of mono-intervals, which characterizes the uncertainty of expert knowledge about the length and location of mono-intervals-estimates of quality indicators. Here with each of the mono-intervals of the set describes the analyzed indicator of the problem with varying degrees of confidence. A peculiarity of the similar comparison tasks is the fact that along with the indicator characterizing the preference of alternatives should be considered on a parity basis an indicator of the objectively existing risk that the alternative recognized as the best at the time of comparison will not be such later, after its completion.

Two main directions can be distinguished in the poly-interval approach: description using the apparatus of fuzzy sets and using the formalism of generalized interval estimations. Comparison of such "interval" alternatives requires the development of special methods. This is especially true in the poly-interval case.

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In practice, one has to compare alternatives with different types of interval quality indicators. Moreover, for the comparability of the results, the comparison should be made using the same, sufficiently universal, methods. Such rather universal methods are the “mean - risk” method [1, 2] and the collective risk estimating method [3].These methods are focused on assessing different types of risk that arise in the process of comparing interval alternatives by effectiveness. Among them the risk as the possibility of obtaining a real outcome that differs from the desired predicted result (including the risk of losses), as well as the risk that the alternative estimated at the time of comparison as effective in their presented set will not be such at the moment of removing the uncertainty. The “mean - risk” method is used to estimate the risk of the first type and the collective risk estimating method for the risk of the second type.

Each interval alternative is considered in the “mean-risk” method separately, independently of the others. Estimates of preference and risk indicators in the method do not depend on the context, i.e. the presence of other comparable alternatives and the influence of this fact on the comparison results. This is the disadvantage of the method and at the same time its advantage since calculations of estimates of preference and risk indicators are here simpler than in the method of collective risk estimating. The dependence of both preference of interval alternatives and associated risk on the context, that is, on a specific set of compared objects, is taken into account in the collective risk estimating method. The chances of the plausibility of the hypothesis that the analyzed alternative will be preferable to the other compared ones are selected here as a measure of preference, and the chances that in reality at least one another alternative will be preferred as a measure of risk. The method has the advantage that it allows to evaluate the “collective” risk, the value of which can significantly exceed the value of risk in case of pairwise comparison. The disadvantage of the method is a consequence of the fact that it compares only the relative effectiveness of interval alternative. That is the alternative recognized as effective in such comparing may in itself be ineffective (unprofitable).

The main difficulty in transferring the “mean – risk” method to the case of poly-interval alternatives, especially to fuzzy ones, was associated with the lack of a regular method for finding one-numerical estimates for interval, generally speaking, estimates of preference and risk indicators used in the method, namely, mathematical expectation and mean semi-deviation [4]. The procedure for finding such one-numerical estimates was proposed in [5]. Namely for fuzzy objects it was proposed to use the simplest defuzzification method by averaging the contributions of mono-intervals that form their set in the object and the center of gravity method. One-numerical estimates of the method indicators were obtained in these papers for triangular and trapezoidal membership functions of the fuzzy theory.

In the case of the generalized interval estimations approach when they are presented in the form of a probabilistic mixture on a mono-interval of maximum range in a set of mono-intervals forming a generalized interval object, analytical relations for calculating the indicated method indicators were established for the so-called generalized uniform distribution of chances. If the expressions for preference indicators of the “mean – risk” method are the same for both types of poly-interval alternatives, then the expressions for the risk indicators are significantly different. All other things being equal, the value of the calculated risk in the approach of generalized interval estimates exceeds the value of the risk for the corresponding fuzzy objects.

The relations for the preference and risk indicators in the case of the collective risk estimating method significantly depend on the localization of interval alternatives (configurations) in their compared pair. Big differences in the approach to comparing fuzzy and generalized interval objects are showed namely in this method. This affects the structure of relations for preference and risk indicators for these objects.

This difference is most clearly manifested when we wish to implement a numerical procedure for calculating effectiveness indicators by the method of statistical tests. If we have deal with fuzzy objects, it is sufficient in each “history” of the statistical test method to play one value of α for the specification of the α-cut and, therefore, to choose the compared mono-intervals that form the poly-object for both poly-alternatives at once [6]. For generalized interval objects the value of α is played out independently for each compared alternative in their pair and, thus, with this approach to comparing of mono-intervals that form poly-objects the point of view on the comparison process as a “game with nature” is more fully realized.

Since all the considered approaches to comparing interval alternatives have and advantages and disadvantages, own for each of the approaches, and the methods for calculating indicators of preference and risk in approaches complement each other, it seems reasonable to combine the advantages of both approaches in a procedure of their joint using.

Since method "mean – risk" was considered by us earlier [5], the purpose of this paper is to develop the method of collective risk estimating to a poly-interval case.

2. Method of Collective Risk Estimating

Compared interval alternatives are considered within the framework of the method of collective risk estimating as a set of interrelated objects. The method takes into account that the risk of choosing the "best" interval alternative depends, among other things, on the number of objects being compared: if other things are equal, the more of objects in the system the greater the risk. Interval alternatives are compared "as a whole" [3]. This is due to the collective effect that is typical for many natural systems, when the properties of the system may differ significantly from the properties of its components.

Let there are K alternatives $I_i,\ i = 1,2,\dots,K$ and let $C\big(I_i \succ (I_1, I_2, \dots, I_{i-1}, I_{i+1}, \dots, I_K)\big)$ is the dimensionless quantity that describes the degree of confidence (chances) in the truth of the tested hypothesis that interval alternative I_i is preferable to all other compared interval alternatives in their existing set $(I_1, I_2, \dots, I_{i-1}, I_{i+1}, \dots, I_K)$. Let $\equiv$ and $\wedge$ are equivalence and conjunction symbols, respectively. Then the term is preferable to "all others" means that

$$I_i \succ (I_1, I_2, \dots, I_{i-1}, I_{i+1}, \dots, I_K) \equiv (I_i \succ I_1) \wedge (I_i \succ I_2) \wedge$$
$$\wedge (I_i \succ I_3) \wedge \dots \wedge (I_i \succ I_{i+1}) \wedge \dots \wedge (I_i \succ I_K).$$

The risk that I_i will not be actually preferred is measured by $R_s\big(I_i \succ (I_1, I_2, \dots, I_{i-1}, I_{i+1}, \dots, I_K)\big)$, the quantity, which complements the chances $C\big(I_i \succ (I_1, I_2, \dots, I_{i-1}, I_{i+1}, \dots, I_K)\big)$ to one. This quantity measures the chances that at least one alternative would be preferable to I_i. Using the entered quantities, the nature of the collective effect can be described by the following relations [9]:

$$C\big(I_1 \succ (I_2, I_3, \dots, I_K)\big) + C\big(I_2 \succ (I_1, I_3, \dots, I_K)\big) +$$
$$+ C\big(I_3 \succ (I_1, I_2, I_4, \dots, I_K)\big) + \dots + C\big(I_K \succ (I_1, I_2, \dots, I_{K-1})\big) = 1,$$
$$R_s\big(I_1 \succ (I_2, I_3, \dots, I_K)\big) + R_s\big(I_2 \succ (I_1, I_3, \dots, I_K)\big) +$$
$$+ R_s\big(I_3 \succ (I_1, I_2, I_4, \dots, I_K)\big) + \dots + R_s\big(I_K \succ (I_1, I_2, \dots, I_{K-1})\big) = K - 1.$$

Here the next important question arises: is the ordering of interval alternatives on preference different for comparing "as a whole" from the results of pairwise comparison? The answer to this question is negative: the order determined by pairwise comparison coincides with the results of the comparison "as a whole". However, only the comparison "as a whole" gives an idea of the true magnitude of the risk. Analytical relations for calculating the value of the preference criterion with a small number of compared mono-interval alternatives (two, three) were obtained in [3, 6] under the assumption that the chances distributions given on the compared intervals are uniform. For a larger number of interval alternatives these relations can be obtained numerically by statistical test method. For poly-interval estimates only the most important case of pairwise comparison is practically realizable. Namely this case is considered further for the two main directions of the poly-interval approach – the description by means of the fuzzy sets apparatus and with the help of the general interval estimations formalism.

As for the mono-interval case, the expressions for the chances of the preference $C(I_1 \succ I_2)$ of the interval alternative I_1 compared with I_2 depend on the relative position of the compared alternatives, i.e. on their configurations. If for a mono case there are, up to a permutation, only four configurations of compared interval alternatives – right-shift configuration, nested intervals, coinciding and non-intersecting intervals, and only two first configurations with non-zero intersection of estimates are of main interest, in the case of poly-interval alternatives a number of configurations for pairs of intersecting estimates are significantly richer.

We will show that, up to a permutation of poly-interval objects in their compared pair, there are six different configurations of intersecting interval alternatives for triangular membership functions of the fuzzy theory and for triangular poly-interval estimations of the general interval formalism.

Each of the two compared objects is defined by three corner points of triangle, which was mentioned above: $L_i < T_i < R_i, i = 1, 2$. It is convenient to classify configurations by the number of intersections of the left l_i and the right r_i sides of the triangles. One may see that

$$l_i(\alpha) = (1-\alpha)L_i + \alpha T_i, r_i(\alpha) = (1-\alpha)R_i + \alpha T_i.$$

In the future, we will need the coordinates α_j of the intersection points of l_i and r_i. Let us assume that the hypothesis is being tested that the first poly-interval object I_1 is preferable to the second I_2.

If $T_2 < T_1$ (configurations 1 and 2). Configuration 1: l_1 once intersects only r_2, and $L_2 < L_1 < < R_2 < R_1$. Line l_1 intersects only r_2 once; there are no other intersections of l_i and r_i. Intersection point coordinate is $\alpha_1 = (R_2 - L_1)/(R_2 - T_2 + T_1 - L_1)$.

Configuration 2: l_1 and r_1 once intersect r_2, and $L_2 < L_1 < R_1 < R_2$. Intersection points of l_1 and r_1 with r_2 have coordinates α_1 and $\alpha_2 = (R_2 - L_1)/(R_2 - T_2 + T_1 - R_1)$.

If $T_2 > T_1$ (configurations 3 and 4). Configuration 3: l_1 and r_1 once intersect l_2, and $L_2 < L_1 < < R_1 < R_2$. Intersection points of l_1 and r_1 with l_2 are $\alpha_3 = (L_1 - L_2)/(T_2 - L_2 - T_1 + L_1)$, $\alpha_4 = (R_1 - L_2)/(R_1 - L_2 + T_2 - T_1)$.

Configurations 4 and 5: l_1 intersects l_2, r_1 once intersect l_2 and r_2, besides $L_2 < L_1 < R_2 < R_1$. The difference between these configurations lies in the location of the intersection points. Their coordinates are set by already known points α_2, α_3, α_4. However, in configuration 4 $\alpha_2 > \alpha_3$, and in configuration 5 $\alpha_2 < \alpha_3$. One can see that configuration 4 takes place, if $\Delta_2 > \Delta_1$, and configuration 5, if $\Delta_1 > \Delta_2$, where $\Delta_1 = R_1 - L_1, \Delta_2 = R_2 - L_2$.

If $T_2 = T_1$ (configuration 6). There are no intersections of the sides, and $L_1 < L_2 < R_2 < R_1$.

3. Collective Risk Estimating: Fuzzy Alternatives

When comparing two interval alternatives I_1 and I_2 by the collective risk estimating method, the values of the preference criterion are calculated as well as the values of the risk size criterion. The value of $C(I_1 \succ I_2)$, i.e. the chances that I_1 is preferable to I_2, is chosen as the first criterion and as the risk size criterion is chosen an indicator $R_s(I_1 \succ I_2) = 1 - C(I_1 \succ I_2) = C(I_2 \succ I_1)$, i.e. an indicator of the possible error of the decision on choosing the alternative I_1 as the preferred one.

To obtain one-numerical characteristics of fuzzy alternatives, which include such characteristics as $C(I_1 \succ I_2)$ and $R_s(I_1 \succ I_2)$, the defuzzification procedure should be used.

In the course of applying the procedure, one-numerical characteristics, such as $C_\alpha(I_1 \succ I_2)$, calculated on mono-intervals $I(\alpha)$, are averaged over all α with one or another weight, depending on the defuzzification method used. Here $C_\alpha(I_1 \succ I_2)$ are the preference chances of mono-intervals forming a poly-estimates for a given value of α. The choice of defuzzification method is determined by an expert. We will use two defuzzification methods in the continuous case.

In the first of them, all $I(\alpha)$, are considered in the process of obtaining the final defuzzified one-numerical characteristic $C_F(I_1 \succ I_2)$ in a parity basis (simple averaging). In the second, in the center of gravity method, the contribution of $I(\alpha)$ to the one-numerical characteristic $C_{FG}(I_1 \succ I_2)$ increases with increasing α.

$$C_F(I_1 \succ I_2) = \int_0^1 d\alpha C_\alpha(I_1 \succ I_2), C_{FG}(I_1 \succ I_2) = 2\int_0^1 d\alpha \alpha C_\alpha(I_1 \succ I_2). \quad (1)$$

Since $0 < \alpha < 1$, the center of gravity method reduces the contribution to the final indicator of mono-intervals of greater range (greater uncertainty and, therefore, a greater contribution to the risk of choosing the preferred alternative). Therefore, one might expect that for the same configurations of pairwise compared alternatives, the numerical estimates for the risk indicator calculated by the center of gravity method will be smaller, and the estimates of the preference indicator, respectively, will be higher than for the defuzzification method by averaging. Further, it will be shown that this is not so: the localization of alternatives in their compared pairs has no less influence on the value of the preference and risk criteria calculated by different defuzzification methods. Which of the estimates will be more (less) is determined by the geometry of each specific configuration, the central role is played by the relationships $(>, <, =)$, between the upper corner points of the triangular membership functions of the compared alternatives.

Recall that we restrict ourselves to considering triangular membership functions. Earlier, six different configurations of intersecting triangular membership functions were indicated, which, up to permutation, exhaust the possible configurations. The regions of integration in (1), due to the different geometry of the intersecting membership functions of the compared alternatives, are divided into several connected subdomains, each of which has its own function $C_\alpha(I_1 \succ I_2)$.

Let us pay attention to the fact that for each of the above-mentioned connected subdomains, delimited by the upper α_u and lower α_d values of the parameter α, the conditions of equality to unity of the sum of the preference and risk indicators are not met. Instead, for each connected subdomain $\alpha_d < \alpha < \alpha_u$ for the corresponding chances $C\big(I_i \succ I_j \big| \alpha_d, \alpha_u\big)$ there is a relation that depends not only on the boundaries of the region, but also on the defuzzification method.

So the equality $C(I_1 \succ I_2|\alpha_d, \alpha_u) + C(I_2 \succ I_1|\alpha_d, \alpha_u) = \alpha_u - \alpha_d$ holds for defuzzification by simple averaging, and for defuzzification by the center of gravity method, a similar equality has the another form: $C(I_1 \succ I_2|\alpha_d, \alpha_u) + C(I_2 \succ I_1|\alpha_d, \alpha_u) = \alpha_u^2 - \alpha_d^2$. But in all such connected subdomains on separate mono-intervals that form a multi-interval estimate the relation $C_\alpha(I_2 \succ I_1) = 1 - C_\alpha(I_1 \succ I_2)$ is preserved. The main reason for distinguishing between the indicated subdomains is the fact that the functions $C_\alpha(I_2 \succ I_1)$, included in (1), change when passing from one subdomain to another.

Let us now turn to the derivation of analytical relations for the indicators $C(I_1 \succ I_2)$ (preference) and risk $R_s(I_1 \succ I_2)$ for all the above configurations. We will base on the formula (1).

One can see that for triangular membership functions at $T_1 > T_2$ it is more convenient to look for the chances $C(I_2 \succ I_1)$ and for $T_1 < T_2$ the chances $C(I_1 \succ I_2)$. Indeed, the chances $C(I_2 \succ I_1)$ are equal to zero in the subdomain of α values between the largest intersection point of the graphs of the membership functions of the compared alternatives and unity, in the first case. This simplifies the integration in (1) and in the second case the same is true for chances $C(I_1 \succ I_2)$.

For all configurations, we will connect preference indicators $C_{FG}(I_1 \succ I_2)$ or $C_{FG}(I_2 \succ I_1)$obtained by defuzzification with the center of gravity method, and indicators $C_{FG}()$, obtained with defuzzification by averaging, through the relation $C_{FG}() = C_{FG}() + DIF$, where the DIF is function of the difference for the same-name one-numerical indicators obtained by the indicated defuzzification methods.

Let us start with configuration 1. Here, as already noted, it is more convenient to look for the chances $C(I_2 \succ I_1)$. The subdomain where the preference $I_2 \succ I_1$ is possible is limited by the band $0 < \alpha < \alpha_1$. In the subdomain $\alpha_1 < \alpha < 1$ the chances of preference for $I_2 \succ I_1$ are 0.

The fact that the mono-intervals of both compared fuzzy objects nested in the graphs of the membership functions are compared for the same α-values and that on all (normal, not fuzzy) intervals $I(\alpha)$, corresponding to α-levels, uniform distributions are given is a distinctive significant feature of the comparison of fuzzy poly-interval objects [6]. Let $i_1(\alpha)$ and $i_2(\alpha)$ be the mono-intervals of the objects I_1 and I_2 respectively, corresponding to some admissibleα. One can see from the geometry of the configuration that in the subdomain of preference $I_2 \succ I_1$ there are only mono configurations of the right shift, when $i_1(\alpha)$ are shifted to the right from $i_2(\alpha)$. This means that at every admissible α-level

$$C\big(i_2(\alpha) \succ i_1(\alpha)\big) = [(1-\alpha)(R_2 - L_1) + \alpha(T_2 - T_1)]^2 / [2(1-\alpha)^2 \Delta_1 \Delta_2]. \quad (2)$$

It follows from this that the possible values of chances lie in a certain interval of values in accordance with changing α. This also applies to other characteristics of fuzzy objects [12]. Communication with expert-practitioners is preferably carried out in their usual language. Namely therefore, it is advisable to move from such interval values to characterizing their one-numeric estimates. As already stated the defuzzification procedure is used for such transformation.

In the first defuzzification method integrating (2) over α from 0 to α_1 we obtain for the risk indicator of the first configuration $R_{sF1}(I_1 \succ I_2) = C_{F1}(I_2 \succ I_1)$:

$$R_{sF1}(I_1 \succ I_2) = \frac{1}{2\Delta_1\Delta_2}\Big[(R_2 - L_1)^2 + 2(R_2 - L_1)(T_1 - T_2) + 2(T_1 - T_2) \times \\ \times (R_2 - L_1 + T_1 - T_2)\ln\frac{T_1 - T_2}{T_1 - T_2 - L_1 + R_2}\Big]. \quad (3)$$

In the second defuzzification method integrating (2) over α from 0 to α_1 with weight 2α we obtain an expression for the risk indicator of the first configuration $R_{sFG1}(I_1 \succ I_2) = C_{FG1}(I_2 \succ I_1)$. After some transformations and comparison of the resulting expression with (3), we have, in accordance with our agreement for the difference between the risk indicators for the two considered defuzzification methods $DIF_1 = R_{sFG1}(I_1 \succ I_2) - R_{sF1}(I_1 \succ I_2)$:

$$DIF_1 = \frac{(T_1 - T_2)}{\Delta_1 \Delta_2}\left\{2(R_2 - L_1) + [R_2 - L_1 + 2(T_1 - T_2)]\ln\frac{T_1 - T_2}{R_2 - L_1 + T_1 - T_2}\right\}. \quad (3A)$$

Let us show that for configuration 1 the variable DIF_1 is negative, so that the risk indicator for defuzzification by averaging is greater than for defuzzification by the center of gravity method, and for the preference indicators we have, therefore, the opposite inequality. One can see that with a fixed alternative I_2 and a given right border R_1 of alternative I_1, the values of risk indicators for both defuzzification methods and, therefore, the values of the difference DIF_1 are determined by the position of the left border L_1 of alternative I_1. Let us examine the behavior of DIF_1 when moving L_1 within the first configuration ($L_2 \le L_1 \le R_2$). We rewrite the expression for DIF_1 as $DIF_1 = S_1 F_1$, where the value $S_1 = \Delta T/(\Delta_1 \Delta_2)$ and $\Delta T = T_1 - T_2\ \Delta T$. The value S_1 is positive and does not affect the sign of DIF_1. For F_1 we have: $F_1(L_1) = 2(R_2 - L_1) - (R_2 - L_1 + 2\Delta T)\ln[1 + (R_2 - L_1)/\Delta T]$.

One can see that $D_1 = \partial F_1/\partial L_1 = -(R_2 - L_1)/(\Delta T + R_2 - L_1) + \ln[1 + (R_2 - L_1)/\Delta T]$, $D_2 = \partial^2 F_1/\partial {L_1}^2 = -(R_2 - L_1)/(\Delta T + R_2 - L_1)^2$ and less than zero, that is on $[L_2, R_2]$ D_1 is a decreasing function of L_1. Since $D_1(L_1 = R_2) = D_2(L_1 = R_2) = F_1(L_1 = R_2) = 0$ and this is the only point on $[L_2, R_2]$ for which this condition is true, then D_1 in the studied configuration is positive, and $F_1(L_1)$ is an upward convex increasing function of L_1. This is only possible if $F_1(L_1)$, and therefore DIF_1, are negative on $[L_2, R_2)$.

Since $DIF_1 < 0$, then $R_{sFG1} < R_{sF1}$ and therefore for the first configuration (with $T_2 < T_1$ the preference estimates obtained with the center of gravity defuzzification method are larger than with the first defuzzification method, the simple averaging method.

In configuration 2, in addition to point α_1, there is one more point of intersection of membership functions, α_1. That is there are two subdomains of possible preference $I_2 \succ I_1$. In the first of them, where $0 < \alpha < \alpha_2$, $i_1(\alpha)$ are embedded in $i_2(\alpha)$. Therefore, for uniform distributions of the chances of preference $C_1\big(i_2(\alpha) \succ i_1(\alpha)\big)$ on mono-intervals in this subdomain, we have:

$$C_1\big(i_2(\alpha) \succ i_1(\alpha)\big) = (2R_2 - R_1 - L_1)/(2\Delta_2) + \alpha(T_2 - T_1)/[(1-\alpha)\Delta_2]. \quad (4)$$

In the second subdomain of possible preference $I_2 \succ I_1$, for which $\alpha_2 < \alpha < \alpha_1$, right shift configurations arise for mono-intervals with mono-intervals $i_1(\alpha)$ shifted to the right relative to $i_2(\alpha)$. Therefore, for the chances of preference $C_2\big(i_2(\alpha) \succ i_1(\alpha)\big)$ in this subdomain we have a relation similar to relation (2). Integrating (4) over α in the range from 0 to α_2 and (2) in the range from α_2 to α_1 and adding the results, after some transformations we get:

$$R_{sF2}(I_1 \succ I_2) = \frac{2(R_2 - T_2 + T_1) - L_1 - R_1}{2\Delta_2} + \frac{T_1 - T_2}{\Delta_2}\left[\ln\frac{T_1 - T_2}{T_1 - T_2 - R_1 + R_2} + \right.$$
$$\left. + \frac{T_2 - T_1 - R_2 + L_1}{\Delta_1}\ln\frac{R_2 + T_1 - T_2 - L_1}{T_1 - T_2 - R_1 + R_2}\right]. \quad (5)$$

This configuration was studied in [8] in relation to the comparison of investment projects by efficiency.

Again, integrating (2) and (4) over α with weight 2α in correspondence with limits, for the difference between the risk indicators for the two considered defuzzification methods, we obtain: $DIF_2 = R_{sFG2}(I_1 \succ I_2) - R_{sF2}(I_1 \succ I_2)$, where

$$DIF_2 = \frac{(T_1 - T_2)}{\Delta_2}\left[2 + \ln\frac{T_1 - T_2}{R_2 - R_1 + T_1 - T_2} + \frac{R_2 - L_1 + 2(T_1 - T_2)}{\Delta_2}\ln\frac{R_2 - R_1 + T_1 - T_2}{R_2 - L_1 + T_1 - T_2}\right]. \quad (5A)$$

It can be shown that in this configuration, as in configuration 1, the difference function DIF_2 of the risk indicators calculated by two defuzzification methods is negative. Let us note that in general, all DIF functions considered here are negative. However, hereinafter, we will not do required proofs for DIF functions, given that, as can be seen from the corresponding simplest proof for DIF_1, these proofs are quite long. They will be given in another publication. Recall that the condition $T_1 > T_2$ serves as a feature that combines configurations 1 and 2.

In configuration 3, for which $T_2 > T_1$, $L_2 \le L_1 < R_1 \le R_2$, there are two new points of intersection of the membership functions α_3 and α_4, which determine the limits of integration. In this case, two subdomains of possible preference $I_1 \succ I_2$ arise: $0 < \alpha < \alpha_3$ and $\alpha_3 < \alpha < \alpha_4$.

Since in the subdomain $\alpha_4 < \alpha < 1$ the function $C(I_1 \succ I_2 | \alpha_4, 1) = 0$, in this configuration it is more convenient to calculate directly the chances of preference $C(I_1 \succ I_2)$ for both defuzzification methods. In the subdomain $0 < \alpha < \alpha_3$ $i_1(\alpha)$ are embedded in $i_2(\alpha)$, and in the subdomain $\alpha_3 < \alpha < \alpha_4$ the mono-intervals $i_2(\alpha)$ are shifted to the right with respect to $i_1(\alpha)$. Hence

$$C_\alpha(I_1 \succ I_2|0 < \alpha < \alpha_3) = (R_1 + L_1 - 2R_2)/(2\Delta_2) + \alpha(T_1 - T_2)/[(1-\alpha)\Delta_2], \quad (6)$$

$$C_\alpha(I_1 \succ I_2|\alpha_3 < \alpha < \alpha_4) = [(1-\alpha)(R_1 - L_2) + \alpha(T_1 - T_2)]^2/[2(1-\alpha)^2\Delta_1\Delta_2]. \quad (6A)$$

Acting as above, in configuration 3 for the preference indicators $C_{F3}()$ and $C_{FG3}()$ we obtain:

$$C_{F3}(I_1 \succ I_2) = \frac{R_1 + L_1 - 2(L_2 - T_2 + T_1)}{2\Delta_2} + \frac{T_1 - T_2}{\Delta_2}\left(\ln\frac{T_2 - T_1}{T_2 - T_1 + L_1 - L_2} + \right.$$
$$\left. + \frac{T_2 - T_1 + R_1 - L_2}{\Delta_1}\ln\frac{T_2 - T_1 + L_1 - L_2}{T_2 - T_1 + R_1 - L_2}\right). \quad (7)$$

$$DIF_3 = C_{FG3}(I_1 \succ I_2) - C_{F3}(I_1 \succ I_2),$$
$$DIF_3 = \frac{T_2 - T_1}{\Delta_2}\left[2 + \ln\frac{T_2 - T_1}{L_1 - L_2 + T_2 - T_1} + \frac{R_1 - L_2 + 2(T_2 - T_1)}{\Delta_1}\ln\frac{L_1 - L_2 + T_2 - T_1}{R_1 - L_2 + T_2 - T_1}\right]. \quad (7A)$$

It is again more convenient to calculate the chances $C(I_1 \succ I_2)$. in configurations 4 and 5. The limits of integration are set by the already known points α_2, α_3, α_4. However, in configuration 4 $\alpha_2 > \alpha_3$, and in configuration 5, $\alpha_2 < \alpha_3$. One can see that configuration 4 occurs if $\Delta_2 > \Delta_1$, and configuration 5 if $\Delta_1 > \Delta_2$, however, in both configurations, the expressions $C(I_1 \succ I_2)$ for the chances obtained by defuzzification do not differ. Therefore, we can restrict ourselves to considering configuration (4).

In the case of this configuration, there are three sub-domains where $I_1 \succ I_2$ is possible. This is a subdomain A_1, where $0 < \alpha < \alpha_3$, A_2 for $\alpha_3 < \alpha < \alpha_2$, and a subdomain A_3 with $\alpha_2 < \alpha < \alpha_4$. In subdomain A_1 the mono-intervals $i_1(\alpha)$ are shifted to the right relative to $i_2(\alpha)$, so that for $C_1 = C_\alpha(I_1 \succ I_2|0 < \alpha < \alpha_3)$ we have:

$$C_1 = 1 - [(1-\alpha)(R_2 - L_1) + \alpha(T_2 - T_1)]^2/[2(1-\alpha)^2\Delta_1\Delta_2]. \quad (8)$$

The mono-intervals $i_2(\alpha)$ are embedded in $i_1(\alpha)$ in the subdomain A_2, so that for preference indicator for mono-intervals $C_2 = C_\alpha(I_1 \succ I_2|\alpha_3 < \alpha < \alpha_2)$ we obtain:

$$C_2 = (2R_1 - L_2 - R_2)/(2\Delta_1) + \alpha(T_1 - T_2)/[(1-\alpha)\Delta_1]. \quad (8A)$$

The mono-intervals $i_2(\alpha)$ are shifted to the right with respect to $i_1(\alpha)$ in the subdomain A_3, so that for $C_3 = C_\alpha(I_1 \succ I_2|\alpha_2 < \alpha < \alpha_4)$ we obtain the relation

$$C_3 = [(1-\alpha)(R_1 - L_2) + \alpha(T_1 - T_2)]^2/[2(1-\alpha)^2\Delta_1\Delta_2]. \quad (8B)$$

After integrating, adding the results for the indicated subdomains and simplifying the resulting expression, we have:

$$C_{F4}(I_1 \succ I_2) = 1 + \frac{(R_2 - L_1)[2(T_2 - T_1) + L_1 - R_2]}{2\Delta_1\Delta_2} + \frac{T_2 - T_1}{\Delta_1}\left(\ln\frac{T_2 - T_1 + L_1 - L_2}{T_2 - T_1 + R_1 - R_2} + \right.$$
$$\left. + \frac{T_2 - T_1 + R_1 - L_2}{\Delta_2}\ln\frac{T_2 - T_1 + R_1 - R_2}{T_2 - T_1 + R_1 - L_2} + \frac{R_2 - T_2 + T_1 - L_1}{\Delta_2}\ln\frac{T_2 - T_1}{T_2 - T_1 + L_1 - L_2}\right). \quad (9)$$

After using the center of gravity defuzzification, integrating and transformations of resulting expressions we obtain for $DIF_4 = C_{FG4}(I_1 \succ I_2) - C_{F4}(I_1 \succ I_2)$:

$$DIF_4 = \frac{T_2 - T_1}{\Delta_1\Delta_2}\left\{2(R_2 - L_1) - \frac{T_2 - T_1}{2}\ln\frac{T_2 - T_1}{L_1 - L_2 + T_2 - T_1} + [R_1 - L_2 + 2(T_2 - T_1)] \times \right.$$
$$\left. \times \ln\frac{R_1 - R_2 + T_2 - T_1}{R_1 - L_2 + T_2 - T_1} + \Delta_2\ln\frac{L_1 - L_2 + T_2 - T_1}{R_1 - R_2 + T_2 - T_1}\right\}. \quad (9A)$$

The region of possible preference of $I_1 \succ I_2$ covers in configuration 6 the entire region $0 < \alpha < 1$, mono-intervals $i_2(\alpha)$ are embedded in $i_1(\alpha)$, and the corner point T is common for membership functions. One can see then that $C(I_1 \succ I_2|\alpha) = (2R_1 - R_2 - L_2)/(2\Delta_1)$ for all α, i.e. does not depend on α. Therefore, integration over α results in unity for both defuzzification methods and

$$C_{F6}(I_1 \succ I_2) = C_{FG6}(I_1 \succ I_2) = (2R_1 - R_2 - L_2)/(2\Delta_1). \quad (10)$$

We now note two facts, firstly, when the upper corner points of the membership functions coincide ($T_1 = T_2 = T$) the functions $C_\alpha(I_1 \succ I_2)$ do not depend on α as can be seen from relations (2, 4, 6, 8), and therefore the indicators of preference and risk coincide in this case for both defuzzification methods for all configurations. Secondly, the negativity of the functions DIF for all considered configurations leads to opposite conclusions for relative values of the preference and risk indicators obtained by different defuzzification ways. If the preference indicators obtained with the center of gravity defuzzification method are larger than with the first defuzzification one in the first two configurations (with $T_2 < T_1$), then the preference estimates obtained with the center of gravity defuzzification method are less than with the first defuzzification one in the next three configurations (with $T_2 > T_1$).

Thus, the decisive role in this behavior of indicators in each specific configuration is played by the relations $(>, <, =)$ between the upper corner points of the triangular membership functions of the compared fuzzy objects.

4. Collective Risk Estimating: Generalized Interval Approach

The general interval estimations approach is a direct generalization of the mono-interval approach to the poly-interval case. In the first of them, to take into account the uncertainty of knowledge about analyzed parameter, its initial point estimate is "blurred", not necessarily symmetrically, filling in a certain interval of possible values of the parameter. To describe the chances of implementation of possible point realizations x of the parameter, the apparatus of distribution functions is used. It is specified on carrier interval by the density of the chances distribution function $f(x)$.

The interval $I_u = [L_u, R_u]$ serves as the initial estimate in the general interval approach and it is already blurred, again not necessarily symmetrically, giving, as a final parameter estimate, a system of intervals with a maximum length interval $I_d = [L_d, R_d]$.Which intervals will be included in the resulting system, delimited by I_u and I_d, is determined by the form of the so-called poly-interval estimate (PIE), i.e. by a curvilinear trapezium containing all the intervals included in their system. To specify chances of implementation of the intervals forming the system, a random variable β is inserted, placed on the ordinate axis of the two-dimensional plane and having a density of chance distribution $f_1(\beta)$. The variable of β serves as a label for the intervals included in their system. The chances of implementations of possible point realizations x on each of the intervals with label β, placed on the x-axis of a two-dimensional plane, are described by a conditional distribution function with a density $f_2(x|\beta)$. Thus general interval estimation is PIE and $f(\beta, x) = f_1(\beta)\, f_2(x|\beta)$, density of joint distribution function, which is given on the PIE. We will further assume that the sides of the PIE are straightforward, the estimate is normalized so that $0 < \beta < 1$, the label $\beta = 0$ corresponds to the interval $[L_d = L, R_d = R]$, $\beta = 1$ to the interval $[L_u, R_u]$, and $L_d < L_u < R_u < R_d$. Such configurations most often arise when expert knowledge of the parameters of the analyzed problems is presented as generalized interval estimations.

Let the PIE has a triangular shape, as in the considered above case of fuzzy objects with triangular membership functions, and be given by three corner points such that $L < T < R$. This corresponds to the situation when the initial point estimate T is replaced by the interval system. Chance distributions $f_1(\beta)$ and $f_2(x|\beta)$ can be any.

Let, however, for simplicity, the distributions of chances on the coordinate axes of the PIE are uniform. Then, integrating over all β and taking into account triangular form of the joint distribution function on the PIE, we obtain on the interval $[L, R]$, with the label $\beta = 0$ the density of the marginal chances distribution function $f(x)$, or the density of the generalized uniform distribution (GUD). GUD on $[L, R]$ is a probability mixture of the distributions $f_2(x|\beta)$ with the mixing function $f_1(\beta)$. The properties of the GUD for trapezoidal and, as a special case, for triangular PIE have been studied by us earlier [9].

Using these results we have for the density $f(x)$ of GUD on PIE of a triangular shape: for $< x < T$ $f(x) = f_l(x)$; for $L < x < R$ $f(x) = f_r(x)$, where $f_l(x)$ and $f_r(x)$ are the left and right branches of the density distribution of the GUD:

$$f_l(x) = \ln[(T - L)/(T - x)]/(R - L);\ f_r(x) = \ln[(R - T)/(x - T)]/(R - L). \quad (11)$$

Let us turn to relations for the chances of preference and risk in the general interval formalism. It was noted above that in the case of general interval estimations it is possible to transform poly-interval estimates into mono-interval ones by passing to probabilistic mixtures. Namely, the system of chances distributions on general interval estimation intervals can be replaced by distribution set on the interval of the greatest extent (base interval) of general interval estimation. This distribution is a probabilistic mixture of chances distributions of the system. The configurations of poly-interval objects defined above are transferred to the configurations of mono-interval objects with preservation of the relationship $<>$ for corner points T_i after such a replacement. Specifically, configurations 1 and 4 are transferred to configuration of the right-shift for a pair of mono-interval objects (the estimate I_1 is shifted to the right), configurations 2 and 3 are transferred to the configuration of nested intervals (the estimate I_1 is embedded in I_2), configuration 6 passes into the configuration of nested intervals (estimate I_2 nested in I_1).

Previously, using simple geometric considerations, relations for chances of preference and corresponding risks were obtained for uniform [6] and triangular distributions of chances on compared mono-interval estimates. We used above such relations for uniform distributions in the process of comparing fuzzy objects. However, distribution (11) differs significantly from uniform one. Thus, we need now the similar relations for arbitrary chances distributions. We will use for this purpose the "integral" comparison method, originally proposed in [10] and developed by us for application to arbitrary distributions.

Let us demonstrate the features of its application for the first configuration of pairs of compared alternatives. In the configuration under consideration, we are dealing, as was indicated above, with a right shift situation for mono-interval estimates. Let i_j be the current point realizations of the values of the quality indicator $I_j, i_j \in I_j, j = 1{,}2$. In the case of a right-shift configuration, it is easier to distinguish events that favor the hypothesis $I_2 \succ I_1$ from the complete system of events. These are events in which point implementations lie in the area $(i_1 \in [L_1, R_2]) \cap (i_2 \in [L_1, R_2])$. Then

$$C(I_2 \succ I_1) = \int_{L_1}^{R_2} f_1(x_1) \int_{x_1}^{R_2} f_2(x_2) d\,x_1 dx_2.$$

We consider here the case $L_2 < T_2 < L_1 < R_2 < T_1 < R_1$ from all the possible options of the relative positions of L_1 and T_2, R_2 and T_1 in the first configuration. Recalling expressions for the GUD density for a triangular PIE, when for $L < x < T$ $f(x) = f_l(x)$ and for $< x < R$ $f(x) == f_r(x)$, we have:

$$C(I_2 \succ I_1) = \int_{L_1}^{R_2} f_{1l}(x_1) \int_{x_1}^{R_2} f_{2r}(x_2) d\,x_1 dx_2.$$

Integrating, we get

$$C(I_2 \succ I_1) = \left[\frac{(R_1^2 - T_1^2)}{2} \ln\frac{T_1 - R_1}{T_1 - L_1} - \frac{(R_1 - L_1)(L_1 + 2T_1 + R_1)}{4} + \right.$$
$$\left. + \int_{L_1}^{R_2} dx(x - T_2) \ln\frac{x - T_2}{R_2 - T_2} \ln\frac{T_1 - x}{T_1 - L_1}\right] \Big/ (\Delta_1 \Delta_2). \qquad (12)$$

Taking in parts, this integral can be simplified to $\int_c^d dx \frac{\ln(ax+b)}{x}$. However, the indefinite integral appearing here cannot be expressed in finite form through elementary functions. If to specify concrete general estimations parameters, values for the chances of preference and the corresponding risks can be obtained by taking this integral by numerical methods.

It might be appropriate to discuss here some aspects of connection between fuzzy theory and general interval approach in interval alternatives comparing. We already noted that the differences in the approaches are that in the case of fuzziness mono-interval comparisons are made for identical α-levels [6], and in the general interval case, due to mixing the component distributions on the base interval of PIE, for arbitrary permissible ("mixed") values of α. This leads to lower values of the risk indicator for fuzzy objects in comparison with general interval ones. So for an example of the configuration, which we just analysed, for $L_1 = 2$; $T_1 = 4$; $R_1 = 5$; $L_2 = 1$; $T_2 = 1.5$; $R_2 = 3$; $R_{sF} = 0.15$, and $R_{sFGE} = 0.275$. Thus, the use of the general interval approach leads to more careful estimates.

Let note another significant fact. One can show that relations (11) for the distribution density of the GUD, $f_l(x)$ and $f_r(x)$ in the general interval approach, are obtained by defuzzification of distributions on α–levels of fuzzy objects by the first defuzzification. Indeed, density $f_U(\alpha)$ of distributions on α–levels is $f_U(\alpha) = 1/[(1-\alpha)\Delta]$. We have for chances densities $f_1(x)$ and $f_2(x)$ (for $L < x < T$ and $T < x < R$ respectively), averaged by the first method of defuzzification:

$$f_1(x) = \int_0^{(x-L)/(T-L)} d\alpha \Big/ [(1-\alpha)\Delta], \qquad f_2(x) = \int_0^{(R-x)/(R-T)} d\alpha \Big/ [(1-\alpha)\Delta].$$

One can see that $f_1(x) = f_l(x), f_2(x) = f_r(x)$ from (11). Using different methods of defuzzification it can receive different general chances distributions corresponding to fuzzy objects. In particular, integrating the above relations for $f_{1(2)}(x)$ with a weight of 2α, we obtain the densities of the chances distributions $f_{1G(2G)}(x)$, corresponding to defuzzification by the center of gravity method.

$$f_{1G}(x) = \frac{2}{\Delta}\left(\ln\frac{T-L}{T-x} - \frac{x-L}{T-L}\right), \qquad f_{2G}(x) = \frac{2}{\Delta}\left(\ln\frac{R-T}{x-T} - \frac{R-x}{R-T}\right).$$

It means that when comparing the comparison results for fuzzy and generalized interval objects, one should pay attention to what method of defuzzification was used in both cases.

One should mention that connections of fuzzy concept and probability theory was outlined in other works [11, 12].

5. Conclusion

Decision-making problems under uncertainty is an established scientific direction, the results of which have numerous practical applications. This direction received a new development with the advent of fuzzy theory and theory of possibilities (Lotfi A. Zadeh, D. Dubois, H. Prade), the rough sets approach (Z. Pawlak [17]), the theory of NON-factors (Narinyani A.S [18]), the theory of evidence (Dempster A., Shafer G. [19]), soft sets (D. Molodtsov [20]), the approach of generalized interval estimates (Shepelev G.).

One of the branches of this direction is the problematic of comparing by effectiveness of alternatives with interval quality indicators. Natural way these studies are associated with filling the theory of information granules and granular computations with mathematical content [13-16].

Numerous, sometimes contradictory, comparison methods for comparing fuzzy objects have been developed earlier [21, 22]. However, due to the need to compare dissimilar objects under conditions of uncertainty, it is advisable to use universal comparison methods, such as the "mean-risk" method and the collective risk estimating method. Since the indicators of these methods calculate risks of different types and these methods complement each other in the process of evaluating alternatives, their joint consistent use increases the validity of decisions.

The results obtained can be used in solving various practical problems. These include, in particular, the problems of analyzing the effectiveness and risk of investments [6, 8].

Bearing in mind that the results of comparing poly-interval objects can find application in intelligent computer systems, research on the development of methods for their comparison needs further development. In particular, the extension of the proposed approaches to objects with trapezoidal membership functions of comparing fuzzy objects, as well as to generalized interval objects deserves attention. In addition, since each of the indicators characterizing an interval object is associated with two criteria, namely, with preference and risk, it is advisable to study the problem of multi-criteria comparison of poly-interval objects with several quality indicators.

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7. References

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