Sahlqvist Correspondence Theory for Second-Order Propositional Modal Logic
(Short Paper)

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1 Introduction

*Second-Order Propositional Modal Logic (*SOMPL*).* Modal logic with propositional quantifiers has been considered in the literature since Kripke [13], Bull [2], Fine [8, 9], and Kaplan [7]. This language is of high complexity: its satisfiability problem is not decidable, and indeed not even analytical. In Kaminski and Tiomkin [12], the authors showed that the expressive power for SOMPL whose modalities are S4.2 or weaker is the same as second-order predicate logic. However, not every second-order formula is equivalent to an SOMPL-formula, since SOMPL-formulas are preserved under generated submodels (see van Benthem [16]). In ten Cate [15], the author proved the analogues of the van Benthem-Rosen theorem (on the model level) and Goldblatt-Thomason theorem (on the frame level) for SOMPL. Therefore, a natural question is: on the frame level, can we find a natural fragment of SOPML-formulas such that each formula in this fragment corresponds to a first-order formula, in the sense of Sahlqvist theory (see [14, 16])? This is what we will answer in the paper.

Correspondence Theory. Typically, modal correspondence theory [16] concerns the correspondence of modal formulas and first-order formulas over Kripke frames, via the tools of standard translation. Syntactic classes (e.g. Sahlqvist formulas [14], inductive formulas [11], etc.) of modal formulas are identified to have first-order correspondents and are canonical, i.e. their validity are closed under taking canonical extensions. In the present paper, we identify the Sahlqvist formulas of SOMPL, which cover and properly extend the Sahlqvist fragment in basic modal logic. We show that there is an SOMPL Sahlqvist formula which corresponds to $\forall x \forall y(Rxy \land Ryx \to Rxx)$, which is not modally definable, and that the SOMPL Sahlqvist formula $\forall q(\forall p(p \to \Diamond p \lor q) \to q)$ is not canonical, which is in contrast to the basic modal logic setting where each Sahlqvist formula is canonical. The present paper use the same methodology as [6, 3]. The Sahlqvist fragment of SOPML is defined in a step-by-step way, and we give an algorithm $\mathsf{ALBA}^{\mathsf{SOPML}}$ (Ackermann Lemma Based Algorithm) which can successfully reduce Sahlqvist formulas in SOPML to first-order formulas and is sound with respect to Kripke semantics.

2 Preliminaries

2.1 Language and semantics

In the present paper we consider the unimodal language. Given a set Prop of propositional variables, the second-order propositional modal formulas are defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \forall p\varphi \mid \exists p\varphi$$

where $p \in \mathsf{Prop}$. We use the boldface notation $\vec{p}$ to denote a set of propositional variables and use $\varphi(\vec{p})$ to indicate that the propositional variables occur in φ are all in $\vec{p}$. We say that an occurrence of a propositional variable p in a formula φ is *positive* (resp. *negative*) if it is in the scope of an even (resp. odd) number of negations (here $\alpha \to \beta$ is regarded as $\neg\alpha \vee \beta$).

The semantics of the second-order propositional modal formulas are defined as follows:

Definition 1. *A* Kripke frame *is a pair* $\mathbb{F} = (W, R)$ *where* $W \neq \varnothing$ *is the* domain *of* $\mathbb{F}$*, the* accessibility relation R *is a binary relation on* W*. A* Kripke model *is a pair* $\mathbb{M} = (\mathbb{F}, V)$ *where* $V : \mathsf{Prop} \to P(W)$ *is a* valuation *on* $\mathbb{F}$*.* V^p_X *denote a valuation which is the same as* V *except that* $V^p_X(p) = X \subseteq W$*.*

Now the satisfaction relation can be defined as follows: given any Kripke model $\mathbb{M} = (W, R, V)$*, any* $w \in W$*, the basic and Boolean cases are standard, and for modalities and propositional quantifiers,*

$\mathbb{M}, w \Vdash \Box\varphi$ *iff for any* v *such that* Rwv*,* $\mathbb{M}, v \Vdash \varphi$*;*
$\mathbb{M}, w \Vdash \Diamond\varphi$ *iff there exists* v *such that* Rwv *and* $\mathbb{M}, v \Vdash \varphi$*;*
$\mathbb{M}, w \Vdash \forall p\varphi$ *iff for all* $X \subseteq W$*,* $(W, R, V^p_X), w \Vdash \varphi$*;*
$\mathbb{M}, w \Vdash \exists p\varphi$ *iff there exists* $X \subseteq W$ *such that* $(W, R, V^p_X), w \Vdash \varphi$*.*

In order to use the algorithm to compute the first-order correspondents of Sahlqvist SOPML formulas, we will need the following *expanded modal language* which is defined as follows:

$$\varphi ::= p \mid \mathbf{i} \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid$$

$$\Box\varphi \mid \Diamond\varphi \mid \blacksquare\varphi \mid \blacklozenge\varphi \mid \forall p\varphi \mid \exists p\varphi \mid \forall \mathbf{i}\varphi \mid \exists \mathbf{i}\varphi \mid \mathbf{l}(\varphi, \varphi)$$

where $p \in \mathsf{Prop}$, $\mathbf{i} \in \mathsf{Nom}$ is a *nominal*, $\blacksquare$ and $\blacklozenge$ are the backward-looking box and diamond respectively, $\forall\mathbf{i}$ and $\exists\mathbf{i}$ are *nominal quantifiers*, and $\mathbf{l}$ is a binary modality. We call a formula *pure* if it does not contain propositional variables or propositional quantifiers (it can contain nominals, nominal quantifiers and the binary modality $\mathbf{l}$).

The interpretation of the expanded modal language is given as follows: For a valuation V, it is defined as $V : \mathsf{Prop} \cup \mathsf{Nom} \to P(W)$ such that $V(\mathbf{i})$ is a singleton for all $\mathbf{i} \in \mathsf{Nom}$. The additional satisfaction clauses are given as follows (here $V^{\mathbf{i}}_v$ denote a valuation which is the same as V except that $V^{\mathbf{i}}_v(\mathbf{i}) = \{v\} \subseteq W$.):

$$
\begin{array}{ll}
\mathbb{M}, w \Vdash \mathbf{i} & \text{iff } V(\mathbf{i}) = \{w\}; \\
\mathbb{M}, w \Vdash \blacksquare\varphi & \text{iff for any } v \text{ such that } Rvw,\ \mathbb{M}, v \Vdash \varphi; \\
\mathbb{M}, w \Vdash \blacklozenge\varphi & \text{iff there exists } v \text{ such that } Rvw \text{ and } \mathbb{M}, v \Vdash \varphi; \\
\mathbb{M}, w \Vdash \forall\mathbf{i}\varphi & \text{iff for all } v \in W,\ (W, R, V^{\mathbf{i}}_{v}), w \Vdash \varphi; \\
\mathbb{M}, w \Vdash \exists\mathbf{i}\varphi & \text{iff there exists } v \in W \text{ such that } (W, R, V^{\mathbf{i}}_{v}), w \Vdash \varphi; \\
\mathbb{M}, w \Vdash \mathbf{l}(\varphi, \psi) & \text{iff for all } v \in W \ (\text{if } \mathbb{M}, v \Vdash \varphi, \text{ then } \mathbb{M}, v \Vdash \psi).
\end{array}
$$

We can extend V to a map from the set of formulas to $P(W)$ in the natural way.

2.2 Inequalities and complex inequalities

We will find it convenient to use the inequality notation $\varphi \leq \psi$ where φ and ψ are formulas. We use Ineq to denote the set of all inequalities in the expanded modal language. We define *complex inequalities* as follows:

$$\mathsf{Comp} ::= \mathsf{Ineq} \mid \mathsf{Comp}\ \&\ \mathsf{Comp} \mid \mathsf{Comp} \Rightarrow \mathsf{Comp} \mid \forall p\mathsf{Comp} \mid \exists p\mathsf{Comp} \mid \forall\mathbf{i}\mathsf{Comp} \mid \exists\mathbf{i}\mathsf{Comp}$$

Here we assume that the quantifiers have a higher precedence than &, and & is higher than $\Rightarrow$.

Complex inequalities are interpreted in models $\mathbb{M} = (W, R, V)$ instead of pointed models $(\mathbb{M}, w)$. The semantics of complex inequalities is defined as follows:

– An inequality is interpreted as follows:

$$(W, R, V) \Vdash \varphi \leq \psi \text{ iff}$$

$$(\text{for all } w \in W,\ \text{if } (W, R, V), w \Vdash \varphi,\ \text{then } (W, R, V), w \Vdash \psi);$$

– $(W, R, V) \Vdash \mathsf{Comp}_1\&\mathsf{Comp}_2$ iff $(W, R, V) \Vdash \mathsf{Comp}_1$ and $(W, R, V) \Vdash \mathsf{Comp}_2$;
– $(W, R, V) \Vdash \mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2$ iff $((W, R, V) \Vdash \mathsf{Comp}_1$ implies $(W, R, V) \Vdash \mathsf{Comp}_2)$;
– $(W, R, V) \Vdash \forall p\mathsf{Comp}$ iff for all $X \subseteq W$, $(W, R, V^{p}_{X}) \Vdash \mathsf{Comp}$;
– $(W, R, V) \Vdash \exists p\mathsf{Comp}$ iff there exists an $X \subseteq W$ such that $(W, R, V^{p}_{X}) \Vdash \mathsf{Comp}$;
– $(W, R, V) \Vdash \forall\mathbf{i}\mathsf{Comp}$ iff for all $v \in W$, $(W, R, V^{\mathbf{i}}_{v}) \Vdash \mathsf{Comp}$;
– $(W, R, V) \Vdash \exists\mathbf{i}\mathsf{Comp}$ iff there exists an $v \in W$ such that $(W, R, V^{\mathbf{i}}_{v}) \Vdash \mathsf{Comp}$.

2.3 Standard translation

In the correspondence language which is second-order due to the existence of propositional quantifiers in SOPML, we have a binary predicate symbol R corresponding to the binary relation, a set of constant symbols i corresponding to each nominal $\mathbf{i}$, a set of unary predicate symbols P corresponding to each propositional variable p.

Definition 2. *The standard translation of the expanded* SOPML *language is defined as follows (for the basic and Boolean case, it is standard):*

- $ST_x(\mathbf{i}) := x = i;$
- $ST_x(\Box\varphi) := \forall y(Rxy \to ST_y(\varphi));$
- $ST_x(\Diamond\varphi) := \exists y(Rxy \wedge ST_y(\varphi));$
- $ST_x(\blacksquare\varphi) := \forall y(Ryx \to ST_y(\varphi));$
- $ST_x(\blacklozenge\varphi) := \exists y(Ryx \wedge ST_y(\varphi));$
- $ST_x(\forall p\varphi) := \forall P ST_x(\varphi);$
- $ST_x(\exists p\varphi) := \exists P ST_x(\varphi);$
- $ST_x(\forall \mathbf{i}\varphi) := \forall i ST_x(\varphi);$
- $ST_x(\exists \mathbf{i}\varphi) := \exists i ST_x(\varphi);$
- $ST_x(\mathbf{l}(\varphi, \psi)) := \forall y(ST_y(\varphi) \to ST_y(\psi)).$

The following proposition states that this translation is correct:

Proposition 1. *For any Kripke model* $\mathbb{M}$*, any* $w \in W$ *and any expanded* SOPML *formula* φ*,*

$$\mathbb{M}, w \Vdash \varphi \text{ iff } \mathbb{M} \vDash ST_x(\varphi)[x := w].$$

For inequalities and complex inequalities, the standard translation is given in a global way:

Definition 3.
- $ST(\varphi \leq \psi) := \forall x(ST_x(\varphi) \to ST_x(\psi));$
- $ST(\mathsf{Comp}_1 \ \& \ \mathsf{Comp}_2) = ST(\mathsf{Comp}_1) \wedge ST(\mathsf{Comp}_2);$
- $ST(\mathsf{Comp}_1 \ \Rightarrow \ \mathsf{Comp}_2) = ST(\mathsf{Comp}_1) \to ST(\mathsf{Comp}_2);$
- $ST(\forall p(\mathsf{Comp})) := \forall P(ST(\mathsf{Comp}));$
- $ST(\exists p(\mathsf{Comp})) := \exists P(ST(\mathsf{Comp}));$
- $ST(\forall \mathbf{i}(\mathsf{Comp})) := \forall i(ST(\mathsf{Comp}));$
- $ST(\exists \mathbf{i}(\mathsf{Comp})) := \exists i(ST(\mathsf{Comp})).$

Proposition 2. *For any Kripke model* $\mathbb{M}$*, any inequality* Ineq*, any complex inequality* Comp*,*

$$\mathbb{M} \Vdash \mathsf{Ineq} \text{ iff } \mathbb{M} \vDash ST(\mathsf{Ineq});$$

$$\mathbb{M} \Vdash \mathsf{Comp} \text{ iff } \mathbb{M} \vDash ST(\mathsf{Comp}).$$

3 Sahlqvist formulas in second-order propositional modal logic

In this section, we define Sahlqvist formulas of second-order propositional modal logic step by step.

We first define (quantifier-free) positive formulas $\mathsf{POS}(\vec{p})$ whose propositonal variables are among $\vec{p}$:

$$\mathsf{POS}(\vec{p}) ::= p \mid \bot \mid \top \mid \mathsf{POS}(\vec{p}) \wedge \mathsf{POS}(\vec{p}) \mid \mathsf{POS}(\vec{p}) \vee \mathsf{POS}(\vec{p}) \mid \Box\mathsf{POS}(\vec{p}) \mid \Diamond\mathsf{POS}(\vec{p})$$

where p is in $\vec{p}$. These positive formulas have similar roles to the positive consequent part in Sahlqvist formulas in basic modal logic, which are going to receive minimal valuations. The reason why we do not allow propositional quantifiers in positive formulas is that we want the formula after receiving the minimal valuations to be translated into a first-order formula, while propositional quantifiers will make it second-order.

3.1 The Π_1-fragment: Sahlqvist formulas in basic modal logic

We define the Π_1-Sahlqvist antecedent $\mathsf{Sahl}_1(\vec{p})$ whose propositonal variables are among $\vec{p}$:

$$\mathsf{Sahl}_1(\vec{p}) ::= \Box^n p \mid \bot \mid \top \mid \neg\mathsf{POS}(\vec{p}) \mid \mathsf{Sahl}_1(\vec{p}) \wedge \mathsf{Sahl}_1(\vec{p}) \mid \Diamond\mathsf{Sahl}_1(\vec{p})$$

where p is in $\vec{p}$.

Then the Π_1-Sahlqvist formulas are defined as $\forall\vec{p}(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p}))$. Indeed, Sahlqvist formulas[1] in the basic modal logic setting can be treated as universally quantified by propositional quantifiers which bind all occurrences of propositional variables, so in this sense the Π_1-Sahlqvist formulas can be taken as the Sahlqvist formulas in basic modal logic.

3.2 The Π_2-fragment

We define the PIA formula $\mathsf{PIA}(\vec{q}, \vec{p})$ as follows:

$$\mathsf{PIA}(\vec{q}, \vec{p}) ::= p \mid \Box\mathsf{PIA}(\vec{q}, \vec{p}) \mid \mathsf{PIA}(\vec{q}, \vec{p}) \wedge \mathsf{PIA}(\vec{q}, \vec{p}) \mid \mathsf{POS}(\vec{q}) \vee \mathsf{PIA}(\vec{q}, \vec{p})$$

where p is in $\vec{p}$. Here the PIA formula has two bunches of propositional variables: $\vec{q}$ is to receive minimal valuations for $\vec{q}$ from somewhere else, and $\vec{p}$ is used to compute minimal valuations for $\vec{p}$. Then it is easy to see that $\mathsf{PIA}(\vec{q}, \vec{p})$ is equivalent to the form $\bigwedge \Box(\mathsf{POS}(\vec{q}) \vee \Box(\mathsf{POS}(\vec{q}) \vee \ldots p))$, where p is in $\vec{p}$.

Now we can define Π_2-Sahlqvist antecedents as follows:

$$\mathsf{Sahl}_2(\vec{p}) ::= \mathsf{Sahl}_1(\vec{p}) \mid \forall\vec{q}(\mathsf{Sahl}_1(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p})) \mid \mathsf{Sahl}_2(\vec{p}) \wedge \mathsf{Sahl}_2(\vec{p}) \mid \Diamond\mathsf{Sahl}_2(\vec{p})$$

Then Π_2-Sahlqvist formulas are defined as $\forall\vec{p}(\mathsf{Sahl}_2(\vec{p}) \to \mathsf{POS}(\vec{p}))$.

It is easy to see that formulas of the form $\forall\vec{p}(\mathsf{Sahl}_1(\vec{p}) \wedge \forall\vec{q}(\mathsf{Sahl}_1(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p})) \to \mathsf{POS}(\vec{p}))$ are in the Π_2-hierarchy.

3.3 The Π_n-fragment

Now for the Π_n-fragment, assume that we have already defined Π_{n-1}-Sahlqvist antecedents $\mathsf{Sahl}_{n-1}(\vec{p})$ and Π_{n-1}-Sahlqvist formulas $\forall\vec{p}(\mathsf{Sahl}_{n-1}(\vec{p}) \to \mathsf{POS}(\vec{p}))$, then we can define Π_n-Sahlqvist antecedents as follows:

$$\mathsf{Sahl}_n(\vec{p}) ::= \mathsf{Sahl}_{n-1}(\vec{p}) \mid \forall\vec{q}(\mathsf{Sahl}_{n-1}(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p})) \mid \mathsf{Sahl}_n(\vec{p}) \wedge \mathsf{Sahl}_n(\vec{p}) \mid \Diamond\mathsf{Sahl}_n(\vec{p})$$

Then Π_n-Sahlqvist formulas are defined as $\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$.

[1] In [1, Chapter 3], what we call Sahlqvist formulas are called Sahlqvist implications.

4 The Algorithm $\mathsf{ALBA}^{\mathsf{SOMPL}}$

In the present section, we define the correspondence algorithm $\mathsf{ALBA}^{\mathsf{SOMPL}}$ for second-order propositional modal logic, in the style of [4, 5]. The algorithm receives a Π_n-Sahlqvist formula $\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$ as input and goes in three stages.

1. **Preprocessing and first approximation**:
   The algorithm receives a Π_n-Sahlqvist formula $\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$ as input, and then apply the rewriting rule:

$$\frac{\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))}{\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}))}$$

   Then apply the first-approximation rule:

$$\frac{\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}))}{\forall\vec{p}\forall\mathbf{i}_0(\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \ \Rightarrow\ \mathbf{i}_0 \leq \mathsf{POS}(\vec{p}))}$$

2. **The reduction stage**:
   In this stage, we aim at reducing $\mathbf{i} \leq \mathsf{Sahl}_n(\vec{p})$ to a complex inequality in which p occurs either in the form $\varphi \leq p$ where φ is pure or in the form $\mathbf{j} \leq \neg\mathsf{POS}(\vec{p})$.
   (a) The commutativity rule and the associativity rule for &;
   (b) The rules for nominals:
      i. Splitting rule:

$$\frac{\mathbf{i} \leq \alpha \wedge \beta}{\mathbf{i} \leq \alpha \ \&\ \mathbf{i} \leq \beta}\ (Spl - Nom)$$

      ii. Separation rule:

$$\frac{\mathbf{i} \leq \alpha \to \beta}{\mathbf{i} \leq \alpha \ \Rightarrow\ \mathbf{i} \leq \beta}\ (Sep - Nom)$$

      iii. Quantifier rule:

$$\frac{\mathbf{i} \leq \forall q\alpha}{\forall q(\mathbf{i} \leq \alpha)}\ (Quant - Nom)$$

      iv. Approximation rule:

$$\frac{\mathbf{i} \leq \Diamond\alpha}{\exists\mathbf{j}(\mathbf{j} \leq \alpha \ \&\ \mathbf{i} \leq \Diamond\mathbf{j})}\ (Approx - Nom)$$

      The nominals introduced by the approximation rule must not occur in the whole complex inequality before applying the rule.
   (c) The residuation rules:

$$\frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta}\ (Res - \Box) \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg\beta \leq \gamma}\ (Res - \vee)$$

(d) The splitting rule:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \ \& \ \alpha \leq \gamma} \ (Splitting)$$

(e) The quantifier rules:

$$\frac{\exists \mathbf{j}(\mathsf{Comp}_1) \ \& \ \mathsf{Comp}_2}{\exists \mathbf{j}(\mathsf{Comp}_1 \ \& \ \mathsf{Comp}_2)} \ (Scope - \&) \qquad \frac{\exists \mathbf{j}(\mathsf{Comp}_1) \ \Rightarrow \ \mathsf{Comp}_2}{\forall \mathbf{j}(\mathsf{Comp}_1 \ \Rightarrow \ \mathsf{Comp}_2)} \ (Scope - \Rightarrow)$$

where Comp_2 does not have free occurrences of $\mathbf{j}$.

$$\frac{\forall q \forall p(\mathsf{Comp})}{\forall p \forall q(\mathsf{Comp})} \ (Ex - pq) \qquad \frac{\forall \mathbf{i} \forall p(\mathsf{Comp})}{\forall p \forall \mathbf{i}(\mathsf{Comp})} \ (Ex - p\mathbf{i})$$

$$\frac{\forall p \forall \mathbf{i}(\mathsf{Comp})}{\forall \mathbf{i} \forall p(\mathsf{Comp})} \ (Ex - \mathbf{i}p) \qquad \frac{\forall \mathbf{i} \forall \mathbf{j}(\mathsf{Comp})}{\forall \mathbf{j} \forall \mathbf{i}(\mathsf{Comp})} \ (Ex - \mathbf{ji})$$

$$\frac{\forall p(\mathsf{Comp}_1 \Rightarrow (\mathsf{Comp}_2 \& \mathsf{Comp}_3))}{\forall p(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2) \ \& \ \forall p(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_3)} \ (Spl - Quant - p)$$

$$\frac{\forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow (\mathsf{Comp}_2 \& \mathsf{Comp}_3))}{\forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2) \ \& \ \forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_3)} \ (Spl - Quant - \mathbf{i})$$

(f) The Ackermann rule:
In this step, we compute the minimal valuation for propositional variables and use the Ackermann rule to eliminate all the propositional variables.

$$\frac{\forall q(\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n \ \& \ \psi_1 \leq q \& \ldots \& \psi_m \leq q \Rightarrow \alpha \leq \beta)}{\alpha_1[\bigvee \psi/q] \leq \beta_1[\bigvee \psi/q] \& \ldots \& \alpha_n[\bigvee \psi/q] \leq \beta_n[\bigvee \psi/q] \Rightarrow \alpha[\bigvee \psi/q] \leq \beta[\bigvee \psi/q]}$$

where:

i. $\varphi[\theta/p]$ means uniformly replace occurrences of p in φ by θ;
ii. $\bigvee \psi = \psi_1 \vee \ldots \vee \psi_m$;
iii. Each α_i is positive, and each β_i negative in q, for $1 \leq i \leq n$;
iv. α is negative in q and β is positive in q;
v. Each ψ_i is pure (therefore q does not occur in ψ_i).

(g) The packing rule:

$$\frac{\forall \mathbf{i}(\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta)}{\exists \mathbf{i}(\mathbf{l}(\alpha_1, \beta_1) \wedge \ldots \wedge \mathbf{l}(\alpha_n, \beta_n) \wedge \alpha) \leq \beta}$$

where β does not contain occurrences of $\mathbf{i}$.

3. **Output**: By the execution of the algorithm, we can guarantee that given a Π_n-Sahlqvist formula as input, we can rewrite it into a pure complex inequality. Then we use standard translation to translate it into a first-order formula.

Theorem 1 (Soundness and Success).

- *If* $\mathsf{ALBA}^{\mathsf{SOPML}}$ *runs successfully on an input* Π_n*-Sahlqvist formula* $\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$ *and outputs a first-order formula* $\mathsf{FO}(\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})))$, *then for any Kripke frame* $\mathbb{F} = (W, R)$,

$$\mathbb{F} \Vdash \forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})) \quad \textit{iff} \quad \mathbb{F} \models \mathsf{FO}(\forall\vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))).$$

- *There is an algorithm such that for any* Π_n*-Sahlqvist formula* φ*, it can be transformed into an equivalent first-order formula.*

5 Examples

We give three examples of Π_2-Sahlqvist formulas to show how the $\mathsf{ALBA}^{\mathsf{SOPML}}$ algorithm works:

Example 1. $\forall p(\Diamond\Box p \wedge \forall q(\Diamond\Box q \to \Box(\Box q \vee \Box p)) \to \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \wedge \forall q(\Diamond\Box q \to \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \mathbf{i} \leq \forall q(\Diamond\Box q \to \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall q(\mathbf{i} \leq \Diamond\Box q \to \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall q(\mathbf{i} \leq \Diamond\Box q \Rightarrow \mathbf{i} \leq \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall q\forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \;\&\; \mathbf{j} \leq \Box q \Rightarrow \mathbf{i} \leq \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall q\forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \;\&\; \blacklozenge\mathbf{j} \leq q \Rightarrow \mathbf{i} \leq \Box(\Box q \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \Rightarrow \mathbf{i} \leq \Box(\Box\blacklozenge\mathbf{j} \vee \Box p)) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \Rightarrow \blacklozenge\mathbf{i} \leq \Box\blacklozenge\mathbf{j} \vee \Box p) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \Rightarrow \blacklozenge\mathbf{i} \wedge \neg\Box\blacklozenge\mathbf{j} \leq \Box p) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \Rightarrow \blacklozenge(\blacklozenge\mathbf{i} \wedge \neg\Box\blacklozenge\mathbf{j}) \leq p) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \forall\mathbf{j}(\mathbf{l}(\mathbf{i}, \Diamond\mathbf{j}) \wedge \blacklozenge(\blacklozenge\mathbf{i} \wedge \neg\Box\blacklozenge\mathbf{j}) \leq p) \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \exists\mathbf{j}(\mathbf{l}(\mathbf{i}, \Diamond\mathbf{j}) \wedge \blacklozenge(\blacklozenge\mathbf{i} \wedge \neg\Box\blacklozenge\mathbf{j})) \leq p \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$

now denote $\exists\mathbf{j}(\mathbf{l}(\mathbf{i}, \Diamond\mathbf{j}) \wedge \blacklozenge(\blacklozenge\mathbf{i} \wedge \neg\Box\blacklozenge\mathbf{j}))$ as φ, then

$\forall p\forall\mathbf{i}(\mathbf{i} \leq \Diamond\Box p \;\&\; \varphi \leq p \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}\forall\mathbf{k}(\mathbf{i} \leq \Diamond\mathbf{k} \;\&\; \mathbf{k} \leq \Box p \;\&\; \varphi \leq p \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall p\forall\mathbf{i}\forall\mathbf{k}(\mathbf{i} \leq \Diamond\mathbf{k} \;\&\; \blacklozenge\mathbf{k} \leq p \;\&\; \varphi \leq p \Rightarrow \mathbf{i} \leq \Box\Diamond\Box p)$
$\forall\mathbf{i}\forall\mathbf{k}(\mathbf{i} \leq \Diamond\mathbf{k} \Rightarrow \mathbf{i} \leq \Box\Diamond\Box(\blacklozenge\mathbf{k} \vee \varphi))$

Then we can use standard translation to get its first-order correspondence.

Example 2. The following example resembles the irreflexivity rule of Gabbay [10]:
$\forall q(\forall p(p \to \Diamond p \vee q) \to q)$
$\forall q\forall\mathbf{i}(\mathbf{i} \leq \forall p(p \to \Diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q\forall\mathbf{i}(\forall p(\mathbf{i} \leq p \to \Diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q\forall\mathbf{i}(\forall p(\mathbf{i} \leq p \Rightarrow \mathbf{i} \leq \Diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q\forall\mathbf{i}(\mathbf{i} \leq \Diamond\mathbf{i} \vee q \Rightarrow \mathbf{i} \leq q)$

$\forall q \forall \mathbf{i}(\mathbf{i} \wedge \neg \Diamond \mathbf{i} \leq q \Rightarrow \mathbf{i} \leq q)$
$\forall \mathbf{i}(\mathbf{i} \leq \mathbf{i} \wedge \neg \Diamond \mathbf{i})$
$\forall \mathbf{i}(\mathbf{i} \leq \neg \Diamond \mathbf{i})$
$\forall x \neg Rxx.$

By [1, Example 2.58], the irreflexive property is not preserved under taking ultrafilter extensions, which means that the validity of $\forall q(\forall p(p \to \Diamond p \vee q) \to q)$ is not preserved under taking canonical extensions, which means that $\forall q(\forall p(p \to \Diamond p \vee q) \to q)$ is not canonical.

Example 3. The following example is not equivalent to any Sahlqvist formula in the basic modal language:
$\forall p(\Box p \wedge \forall q(q \to \Diamond\Diamond q \vee p) \to p)$
$\forall p \forall \mathbf{i}(\mathbf{i} \leq \Box p \wedge \forall q(q \to \Diamond\Diamond q \vee p) \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\mathbf{i} \leq \Box p \ \& \ \mathbf{i} \leq \forall q(q \to \Diamond\Diamond q \vee p) \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \& \ \mathbf{i} \leq \forall q(q \to \Diamond\Diamond q \vee p) \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \& \ \forall q(\mathbf{i} \leq q \to \Diamond\Diamond q \vee p) \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \& \ \forall q(\mathbf{i} \leq q \Rightarrow \mathbf{i} \leq \Diamond\Diamond q \vee p) \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \& \ \mathbf{i} \leq \Diamond\Diamond \mathbf{i} \vee p \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \& \ \mathbf{i} \wedge \neg \Diamond\Diamond \mathbf{i} \leq p \Rightarrow \mathbf{i} \leq p)$
$\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \vee (\mathbf{i} \wedge \neg \Diamond\Diamond \mathbf{i}) \leq p \Rightarrow \mathbf{i} \leq p)$
$\forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i} \vee (\mathbf{i} \wedge \neg \Diamond\Diamond \mathbf{i}))$
$\forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i}$ or $\mathbf{i} \leq \mathbf{i} \wedge \neg \Diamond\Diamond \mathbf{i})$
$\forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i}$ or $\mathbf{i} \leq \neg \Diamond\Diamond \mathbf{i})$
$\forall \mathbf{i}(\mathbf{i} \leq \Diamond\Diamond \mathbf{i} \to \blacklozenge \mathbf{i})$
$\forall x \forall y(Rxy \wedge Ryx \to Rxx)$

One can show that this property is not modally definable:

Consider $\mathbb{F}_1 = (W_1, R_1)$ where W_1 is the set of all integers, $R_1 = \{(x, x+1) \mid x \in W_1\}$, $\mathbb{F}_2 = (W_2, R_2)$ where $W_2 = \{w_0, w_1\}$, $R_2 = \{(w_0, w_1), (w_1, w_0)\}$, then $\mathbb{F}_2$ is a bounded morphic image of $\mathbb{F}_1$, $\mathbb{F}_1 \vDash \forall x \forall y(Rxy \wedge Ryx \to Rxx)$, while $\mathbb{F}_2 \nvDash \forall x \forall y(Rxy \wedge Ryx \to Rxx)$.

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