Symbol Elimination and Applications to Parametric Entailment Problems
(Abstract)

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Abstract. We analyze possibilities of second-order quantifier elimination for formulae containing parameters – constants or functions. For this, we use a constraint resolution calculus obtained from specializing the hierarchical superposition calculus. If the saturated set of formulae has a finite representation, we analyze possibilities of obtaining weakest constraints on parameters which guarantee satisfiability. We identify situations in which entailment between formulae expressed using second-order quantification can be effectively checked. We illustrate the ideas on an example from wireless network research.

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1 Introduction

The motivation for this work was a study of models for graph classes naturally occurring in wireless network research – in which nodes that are close are always connected, nodes that are far apart from each other are never connected and any other node pairs can, but do not need to be connected. Transformations can be applied to such graphs to make them symmetric; this leads to further graph classes. When checking inclusion between graph classes described using transformations we need to check entailment of second-order formulae. If inclusion cannot be proved and the graph class descriptions are parametric we want to obtain (weakest) conditions on these parameters that guarantee that inclusions hold. This can be achieved by eliminating "non-parametric" constants or function symbols used in the description of such classes.

We show that it is possible to combine methods for general symbol elimination (for eliminating existentially quantified predicates) with methods for property-directed symbol elimination (for obtaining conditions on "parameters" under which formulae are satisfiable or second-order entailment holds). For general second-order quantifier elimination we use a form of ordered resolution similar to that proposed in [11]. For property-directed symbol elimination we use a

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method we proposed in [24]. The advantage of using such a two-layered approach is that it avoids non-termination that might occur if using only general symbol elimination methods. The main application area we consider in this paper is the analysis of inclusions between graph classes arising in wireless network research. The study of second-order quantifier elimination goes back to [7,1,2]. Most of its known applications are in the study of modal logics or knowledge representation [12,13]. In [11], Gabbay and Ohlbach proposed a resolution-based algorithm for second-order quantifier elimination which is implemented in the system SCAN. In [4], Bachmair et al. mention that hierarchical superposition (cf. [5,6] for further refinements) can be used for second-order quantifier elimination modulo a theory. In [19,14], Hoder et al. study possibilities of symbol elimination in inference systems (e.g. the superposition calculus and its extension with ground linear rational arithmetic and uninterpreted functions). Since the saturated sets might be infinite, possibilities of finding finite representations were investigated in the context of superposition [15] or in verification, in relationship to acceleration [9,10,3,8]. Orthogonal to this direction of study is the "property-directed" symbol elimination: Given a theory $\mathcal{T}$ and a ground formula G satisfiable w.r.t. $\mathcal{T}$, the goal is to derive a (weakest) universal formula Γ over a subset of the signature, such that $\Gamma \wedge G$ is unsatisfiable w.r.t. $\mathcal{T}$. In [24] we proved that this is possible for local extensions of theories allowing quantifier elimination.

Structure of the paper. In Sect. 2 we introduce the theoretical results needed in the paper. In Sect. 3 we present an ordered hierarchical resolution calculus, $\mathit{HRes}_{\succ}^{P}$, which can be used for eliminating predicate P, and mention possibilities of giving finite representations for infinite saturated sets and of investigating the satisfiability of the saturated sets. In Sect. 4 we use these ideas for checking entailment and illustrate the method with an example.

This is an abstract containing results published in [21]. Details of the proofs and additional examples can be found in [20] (which is the extended version of [21]).

2 Local Extensions; Hierarchical Symbol Elimination

We assume known the basic notions in (many-sorted) first-order logic. We consider signatures of the form $\Pi = (S, \Sigma, \mathsf{Pred})$, where S is a set of sorts, Σ is a family of function symbols and Pred a family of predicate symbols, such that for every function symbol f (resp. predicate symbol p) their arity $a(f) = s_1 \dots s_n \rightarrow s$ (resp. $a(p) = s_1 \dots s_m$), where $s_1, \dots, s_n, s \in S$, is specified. If C is a fixed countable set of fresh constants, we denote by Π^C the extension of Π with constants in C. A Π-structure $\mathcal{A}$ is a tuple $(\{A_s\}_{s \in S}, \{f_{\mathcal{A}}\}_{f \in \Sigma}, \{p_{\mathcal{A}}\}_{p \in \mathsf{Pred}})$ where for every function symbol f with arity $a(f) = s_1 \dots s_n \rightarrow s$, $f_{\mathcal{A}} : A_{s_1} \times \dots \times A_{s_n} \rightarrow A_s$, and for every predicate symbol p with $a(p) = s_1 \dots s_m$, $p_{\mathcal{A}} \subseteq A_{s_1} \times \dots \times A_{s_m}$. If $\Pi \subseteq \Pi'$ and $\mathcal{A}$ is a Π'-structure, we denote its reduct to Π by $\mathcal{A}_{|\Pi}$.

A theory $\mathcal{T}$ can be defined by specifying a set of axioms, or by specifying a class of structures (the models of the theory). If F and G are formulae we write $F \models G$ (resp. $F \models_{\mathcal{T}} G$ – also written as $\mathcal{T} \cup F \models G$) to express the fact that every model of F (resp. every model of F which is also a model of $\mathcal{T}$) is a model of G. We write $F \models \perp$ (resp. $F \models_{\mathcal{T}} \perp$) to express the fact that F has no model (resp. that there is no model of $\mathcal{T}$ which is a model of F). A theory $\mathcal{T}$ over a signature Π *allows quantifier elimination* (QE) if for every formula ϕ over Π there exists a quantifier-free formula ϕ^* over Π which is equivalent to ϕ modulo $\mathcal{T}$. Examples of theories which allow quantifier elimination are rational and real linear arithmetic ($\mathsf{LI}(\mathbb{Q})$, $\mathsf{LI}(\mathbb{R})$) and the theory of real closed fields.

Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ be a signature, and $\mathcal{T}_0$ be a "base" theory with signature Π_0. We consider extensions $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ with new function symbols Σ (*extension functions*) whose properties are axiomatized using a set $\mathcal{K}$ of (universally closed) clauses in the extended signature $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$, such that each clause in $\mathcal{K}$ contains function symbols in Σ. Especially well-behaved are the *Ψ-local theory extensions*, where Ψ is a suitable closure operator (for details on the properties of the closure operators we consider we refer to [18]). Ψ-local theory extensions are extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfying the following condition:

(Loc_f^Ψ) For every finite set G of ground Π^C-clauses (for an additional set C of constants) it holds that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ is unsatisfiable.

where, for every set G of ground Π^C-clauses, $\mathcal{K}[\Psi_{\mathcal{K}}(G)]$ is the set of instances of $\mathcal{K}$ in which the terms starting with a function symbol in Σ are in $\Psi_{\mathcal{K}}(G) = \Psi(\mathsf{est}(\mathcal{K}, G))$, where $\mathsf{est}(\mathcal{K}, G)$ is the set of ground terms starting with a function in Σ occurring in G or $\mathcal{K}$. Ψ-local extensions can be recognized by showing that certain partial models embed into total ones [22,18]. Especially well-behaved are theory extensions with the property (Comp_f^Ψ) which requires that every partial model of $\mathcal{T}$ whose reduct to Π_0 is total and the "set of defined terms" is finite and closed under Ψ embeds into a total model of $\mathcal{T}$ *with the same support* (cf. [16,18]). If Ψ is the identity, we denote Loc_f^Ψ by Loc_f and Comp_f^Ψ by Comp_f. In (Ψ)-local theory extensions hierarchical reasoning is possible. If the base theory allows quantifier elimination, a form of property-directed symbol elimination is also possible: the symbol elimination problem is hierarchically reduced to a quantifier elimination problem w.r.t. the base theory.

Theorem 1 ([23,24]) *Let $\mathcal{T}_0$ be a Π_0-theory allowing quantifier elimination, Σ_{par} be a set of parameters (function and constant symbols) and $\Pi = (S, \Sigma, \mathsf{Pred})$ be such that $\Sigma \cap (\Sigma_0 \cup \Sigma_{\mathsf{par}}) = \emptyset$. Let $\mathcal{K}$ be a set of clauses in the signature $\Pi_0 \cup \Sigma_{\mathsf{par}} \cup \Sigma$ in which all variables occur also below functions in $\Sigma_1 = \Sigma_{\mathsf{par}} \cup \Sigma$. Assume $\mathcal{T} \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies condition (Comp_f^Ψ) for a suitable closure operator Ψ with $\mathsf{est}(G) \subseteq \Psi_{\mathcal{K}}(G)$ for every set G of ground Π^C-clauses. Then Algorithm 1 can be used to construct a universal $\Pi_0 \cup \Sigma_{\mathsf{par}}$-formula $\forall \overline{x} \Gamma_T(\overline{x})$ such that $\mathcal{T}_0 \cup \forall \overline{x} \Gamma_T(\overline{x}) \cup \mathcal{K} \cup G \models \perp$ which is entailed by every universal formula Γ with $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup G \models \perp$.*

Algorithm 1 Symbol elimination in theory extensions [23,24]

Input: Theory extension $\mathcal{T}_0 \cup \mathcal{K}$ with signature $\Pi = \Pi_0 \cup \Sigma_1$, where $\Sigma_1 = \Sigma \cup \Sigma_{\mathsf{par}}$ where Σ_{par} is a set of parameters
G: set of ground Π^C-clauses; T: set of ground Π^C-terms with $\Psi_{\mathcal{K}}(G) \subseteq T$
Output: $\forall \overline{y} \Gamma_T(\overline{y})$ (universal $\Pi_0 \cup \Sigma_{\mathsf{par}}$-formula)

Step 1 Purify $\mathcal{K}[T] \cup G$ (with set of extension symbols Σ_1). Let $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$ be the set of Π_0^C-clauses obtained this way.

Step 2 Let $G_1 = \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$. Among the constants in G_1, we identify
(i) the constants c_f, $f \in \Sigma_{\mathsf{par}}$, where c_f is a constant parameter or c_f is introduced by a definition $c_f \approx f(c_1, \dots, c_k)$ in the hierarchical reasoning method,
(ii) all constants $\overline{c}_p$ occurring as arguments of functions in Σ_{par} in such definitions.
Replace all the other constants $\overline{c}$ with existentially quantified variables $\overline{x}$ (i.e. replace $G_1(\overline{c}_p, \overline{c}_f, \overline{c})$ with $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$).

Step 3 Construct a formula $\Gamma_1(\overline{c}_p, \overline{c}_f)$ equivalent to $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ w.r.t. $\mathcal{T}_0$ using a method for quantifier elimination in $\mathcal{T}_0$.

Step 4 Replace each constant c_f introduced by definition $c_f = f(c_1, \dots, c_k)$ with the term $f(c_1, \dots, c_k)$ in $\Gamma_1(\overline{c}_p, \overline{c}_f)$. Let $\Gamma_2(\overline{c}_p)$ be the formula obtained this way. Replace $\overline{c}_p$ with existentially quantified variables $\overline{y}$.

Step 5 Let $\forall \overline{y} \Gamma_T(\overline{y})$ be $\forall \overline{y} \neg \Gamma_2(\overline{y})$.

3 Second-Order Quantifier Elimination

We consider only the elimination of one predicate; for formulae of the form $\exists P_1 \dots P_n\ F$ the process can be iterated.

Let $\mathcal{T}$ be a theory over a many-sorted signature $\Pi = (S, \Sigma, \mathsf{Pred})$ where the set of sorts $S = S_i \cup S_u$ consists of a set S_i of interpreted sorts and a set S_u of uninterpreted sorts. The models of the theories are Π-structures $\mathcal{A} = (\{A_s\}_{s \in S}, \{f_{\mathcal{A}}\}_{f \in \Sigma}, \{p_{\mathcal{A}}\}_{p \in \mathsf{Pred}})$, where each support of interpreted sort is considered to be fixed. Following the terminology used in [5,6], we will refer to elements in the fixed domain of sort $s \in S_i$ as *domain elements of sort s*.

Let $\Pi' = (S, \Sigma, \mathsf{Pred} \cup \{P\})$, where $P \notin \mathsf{Pred}$. Let F be a universal first-order Π'-formula. We can assume, without loss of generality, that F is a set of clauses of the form $\forall \overline{x}\ D(\overline{x}) \vee C(\overline{x})$, where $D(\overline{x})$ is a clause over the signature Π and $C(\overline{x})$ is a clause containing literals of the form $(\neg)P(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are variables[1]. Such clauses can also be represented as *constrained clauses* in the form $\forall \overline{x}\ \phi(\overline{x}) \mid\mid C(\overline{x})$, where $\phi(\overline{x}) := \neg D(\overline{x})$. We will refer to clauses of this form as constrained P-clauses.

Our goal is to compute, if possible, a first-order Π-formula G such that $G \equiv_{\mathcal{T}} \exists P\ F$.

Let $\succ$ be a strict, well-founded ordering on terms that is compatible with contexts and stable under substitutions, total on ground terms and with the property that

[1] We can bring the clauses to this form using variable abstraction.

$t \succ d$ for every domain element d of interpreted sort s and every ground term t that is not a domain element. Let $HRes^P_\succ$ be the calculus containing the following ordered resolution and factorization rules for constrained P-clauses:

$$\frac{\phi_1 \mid\mid P(\overline{x}) \vee C \qquad \phi_2 \mid\mid \neg P(\overline{y}) \vee D}{(\phi_1 \wedge \phi_2)\sigma \mid\mid (C \vee D)\sigma} \qquad\qquad \frac{\phi \mid\mid P(\overline{x}) \vee P(\overline{y}) \vee C}{\phi\sigma \mid\mid (P(\overline{x}) \vee C)\sigma}$$

where (i) $\sigma = \mathsf{mgu}(P(\overline{x}), P(\overline{y}))$
(ii) $P(\overline{x})\sigma$ is strictly maximal in $(P(\overline{x}) \vee C)\sigma$
(iii) $\neg P(\overline{y})\sigma$ is maximal in $(\neg P(\overline{y}) \vee D)\sigma$.

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The inference rules are supplemented by a redundancy criterion $\mathcal{R} = (\mathcal{R}_c, \mathcal{R}_i)$ meant to specify a set $\mathcal{R}_c$ of redundant clauses (which can be removed) and a set $\mathcal{R}_i$ of redundant inferences (which do not need to be computed). The following notion of redundancy $\mathcal{R}^0_c$ for clauses is often used: A (constrained) clause is redundant w.r.t. a set N of clauses if all its ground instances are entailed w.r.t. $\mathcal{T}$ by ground instances of clauses in N which are strictly smaller w.r.t. $\succ$. We will use the following notion of redundancy for inferences: If $\mathcal{R}_c$ is a redundancy criterion for clauses, we say that an inference ι on ground clauses is redundant w.r.t. N if either one of its premises is redundant w.r.t. N and $\mathcal{R}_c$ or, if C_0 is the conclusion of ι then there exist clauses $C_1, \dots, C_n \in N$ that are smaller w.r.t. $\succ$ than the maximal premise of ι and $C_1, \dots, C_n \models C_0$. A non-ground inference is redundant if all its ground instances are redundant.

We say that a set of clauses N^* is saturated up to $\mathcal{R}$-redundancy w.r.t. $HRes^P_\succ$ if every $HRes^P_\succ$ inference with premises in N^* is redundant.

Theorem 2 ([4,21]) *Let N be a set of constrained P-clauses over background theory $\mathcal{T}$, N^* its saturation (up to $\mathcal{R}$-redundancy) under $HRes^P_\succ$, and N^*_0 the set of clauses in N^* not containing P. For every model $\mathcal{A}$ of $\mathcal{T}$, $\mathcal{A}$ is a model of N^*_0 iff there exists a Π'-structure $\mathcal{B}$ with $\mathcal{B} \models N$ and $\mathcal{B}_{|\Pi} = \mathcal{A}$.*

If the saturation N^* of N under $HRes^P_\succ$ (up to $\mathcal{R}$-redundancy) is finite, the universal closure of the conjunction of the clauses in N^*_0 is equivalent to $\exists P\ N$.

Theorem 3 ([21]) *Let $\mathcal{T}$ be a theory with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, N a set of constrained P-clauses. Assume that the saturation N^* of N (up to $\mathcal{R}$-redundancy) w.r.t. $HRes^P_\succ$ is finite; let N^*_0 be the set of clauses in N^* not containing P. Let $\Sigma_{\mathsf{par}} \subseteq \Sigma$ be a set of parameters. If (i) $\mathcal{T}$ allows quantifier elimination or (ii) $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is a local theory extension satisfying condition (Comp^Ψ_f) and $\mathcal{T}_0$ allows quantifier elimination, then we can use Algorithm 1 to obtain a (weakest) universal constraint Γ on the parameters such that* every *model $\mathcal{A}$ of $\mathcal{T} \cup \Gamma$ is a model of (the universal closure of) N^*_0, hence $\mathcal{A} \models \exists P\ N$.*

Since the implementations of the hierarchical superposition calculus we are aware of have as background theory linear arithmetic and in our examples we had more complex theories, in [21] and [20] we used a form of abstraction first: We renamed the constraints over more complex theories with new predicate symbols, and used SCAN [11] for second-order quantifier elimination.

The saturation of a set N of constrained P-clauses up to redundancy under $HRes^P_\succ$ might be infinite. In [21] we discuss two possibilities of obtaining finite representations for it: using an encoding of the constraints as minimal models of suitable sets of constrained Horn clauses [8] or using acceleration [9,10].

4 Checking Entailment

Let $\mathcal{T}$ be a theory with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, and let $\overline{P}_1 = P^1_1, \dots, P^1_{n_1}$ and $\overline{P}_2 = P^2_1, \dots, P^2_{n_2}$ be finite sequences of different predicate symbols with $P^i_j \not\in \mathsf{Pred}$, and $\Pi_i = (\Sigma, \mathsf{Pred} \cup \{P^i_j \mid 1 \leq j \leq n_i\})$ for $i = 1, 2$.
Let F_1 be a universal Π_1-formula and F_2 be a universal Π_2-formula. If there exist Π-formulae G_1 and G_2 such that $G_1 \equiv_\mathcal{T} \exists \overline{P}_1 F_1$ and $G_2 \equiv_\mathcal{T} \exists \overline{P}_2 F_2$ (which can be found either by saturation or by using acceleration techniques or other methods) then $\exists \overline{P}_1\ F_1 \models_\mathcal{T} \exists \overline{P}_2\ F_2$ iff $G_1 \models_\mathcal{T} G_2$ (which is the case iff $G_1 \wedge \neg G_2 \models_\mathcal{T} \perp$). Below are some situations in which this can be effectively decided.

Theorem 4 ([21]) *Assume that there exist Π-formulae G_1 and G_2 such that $G_1 \equiv_\mathcal{T} \exists \overline{P}_1 F_1$ and $G_2 \equiv_\mathcal{T} \exists \overline{P}_2 F_2$. If $\mathcal{T}$ is a decidable theory then we can effectively check whether $\exists \overline{P}_1\ F_1 \models_\mathcal{T} \exists \overline{P}_2\ F_2$. If $\mathcal{T}$ has quantifier elimination and the formulae F_1, F_2 contain parametric constants, we can use quantifier elimination in $\mathcal{T}$ to derive conditions on these parameters under which $\exists \overline{P}_1\, F_1 \models_\mathcal{T} \exists \overline{P}_2\, F_2$.*

Theorem 5 ([21]) *Assume that there exist* universal *Π-formulae G_1 and G_2 such that $G_1 \equiv_\mathcal{T} \exists \overline{P}_1 F_1$ and $G_2 \equiv_\mathcal{T} \exists \overline{P}_2 F_2$, and that $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$, where $\mathcal{T}_0$ is a decidable theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred}_0)$ where S_0 is a set of interpreted sorts and $\mathcal{K}$ is a set of (universally quantified) clauses over $\Pi = (S_0 \cup S_1, \Sigma_0 \cup \Sigma_1, \mathsf{Pred}_0 \cup \mathsf{Pred}_1)$, where (i) S_1 is a new set of uninterpreted sorts, (ii) $\Sigma_1, \mathsf{Pred}_1$ are sets of new function, resp. predicate symbols which have only arguments of uninterpreted sort $\in S_1$, and all function symbols in Σ_1 have interpreted output sort $\in S_0$. Assume, in addition, that all variables and constants of sort $\in S_1$ in $\mathcal{K}, G_1$ and $\neg G_2$ occur below function symbols in Σ_1.*

We can use the decision procedure for $\mathcal{T}_0$ to effectively check whether $G_1 \wedge \neg G_2 \models_\mathcal{T} \perp$ (hence whether $\exists \overline{P}_1\ F_1 \models_\mathcal{T} \exists \overline{P}_2\ F_2$). If $\mathcal{T}_0$ allows quantifier elimination and the formulae F_1, F_2 (hence also G_1, G_2) contain parametric constants and functions, we can use the property-directed symbol elimination in Algorithm 1 (cf. Theorem 1) for obtaining a universal formula Γ representing weakest universal constraints on the parameters under which $\exists \overline{P}_1\ F_1 \models_\mathcal{T} \exists \overline{P}_2\ F_2$.

We illustrate how the previous results can be used for checking an inclusion between two classes of graphs of interest in wireless network theory.

Example 1. Let $\mathbf{QUDG}(r) = (\mathbf{MinDG}(r) \cap \mathbf{MaxDG}(1))^-$ be axiomatized by $\mathsf{MinDG}(r) \wedge \mathsf{MaxDG}(1) \wedge \mathsf{Tr}^-(E, F)$, where r is a function symbol (where $r(v)$ models the maximum communication distance of node v), and:

$\mathsf{MinDG}(r): \forall x, y\ \pi^i_r(x,y) \rightarrow E(x,y)$ where $\pi^i_r(x,y) = x \neq y \wedge d(x,y) \leq r(x)$
$\mathsf{MaxDG}(1): \forall x, y\ \pi^e(x,y) \rightarrow \neg E(x,y)$ where $\pi^e(x,y) = d(x,y) > 1$
$\mathsf{Tr}^-(E,F): \forall x, y\ (F(x,y) \leftrightarrow E(x,y) \wedge E(y,x))$
$\mathsf{Tr}^+(E,F): \forall x, y\ (F(x,y) \leftrightarrow E(x,y) \vee E(y,x))$.

We want to check[2] whether $\mathbf{A}(r) \subseteq \mathbf{B}(r)$, where $\mathbf{A}(r) = \mathbf{QUDG}(r)$ and $\mathbf{B}(r) = (\mathbf{MinDG}(r) \cap \mathbf{MaxDG}(1))^+$ is described by $\mathsf{MinDG}(r) \wedge \mathsf{MaxDG}(1) \wedge \mathsf{Tr}^+(E,F)$.

We obtain axiomatizations $G_1 \equiv \exists E(\mathsf{MinDG}(r) \wedge \mathsf{MaxDG}(1) \wedge \mathsf{Tr}^-(E,F))$ for $\mathbf{A}(r)$ and $G_2 \equiv \exists E(\mathsf{MinDG}(r) \wedge \mathsf{MaxDG}(1) \wedge \mathsf{Tr}^+(E,F))$ for $\mathbf{B}(r)$ by eliminating E. As mentioned before, the implementations of the hierarchical superposition calculus we are aware of allow only linear arithmetic as a background theory, whereas in our examples we had more complex theories. This is why we renamed the constraints over more complex theories with new predicate symbols π^i_r, π^e, π^t and used SCAN [11] for second-order quantifier elimination in first-order logic. We obtained the following formulae:

	G_1					G_2		
$\forall x, y$	$\pi^i_r(x,y) \wedge \pi^e(x,y)$	$\rightarrow$	$\perp$		$\forall x, y$	$\pi^i_r(x,y) \wedge \pi^e(x,y)$	$\rightarrow$	$\perp$
$\forall x, y$	$\pi^i_r(x,y) \wedge \pi^i_r(y,x)$	$\rightarrow$	$F(y,x)$		$\forall x, y$	$\pi^e(x,y) \wedge \pi^e(y,x)$	$\rightarrow$	$\neg F(y,x)$
$\forall x, y$	$\pi^e(x,y)$	$\rightarrow$	$\neg F(x,y)$		$\forall x, y$	$\pi^i_r(x,y)$	$\rightarrow$	$F(x,y)$
$\forall x, y$	$\pi^e(x,y)$	$\rightarrow$	$\neg F(y,x)$		$\forall x, y$	$\pi^i_r(x,y)$	$\rightarrow$	$F(y,x)$
$\forall x, y$	$F(x,y)$	$\rightarrow$	$F(y,x)$		$\forall x, y$	$F(x,y)$	$\rightarrow$	$F(y,x)$
					$\forall x$	$\pi^e(x,x)$	$\rightarrow$	$\neg F(x,x)$

The task is now to check whether $G_1 \models_{\mathcal{T}} G_2$, i.e. whether $G_1 \wedge \neg G_2$ is unsatisfiable w.r.t. $\mathcal{T}$, where $\neg G_2$ is the disjunction of the following ground formulae (we ignore the negation of the first clause obviously implied by G_1).

$(g_1)\ \pi^e(a,b) \wedge \pi^e(b,a) \wedge F(b,a)$ $(g_2)\ \pi^e(a,a) \wedge F(a,a)$ $(g_3)\ F(a,b) \wedge \neg F(b,a)$
$(g_4)\ \pi^i_r(a,b) \wedge \neg F(a,b)$ $(g_5)\ \pi^i_r(a,b) \wedge \neg F(b,a)$

Here $\mathcal{T} = \mathcal{T}_d \cup UIF_r$, where $\mathcal{T}_d$ is a theory describing the properties of d and r is considered to be an uninterpreted function symbol. In [21] we analyzed the situations in which $\mathcal{T}_d$ is one of the theories $\mathcal{T}^m_d = \mathcal{T}_0 \cup \mathcal{K}_m$, where $\mathcal{K}_m$ are axioms of a metric, $\mathcal{T}^s_d$, the extension of $\mathcal{T}_0$ with a function d satisfying symmetry, $\mathcal{T}^p_d = \mathcal{T}_0 \cup \mathcal{K}_p$, where $\mathcal{K}_p = \forall x, y\ d(x,y) \geq 0$, and $\mathcal{T}^u_d$, the extension of $\mathcal{T}_0$ with an uninterpreted function d – where $\mathcal{T}_0$ is the disjoint combination of the theory $\mathcal{E}$ of pure equality (sort p) and linear real arithmetic (sort num).

In [21] we proved that all these theories satisfy suitable locality properties. For testing entailment, by Theorem 5, we can consider the set of all instances of G_1 in which the variables of sort p are replaced with the constants a, b, then use a method for checking ground satisfiability of $G_1[T] \wedge g_i$ w.r.t. $\mathcal{T}_d \cup UIF_r$, where $\mathcal{T}_d \in \{\mathcal{T}^u_d, \mathcal{T}^p_d, \mathcal{T}^s_d, \mathcal{T}^m_d\}$. For this, we use H-PILoT [17]. This allows us to check that $G_1[T] \wedge g_i$ is unsatisfiable for $i \in \{1,2,3\}$, but satisfiable for $i \in \{4,5\}$ (this is so for all four theories). For cases 4 and 5 we can use Algorithm 1 to derive conditions on parameters under which $G_1[T] \wedge g_i$ is unsatisfiable. If e.g. we consider d and r to be parameters, for $\mathcal{T}^m_d$ we obtain condition

[2] To check that the inclusion holds in one given model $\mathcal{A}$ we can choose $\mathcal{T} = \mathsf{Th}(\mathcal{A})$.

$$C^{d,r} = \forall x, y (x \neq y \wedge d(x, y) \leq 1 \wedge d(x, y) \leq r(x) \rightarrow d(y, x) \leq r(y))$$

(which is true in any model of $\mathcal{T}$ in which r is interpreted as a constant function). For further details cf. [20]. ■

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