Duality and Outermost Boundaries in Generalized Percolation Lattices

Ghurumuruhan Ganesan[1]

[1]Institute of Mathematical Sciences, HBNI, Chennai

Abstract

In this paper we consider a connected planar graph G and impose conditions that results in G having a percolation lattice-like cellular structure. Assigning each cell of G to be either occupied or vacant, we describe the outermost boundaries of star and plus connected components in G. We then consider the dual graph of G and impose conditions under which the dual is also a percolation lattice. Finally, using G and its dual, we construct vacant cell cycles surrounding occupied components and study left right crossings and bond percolation in rectangles.

Keywords

Percolation lattices, Star and plus connected components, Outermost boundaries, Duality, Left-right crossings,

1. Introduction

The structure of the outermost boundary of finite components is crucial for contour analysis problems of percolation [3] and random graphs [5]. For the square lattice, self-duality plays a crucial role in determining the properties of star and plus connected components and we refer to Chapter 3 [2] for a detailed discussion of combinatorial properties of percolation in regular lattices. For general graphs, [6] uses separating sets in equivalence class of infinite paths to study duality in locally finite graphs and [5] uses unicoherence and topological arguments to investigate plus connected components.

In many applications, it might happen that the lattice on which percolation occurs is not necessarily regular, like for example percolation in Voronoi tessellations [2]. It would therefore be interesting to study the duality properties of such irregular lattices and determine conditions under these lattices have behaviour similar to the regular lattices. In this paper we study outermost boundaries in generalized percolation lattices and prove duality properties analogous to regular lattices. We first consider an arbitrary planar graph G and impose certain cyclic conditions on G that results in a cellular structure analogous to regular lattices. We then define the dual graph of G and determine necessary and sufficient conditions for the dual to be a percolation lattice, analogous to G. Using G and its dual, we study outermost boundaries, occupied components and rectangular left-right crossings in generalized percolation lattices.

The paper is organized as follows: In Section 2, we define generalized percolation lattices and describe the cellular structure of such lattices in Theorem 1 and in Section 3 we study outermost

Algorithms, Computing and Mathematics Conference (ACM 2021), Chennai, India
gganesan82@gmail.com (G. Ganesan) ORCID: 0000-0001-5857-5411 (G. Ganesan)

boundaries of star and plus connected components in generalized percolation lattices. Next, in Section 4, we define the dual graph to a percolation lattice and describe conditions under which the dual graph has the properties of a percolation lattice. Following this, we use dual lattices to study vacant cycles of cells surrounding plus and star connected components. Finally in Section 5, we study left right crossings of generalized percolation lattices in rectangles.

2. Percolation lattices

Let $G = (V, E)$ be any connected finite planar graph in $\mathbb{R}^2$ where each edge is a straight line segment. Two vertices v_1 and v_2 are said to be adjacent if they share an edge in common. Two edges e_1 and e_2 are said to be adjacent if they share a vertex in common. A subgraph $P = (v_1, \ldots, v_l) \subseteq G$ is said to be a *walk* if v_i is adjacent to v_{i+1} for $1 \leq i \leq l-1$. If $e_i = (v_i, v_{i+1})$ is the edge containing end-vertices v_i and v_{i+1}, we also represent $P = (e_1, \ldots, e_{l-1})$ and say that v_1 and v_l are *end-vertices* of P. We say that P is a *circuit* if P is a walk and $v_1 = v_l$. We say that P is a *path* if P is a walk and all the l vertices in P are distinct. Finally, we say that P is a *cycle* if P is a path and $v_1 = v_l$.

For a cycle $C \in G$, let $A = A(C)$ be the bounded open set whose boundary is C. We define the *interior* of C to be A and the *closed interior* of C to be $A \cup C$. We also define the *exterior* of C to be $(C \cup A)^c$ and the *closed exterior* of C to be A^c. We say that the graph G is a *percolation lattice* if every edge in G belongs to a cycle. We say that a cycle C in G is a *cell* if there exists no point of an edge of G in the interior of C. By definition any two distinct cells C_1 and C_2 have mutually disjoint interiors and the intersection $C_1 \cap C_2$ is either empty or a union of vertices and edges in G. Any edge of e belongs to at most two cells and we say that e is *unicellular* if there is at most one cell containing e as an edge.

The following intuitive result captures the main features of percolation lattices as studied in [2].

Theorem 1. *If G is a percolation lattice, then there are distinct cells $Q_1, Q_2, \ldots, Q_T$ such that*

$$G = \bigcup_{i=1}^{T} Q_i. \tag{2.1}$$

Moreover, the representation (2.1) is unique in the sense that if $V_1, \ldots, V_W$ are cells such that $G = \bigcup_{j=1}^{W} V_j$, then $T = W$ and $\{V_i\}_{1 \leq i \leq T} = \{Q_i\}_{1 \leq i \leq T}$.

The following additional properties hold:

$(x1)$ *For every edge e, there are at most two cells containing e as an edge.*

$(x2)$ *If e is contained in the closed interior of a cycle $C \in G$, then there are two cells containing e as an edge and both cells lie in the closed interior of C. If e is contained in the closed exterior of C, then all cells containing e lie in the closed exterior of C.*

$(x3)$ *There are cycles $\Delta_1, \Delta_2, \ldots, \Delta_B$ with mutually disjoint interiors such that every cell in G is contained in the closed interior of one of these cycles. For any $i \neq j$, the cycles Δ_i and Δ_j have at most one vertex in common and an edge $e \in G$ is unicellular if and only if e belongs to some cycle in $\{\Delta_l\}$.*

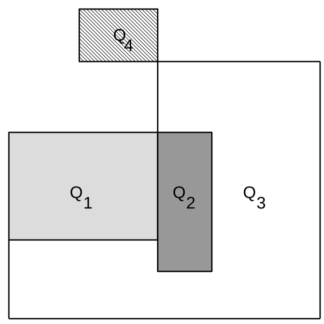


Figure 1: Example of a percolation lattice G along with the corresponding cellular decomposition $\bigcup_{i=1}^{4} Q_i$.

In Figure 1(a), we illustrate the above result using a percolation lattice containing 4 cells Q_1, Q_2, Q_3 and Q_4.

For completeness, we prove Theorem 1 using the following auxiliary result regarding merging of two cycles that is of independent interest and used throughout the paper.

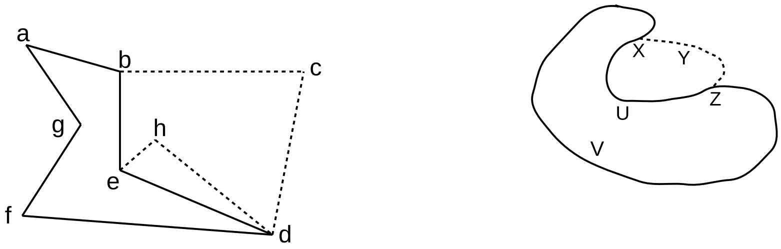
Proposition 1. *Let C and D be cycles in the graph G that have more than one vertex in common. There exists a unique cycle E consisting only of edges of C and D with the following properties:*
(i) *The closed interior of E contains the closed interior of both C and D.*
(ii) *If an edge e belongs to C or D, then either e belongs to E or is contained in its closed interior.*

Moreover, if D contains at least one edge in the closed exterior of C, then the cycle E also contains an edge of D that lies in the closed exterior of C.

The above result essentially says that if two cycles intersect at more than one point, there is an innermost cycle containing both of them in its interior. For illustration, we refer to Figure 2(a) where two cycles $abedfga$ and $behdcb$ have the edge be and the vertex d in common. The merged cycle $abcdfga$ contains both the smaller cycles in its closed interior. Analogous to [4], we use an iterative piecewise algorithmic construction for obtaining the merged cycle.

Proof of Proposition 1: Let $P \subset D$ be any path that has its end-vertices in the cycle $D_0 := C$ and lies in the exterior of D_0 (for illustration see Figure 2(b) where $P = XYZ$ and $C = XUZVX$). Letting $Q = XVZ \subset C$ the cycle $D_1 := P \cup Q$ then contains the cycle $C = D_0$ in its interior. We then repeat the above procedure with the cycle D_1 and look for another path $P_1 \subset D$ that lies in the exterior of D_1 and has end-vertices in D_1. Arguing as before, we get a cycle D_2 that contains P_1 as a subpath and has the cycle C in its closed interior. This procedure continues until we obtain a cycle D_n that does not contain any edge of D in its exterior.

We argue that D_n is the desired cycle E. By construction both C and D are contained in the closed interior of D_n and D_n consists of only edges of C and D so (i) and (ii) are true. Moreover, each cycle $D_i, 1 \leq i \leq n$ contains an edge of D that lies in the closed exterior of C. The uniqueness of E is also true since if $E_1 \neq E$ is any edge satisfying $(i) - (ii)$ and E_1 contains an edge of E in its closed exterior, then some edge of C or D is present in the closed exterior of E_1, a contradiction. $\square$



(a) (b)

Figure 2: (a) The cycle $abcdfga$ formed by merging the cycles $abedfga$ and $behdcb$. (b) The path $P = XYZ \subset D$ lies in the exterior cycle $C = XUZVX$.

Proof of Theorem 1: Let $C_1, \ldots, C_T$ be the set of all cycles containing $e = (u, v)$ as an edge. We shrink the cycles in an iterative manner as follows. Let $D_0 := C_1$ and suppose there exists some point of an edge of the cycle $C_j, j \geq 2$ in the *interior* of D_0. Because the graph G is planar, there exists a path $P_1 \subset C_j$ present in the closed interior of D_0. We shrink the cycle D_0 to the cycle D_1 containing the edge e and the path P_1. For illustration we again use Figure 2(b) with $D_0 = C_1 = XYZVX$ and $C_2 = XYZVX$. In this case $P_1 = XUZ$ and $D_0 = XYZVX$. The "shrinked" cycle $D_1 = XUZVX$ contains every edge P_1. We now repeat the above procedure with the cycle D_1 and proceed iteratively to finally obtain a cycle D_{fin} that does not contain any point of $\bigcup_{j=1}^{T} C_j$ in its interior.

The cycle $D_{fin} =: Q_e$ is a cell containing the edge e. If there exists a cycle C_j such that C_j and Q_e have mutually disjoint interiors, then we repeat the above procedure starting with the cycle C_j and obtain another cell R_e, also containing e as an edge. By construction Q_e and R_e have mutually disjoint interiors and for any cycle C_i we have that either Q_e or R_e is contained in the closed interior of C_i. The set of all distinct cells in $\{Q_e\}_{e \in G} \bigcup \{R_e\}_{e \in G}$ is the desired cellular decomposition (2.1) of G. The decomposition (2.1) is unique since every V_i must necessarily be one of $Q_1, \ldots, Q_T$.

The proof of $(x1)$ and the proof of $(x2)$ for e present in the closed exterior of C, follow from the above construction. To prove the remaining part of $(x2)$, suppose that e is present in the closed interior of C. Since G is a percolation lattice, there exists a cycle $C_e \in G$ containing e as an edge. This cycle C_e contains an edge e in the closed interior of C and therefore a path $P_e \subset C_e$ with end-vertices $a, b \in C$ contained in the closed interior of C. Let F_1 and F_2 be the two paths with end-vertices a, b that form the cycle C. For reference, in Figure 2(b), we have $X = a, Z = b, F_1 = XYZ, F_2 = XVZ$ and $P_e = XUZ$. The cycles $P_e \cup F_1$ and $P_e \cup F_2$ have mutually disjoint interiors and so arguing as before, we obtain two cells Q_e and R_e containing e as an edge. By construction both Q_e and R_e are contained in the interior of C.

The cycles in $(x3)$ are obtained by repeatedly merging cells in G as follows: We first pick one cell Q_1 and using Proposition 1, merge Q_1 with another cell, say Q_2, that shares an edge with Q_1 to get a new cycle Q_{12}. We then pick another cell, say Q_3, that shares an edge with Q_{12} and lies in the exterior of Q_{12} and merge these together to get a new cycle Q_{123}. Continuing this way, we get a cycle Δ_1 satisfying the property that no cell in $\{Q_j\}$ lying in the exterior of Δ_1 shares an edge with Δ_1. If there still exists cells in the exterior of Δ_1, then because G is connected, one of these exterior cells (call it Q_{21}) necessarily shares a vertex with Δ_1. We then repeat the above procedure starting with the cell Q_{21}. Continuing this way until all cells are exhausted, we get the desired cycles $\Delta_l, 1 \leq l \leq B$.

Finally, if e is unicellular and is contained within Δ_l, then the cell Q_e necessarily shares the edge e with Δ_l. This completes the proof of $(x3)$. $\square$

3. Outermost boundaries

Let G be a percolation lattice with cellular decomposition $G = \bigcup_{k=1}^{T} Q_i$ as in (2.1). We have the following definition of star and plus adjacency.

Definition 1. *We say that two cells Q_1 and Q_2 in G are* star adjacent *if $Q_1 \cap Q_2$ contains a vertex in G and* plus adjacent *if $Q_1 \cap Q_2$ contains an edge in G.*

We assign every cell $Q_k, 1 \leq k \leq T$, one of the two states, occupied or vacant and assume that there exists an occupied cell Q_0 containing the origin. We say that the cell Q_i is connected to the cell Q_j by a *star connected $S-$path* if there is a sequence of distinct cells $(Y_1, Y_2, ..., Y_t), Y_l \subset \{Q_k\}, 1 \leq l \leq t$ such that Y_l is star adjacent to Y_{l+1} for all $1 \leq l \leq t-1$ and $Y_1 = Q_i$ and $Y_t = Q_j$. If all the cells in $\{Y_l\}_{1 \leq l \leq t}$ are occupied, we say that Q_i is connected to Q_j by an *occupied* star connected $S-$path.

Let $C(0)$ be the collection of all occupied cells in $\{Q_k\}_{1 \leq k \leq T}$ each of which is connected to the cell Q_0 by an occupied star connected $S-$path. We say that $C(0)$ is the star connected occupied component containing the origin and let $\{J_k\}_{1 \leq k \leq M} \subset \{Q_j\}$ be the set of all the occupied cells belonging to the component $C(0)$.

To define the outermost boundary of $C(0)$, we begin with a few preliminary definitions. Let G_0 be the graph with vertex set and edge set, respectively being the vertex set and edge set of the cells $\{J_k\}_{1 \leq k \leq M} = C(0)$. An edge $e \in G_0$ is a said to be *boundary edge* if e is unicellular or e is adjacent to a vacant cell. (By definition, e is already adjacent to an occupied cell of the component $C(0)$). We have the following definition.

Definition 2. *We say that the edge e in the graph G_0 is an* outermost boundary *edge of the component $C(0)$ if the following holds true for every cycle C in G_0 : either e is an edge of C or e is in the closed exterior of C.*

We define the outermost boundary ∂_0 of $C(0)$ to be the set of all outermost boundary edges of G_0.

Thus outermost boundary edges cannot be contained in the interior of any cycle in the graph G_0. We have the following result regarding the outermost boundary of the star component $C(0)$.

Theorem 2. *There are cycles $C_1, C_2, \ldots, C_n \subseteq G$ satisfying the following properties:*
(i) An edge $e \in \partial_0$ if and only if $e \in \bigcup_{i=1}^{n} C_i$.
(ii) The graph $\bigcup_{i=1}^{n} C_i$ is a connected subgraph of G_0.

(iii) *If $i \neq j$, the cycles C_i and C_j have disjoint interiors and have at most one vertex in common.*
(iv) *Every occupied cell $J_k \in C(0)$ is contained in the interior of some cycle C_j.*
(v) *If $e \in C_j$ for some j, then e is a boundary edge belonging to an occupied square of $C(0)$ contained in the interior of C_j. If e is not unicellular, then e also belongs to a vacant cell lying in the exterior of all the cycles in ∂_0.*
Moreover, there exists a circuit C_{out} containing every edge of $\cup_{1\leq i\leq n} C_i$.

The outermost boundary ∂_0 is a connected union of cycles satisfying properties $(i) - (v)$ and is therefore an Eulerian graph with C_{out} denoting the corresponding Eulerian circuit (see Chapter 1, [1]). As an illustration, Figure 3(a) describes a percolation lattice with six cells. The cells with circle inside them are occupied and the rest are vacant. The occupied cells form a star connected component and the outermost boundary consists of two cycles $C_1 = abcdefga$ and $C_2 = fhkf$.

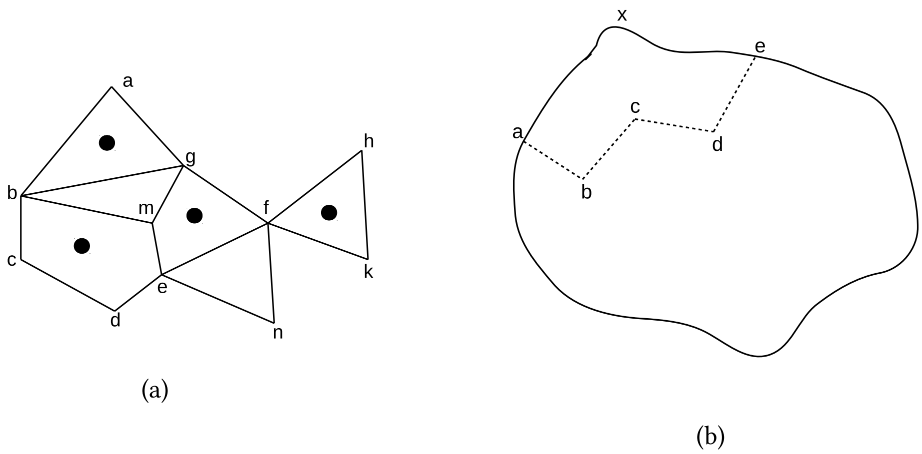


Figure 3: (a) A star connected component formed by the cells with circle inside them. The outermost boundary consists of two cycles $C_1 = abcdefga$ and $C_2 = fhkf$. (b) Replacing an interior path $P = abcde$ in the cycle D_k (denoted by the wavy curve) with a path $Q = axe \subseteq D_k$.

To prove Theorem 2, we use the following Proposition also of independent interest.

Proposition 2. *For every occupied cell $J_k \in C(0), 1 \leq k \leq M$, there exists a unique cycle D_k in G_0 satisfying the following properties.*
(a) *The cell J_k is contained in the closed interior of D_k.*
(b) *Every edge $e \in D_k$ is a boundary edge adjacent to an occupied cell of $C(0)$ present in the closed interior of D_k. If e is not unicellular, then e is also adjacent to a vacant cell present in the closed exterior of D_k.*
(c) *If C is any cycle in G_0 that contains J_k in the interior, then every edge in C either belongs to D_k or is contained in the interior.*

Every edge of D_k is also an outermost boundary edge in the graph G_0 and so we denote D_k to be the *outermost boundary cycle* containing the cell $J_k \in C(0)$. For example in Figure 3(a), the outermost boundary cycle containing the cell $efgm$ is the cycle $C_1 = abcdefga$.

Below we prove Proposition 2 and Theorem 2 in that order.

Proof of Proposition 2: Let $\mathcal{E} \neq \emptyset$ be the set of all cycles in the graph G_0 satisfying property (a); i.e., if $C \in \mathcal{E}$ then J_k in present in the closed interior of C. The set $\mathcal{E}$ is not empty since J_k is itself a cycle containing J_k in its closed interior and so belongs to $\mathcal{E}$. Moreover if C_1 and C_2 are any two cycles in $\mathcal{E}$, then it cannot be the case that C_1 and C_2 have mutually disjoint interiors since both C_1 and C_2 contain the cell J_k in its closed interior. Thus it is possible to merge C_1 and C_2 using Proposition 1 to get a new cycle $C_3 \in \mathcal{E}$ that contains both C_1 and C_2 in its closed interior. Continuing this way, we obtain an "outermost cycle" C_{fin} that contains all the cycles of $\mathcal{E}$ in its closed interior.

By construction, the cycle C_{fin} satisfies properties (a) and (c). To see that (b) also holds, suppose there exists an edge $e \in C_{fin}$ that is not a boundary edge. Since e belongs to the graph G_0, the edge e is adjacent to an occupied cell $A_1 \in \{J_i\}$. But since e is not a boundary edge, there exists one other cell $A_2 \in \{J_i\} \setminus A_1$ containing e as an edge and moreover A_2 is also occupied. One of these cells, say A_1, is contained in the interior of C_{fin} and the other cell A_2, is contained in the exterior.

The cell A_2 and the cycle C_{fin} have the edge e in common and thus more than one vertex in common. We then use Proposition 1 to obtain a larger cycle $C_{lar} \neq C_{fin}$ containing both C_{fin} and A_2 in its closed interior. This is a contradiction to the fact that C_{fin} satisfies property (c). Thus every edge e of C_{fin} is a boundary edge. By the same argument above, we also get that the edge e cannot be adjacent to an occupied cell in the exterior of the cycle C_{fin}. Thus e is adjacent to an occupied cell in the interior of C_{fin} and a vacant cell (if it exists) in the exterior. $\square$

Proof of Theorem 2: We argue that the set of distinct cycles in the set $\mathcal{D} := \cup_{1 \leq k \leq M} \{D_k\}$ obtained in Proposition 2 is the desired outermost boundary ∂_0 and satisfies the properties $(i) - (v)$ mentioned in the statement of the theorem.

To prove (i), it suffices to see that every edge in the union of the cycles $\mathcal{D} = \cup_{1 \leq k \leq M} \{D_k\}$ is an outermost boundary edge. This is because, by definition, no edge with an end-vertex present in the interior of some cycle in $\mathcal{D}$ can be an outermost boundary edge. Now, suppose some edge $e \in D_k$ has an end-vertex in the interior of some cycle $C \subseteq G_0$. This necessarily implies that at least one edge of C is present in the exterior of D_k and moreover C and D_k cannot have a single common vertex. Therefore it is possible to merge C and D_k using Proposition 1 to get a bigger cycle $F_k \subset G$ containing D_k in its closed interior, a contradiction to the construction of D_k. This completes the proof of (i).

To prove (ii), we first see that the graph G_0 formed by the vertices and edges of the component $C(0)$, is connected. Indeed, let u_1 and u_2 be vertices in G_0 so that each $u_i, i = 1, 2$ is a corner of an occupied cell $J_i \in C(0)$. By definition, there is a star connected cell path connecting J_1 and J_2, consisting only of cells in $C(0)$ and consequently, there exists a path in G_0 from u_1 to u_2. Now let v_1 and v_2 be vertices in $\mathcal{D}$ belonging to cycles D_{r_1} and D_{r_2}, respectively, for some $1 \leq r_1, r_2 \leq M$. By previous discussion, there exists a path P_{12} from v_1 to v_2 containing only edges of G_0 and by construction, each such edge lies in the closed interior of some cycle in $\{D_k\}$. This is true since all the occupied cells are present in some cycle in $\{D_k\}$. For each sub-path $P \subseteq P_{12}$ that contains a point in the interior of some cycle D_k and has end-vertices in D_k, we replace P with a path $Q \subseteq D_k$ (see Figure 3(b)). Iteratively replacing all such interior paths, we get a path from v_1 to v_2 containing only edges of $\{D_k\}$. Thus the union of cycles $\mathcal{D}$ is connected and this proves (ii).

Property (iii) is true since otherwise we could merge the cycles D_i and D_j obtained in Proposition 2 to get a larger cycle D_{tot} containing both D_i and D_j in its closed interior, a contradiction to the fact that D_i satisfies property (c) in Proposition 2. Indeed for any occupied cell $J_k \in C(0)$ the corresponding outermost boundary cycle D_k satisfies property (a) of Proposition 2 and so (iv) is true. Moreover if $e \in D_k$ is any edge, then using the fact that D_k satisfies property (b) of Proposition 2, we get that the edge e satisfies property (v).

Finally, to obtain the circuit C_{out} containing the outermost boundary, we first compute the cycle graph T_{cyc} as follows. Let $E_1, E_2, ..., E_n$ be the distinct outermost boundary cycles in $\mathcal{D}$. Represent E_i by a vertex i in T_{cyc}. If E_i and E_j share a corner, we draw an edge $e(i,j)$ between i and j. Since the union of cycles $E_i, 1 \leq i \leq n$ is connected, we get that T_{cyc} is connected as well.

Let H_{cyc} be any spanning tree of T_{cyc} and consider an increasing sequence of tree subgraphs $\{1\} = H_1 \subset H_2 \subset \ldots H_n = H_{cyc}$. The graph H_1 contains a single vertex $\{1\}$ and so we set $\Pi_1 = E_1$ to be the circuit obtained at the end of the first iteration. Having obtained the circuit Π_i, let $q_{i+1} \in H_{i+1} \setminus H_i$ be adjacent to some leaf $v_i \in H_i$. This implies that the cycle $E_{q_{i+1}}$ shares a vertex w_i with the cycle E_{v_i}. Since the circuit Π_i contains w_i, we assume that w_i is the starting and ending vertex of Π_i and also of $E_{q_{i+1}}$. The concatenation of Π_i and $E_{q_{i+1}}$ is the desired circuit Π_{i+1}. Continuing this way, the final circuit Π_n obtained is the desired circuit C_{out}. $\square$

Plus connected components

The techniques used in the previous sections also allows us to obtain the outermost boundary for plus connected components. We recall that cells Q_i and Q_j are said to be *plus adjacent* if they share an edge between them. We say that the cell Q_i is connected to the cell Q_j by a *plus connected* $S-$*path* if there is a sequence of distinct cells $(Q_i = Y_1, Y_2, ..., Y_t = Q_j) \subseteq \{Q_k\}_{1 \leq k \leq T}$ such that Y_l is plus adjacent to Y_{l+1} for all $1 \leq l \leq t-1$. If all the cells in $\{Y_l\}_{1 \leq l \leq t}$ are occupied, we say that Q_i is connected to Q_j by an *occupied* plus connected $S-$path.

Let $C^+(0)$ be the collection of all occupied cells in $\{Q_k\}_{1 \leq k \leq T}$ each of which is connected to the occupied cell Q_0 containing the origin, by an occupied plus connected $S-$path. We say that $C^+(0)$ is the plus connected occupied component containing the origin. Further we also define G_0^+ to be the graph with vertex set being the set of all vertices of the cells of $\{Q_k\}$ present in $C^+(0)$ and edge set consisting of the edges of the cells of $\{Q_k\}$ present in $C^+(0)$.

Every plus connected component is also a star connected component and so the definition of outermost boundary edge in Definition 2 holds for the component $C^+(0)$ with G_0 replaced by G_0^+. We have the following result.

Theorem 3. *The outermost boundary ∂_0^+ of $C^+(0)$ is unique cycle in G_0^+ with the following properties:*
(i) All cells of $C^+(0)$ are contained in the interior of ∂_0^+.
(ii) Every edge in ∂_0^+ is a boundary edge adjacent to an occupied cell of $C^+(0)$ contained in the interior of ∂_0^+ and a vacant cell in the exterior.

This is in contrast to star connected components which may contain multiple cycles in the outermost boundary. In Figure 3(a) for example, the union of the cells $bcdemb$ and $efgme$ forms a plus connected component whose outermost boundary is the cycle $bcdefgmb$.

Proof of Theorem 3: Proposition 2 holds with $C(0)$ replaced by $C^+(0)$. Since $C^+(0)$ is plus connected, the outermost boundary cycle D_0 the cell Q_0 in its interior also contains all the cells of $C^+(0)$ in its interior. Therefore cycle D_0 satisfies the conditions (i) and (ii) in the statement of the theorem, is unique and so $\partial_0^+ = D_0$. $\square$

4. Vacant cell cycles surrounding occupied components

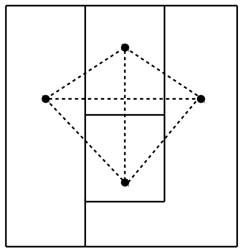
In this section, we study vacant cycles of cells surrounding occupied star and plus components. We therefore begin with a discussion of the dual lattice. Let G be any percolation lattice and let $G = \bigcup_{k=0}^{N} S_k$ be the cellular decomposition of G.

Definition 3. *We say that a graph G_d is* dual *to G if the following conditions hold:*
$(d1)$ *Every cell in G contains exactly one vertex of G_d in its interior.*
Suppose vertex $w \in G_d$ is the present in the interior of the cell $Q(w)$ of G.
$(d2)$ *Vertices $w_1, w_2 \in G_d$ are adjacent if and only if the cells $Q(w_1)$ and $Q(w_2)$ are plus adjacent.*

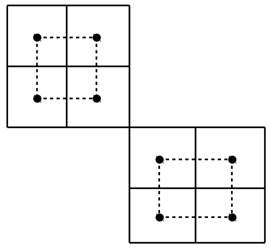
To study the similarity between G and G_d, we would like to first ensure that the dual graph G_d is also a percolation lattice, i.e., we prefer that G_d is connected, planar and each edge of G_d belongs to a cycle. This is because, there are in fact dual graphs that satisfy exactly two of these three properties. For example, consider two plus adjacent squares S_1 and S_2 of same side length, to be the graph G. If we let the centres of S_1 and S_2 be the vertex set of G_d, then the edge set of G_d is the single edge joining the centres of S_1 and S_2. The graph G_d is connected and planar but acyclic. In Figure 4 (a), we have an example of a graph G (denoted by solid lines) and the corresponding dual graph G_d (denoted by the dotted lines). The graph G_d is connected and every edge of G_d belongs to a cycle but G_d is not planar. In Figure 4 (b), the dual graph G_d is planar and every edge of G_d belongs to a cycle but G_d is not connected.

Throughout the paper, we assume that the graph G admits a dual graph G_d satisfying the following properties:
$(a1)$ (Niceness property) The percolation lattice G is nice in the sense that any two plus adjacent cells in G share exactly one edge in common and no other vertex.
$(a2)$ (Interior edge property) Any edge $(w_1, w_2) \in G_d$ is present in the interior of the cycle formed by merging the plus adjacent cells $Q(w_1)$ and $Q(w_2)$.
$(a3)$ The dual graph G_d is a connected and planar percolation lattice and G is dual to G_d.
$(a4)$ The dual graph G_d satisfies the niceness and interior edge property.
$(a5)$ If z is a vertex of the dual cell $R(v)$ and $Q(z)$ is the cell in G containing z, then v is a vertex of $Q(z)$.
Thus G and G_d have similar cellular structure. For example, the usual square lattice on the plane satisfies properties $(a1) - (a5)$.

Let G be a percolation lattice with cellular decomposition $\bigcup_{k=0}^{N} S_k$ and let G_d be a lattice dual to G satisfying properties $(a1) - (a5)$ and having cellular decomposition $G_d = \bigcup_{l=0}^{L} W_l$. We say that the sequence $L_S = (S_1, ..., S_m)$ is a *plus connected cell path* in G if for each $1 \leq i \leq m-1$, the cell S_i is plus adjacent with the cell S_{i+1}. We say that L_S is a *plus connected cell cycle* if S_i



(a)



(b)

Figure 4: (a) An example where the dual graph G_d is non-planar but connected and every edge of G_d belongs to a cycle. (b) An example where G_d is planar and every edge of G_d belongs to a cycle but G_d is not connected.

is adjacent to S_{i-1} and S_{i+1} for each $1 \leq i \leq m$ with the notation that $S_{m+1} = S_1$. Analogous definition holds for star connected paths and cycles.

Let $C(0) = \bigcup_{k=0}^{T} J_k$ be the star connected occupied cell component containing the cell S_0 with origin in its interior. By definition, every cell in $C(0)$ is connected to S_0 by a star connected cell path. We have the following result.

Theorem 4. *Suppose properties $(a1) - (a5)$ hold and suppose every vertex in the component $C(0)$ is present in the interior of some dual cell of G_d.*

There exists a unique cycle $\mathcal{P}_{out} = (w_1, \ldots, w_s) \subset G_d$ such that each vertex w_i is present in the interior of a vacant cell V_i and satisfies the following properties:
(i) For every $i, 1 \leq i \leq s$, the cell V_i is vacant and star adjacent to some occupied cell in $C(0)$.
(ii) All occupied cells in $C(0)$ are contained in the interior of $\mathcal{P}_{out}$.
(iii) If $F_{out} \neq \mathcal{P}_{out}$ is any other cycle in G_d that satisfies $(i) - (ii)$ above, then F_{out} is contained in the closed interior of $\mathcal{P}_{out}$.

The sequence of cells in $(V_1, \ldots, V_s)$ form a plus connected cycle of vacant cells surrounding the star connected component $C(0)$. In Figure 5, the two cells containing the circles form the star connected occupied component. Every other cell is vacant. The two dual cycles 12345671 and 1234598671 both satisfy $(i) - (ii)$ and $\mathcal{P}_{out} = 12345671$.

Proof of Theorem 4: Let ∂_0 denote the outermost boundary of the star connected component $C(0)$ in the percolation lattice G. From Theorem 2 we have that $\partial_0 = \cup_{1 \leq i \leq n} C_i$ is a connected union of cycles $\{C_i\}$ with mutually disjoint interiors and moreover, for $i \neq j$, the cycles C_i and C_j have at most one common vertex.

If vertices $v_1, v_2 \in G$ are adjacent in the outermost boundary ∂_0 of the component $C(0)$, then the corresponding dual cells $R(v_1)$ and $R(v_2)$ are plus adjacent (property $(a4)$. Also, because ∂_0 is connected (see property (ii) Theorem 2), the union of dual cells

$$C_V(\partial_0) := \bigcup_{v \in \partial_0} R(v) \tag{4.1}$$

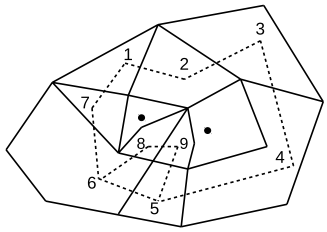


Figure 5: Cycle of vacant cells surrounding a star connected occupied component.

is a plus connected component in the dual graph G_d. Moreover, each edge of ∂_0 is present in the interior of closed interior of the union of cells of $C_V(\partial_0)$. Using Theorem 3 we therefore have that the outermost boundary $\partial_V(\partial_0)$ of $C_V(\partial_0)$ is a single cycle in G_d containing all cells of $C_V(\partial_0)$ in its closed interior and all edges of ∂_0 in its interior. Here the dual outermost boundary $\partial_V(\partial_0)$ is obtained as follows. Every dual cell belonging to $C_V(\partial_0)$ is labelled 1 and every dual cell sharing an edge with a cell in $C_V(\partial_0)$ and not belonging to $C_V(\partial_0)$, is labelled 0. We then apply Theorem 3 with label 1 cells as occupied and label 0 cells as vacant.

Suppose $z_1, z_2, \ldots, z_t$ are the vertices of the dual cycle $\partial_V(\partial_0)$ encountered in that order; i.e., the vertex z_1 is adjacent to z_2, the vertex z_2 is adjacent to z_3 and so on. Each vertex z_i is a vertex of the dual cell $R(v)$ for some $v \in \partial_0$. Therefore if $Q(z_i)$ is the cell in G containing z_i in its interior, then v is a vertex of $Q(z_i)$, by property $(a5)$. Moreover $Q(z_i)$ lies in the exterior of ∂_0 and is adjacent to the vertex v of $C(0)$. This implies that $Q(z_i)$ must necessarily be vacant. This proves that the cycle $\partial_V(\partial_0)$ satisfies properties $(i) - (ii)$.

To get a unique dual cycle satisfying properties $(i) - (iii)$, we merge all dual cycles satisfying properties $(i) - (ii)$. This is possible since any two dual cycles satisfying $(i) - (ii)$ both contain the cell S_0 in their respective interiors and therefore cannot have mutually disjoint interiors. $\square$

Plus connected components

In this subsection we let $C^+(0) = \bigcup_{k=0}^{T} J_k$ be the plus connected occupied cell component containing the cell S_0 with origin in its interior. By definition, every cell in $C^+(0)$ is connected to S_0 by a plus connected cell path and the outermost boundary ∂_0^+ of $C^+(0)$ is a single cycle containing all the cells of $C^+(0)$ in its interior. We have the following result.

Theorem 5. *There exists a star connected cell cycle $\mathcal{M}_{out} = (Y_1, \ldots, Y_t) \subset G$ such that:*
(i) *Each cell Y_i is vacant, lies in the exterior of ∂_0^+ and is plus adjacent to some occupied cell of $C^+(0)$.*
(ii) *The outermost boundary of $\mathcal{M}_{out}$ is a single cycle containing all the cells of $\mathcal{M}_{out} \cup C^+(0)$ in its interior.*

The sequence of cells in $(Y_1, \ldots, Y_t)$ form a star connected cell cycle of vacant cells surrounding the plus connected component $C^+(0)$.

Proof of Theorem 5: From Theorem 3, we have that the outermost boundary $\partial_0^+ = (e_1, \ldots, e_t)$ of $C^+(0)$ is a single cycle containing all cells of $C^+(0)$ in its interior. Moreover every edge e_i

of $\partial^+(0)$ belongs to a occupied cell of $C^+(0)$ and also to a vacant cell Z_i lying in the exterior of ∂_0^+. It is possible that multiple edges in ∂_0^+ belong to the same cell Z_i and so the sequence $(Z_1, Z_2, \ldots, Z_m)$ could have repetitions. In such a case, we remove recurring entries and assume without loss of generality that Z_i is star adjacent to Z_{i+1} for $1 \leq i \leq m$ with the notation that $Z_{m+1} = Z_1$.

The set of cells in $\Gamma := (Z_1, \ldots, Z_m)$ form a star connected component and we suppose that the sequence of cells $(Z_1, \ldots, Z_n)$ form a star connected cycle L_Z for some $n \leq m$. The outermost boundary ∂_Z of L_Z is a connected union of cycles $\bigcup_{1\leq i\leq T} D_i$ and contains every cell of L_Z in the interior of some D_i. If some cell of $C^+(0)$ is contained in the interior of D_i, then because $C^+(0)$ is plus connected, every cell of $C^+(0)$ is contained in the interior of D_i. Moreover, since D_i and D_j share at most one vertex in common, it must the case that $T = 1$ and so the outermost boundary of the star cycle $(Z_1, \ldots, Z_n)$ is the single cycle D_1.

If on the other hand, every cell of $C^+(0)$ is contained in the exterior of every cycle of ∂_Z, then *every* edge of ∂_Z is the edge of some occupied cell of $C^+(0)$ that lies in the exterior of ∂_Z. This implies that the outermost boundary ∂_0^+ of $C^+(0)$ is contained in the *strict* exterior of every cycle in ∂_Z. But this contradicts the fact that each Z_i contains at least one edge of ∂_0^+ and so there is at least one edge of ∂_0^+ contained in the closed interior of some cycle of ∂_Z. $\square$

5. Left right and top bottom crossings in rectangles

In this section, we study the mutual exclusivity of left right and top down crossings in a rectangle. As before we assume that the percolation lattice $G = \bigcup_{k=0}^{N} S_k$ and the dual lattice $G_d = \bigcup_{l=0}^{M} W_l$ satisfy properties $(a1) - (a5)$ and the origin is the present in the interior of the cell S_0.

For a fixed rectangle R, we assume that the sides of R are *nicely covered* by cells of G as shown in Figure 6. Consider the edges a_1b_1 and a_2b_2 intersecting the left side of R. The vertices a_1 and a_2 are connected by a path A_1 (shown by dotted line) in the interior of R. Similarly the vertices b_1 and b_2 are connected by a path B_1 in the exterior of R. The union $A_1 \cup B_1 \cup \{a_1b_1, a_2b_2\}$ forms the cell L_1. We define $L_1, \ldots, L_l$ to be the left cells. Similarly, the cells $T_1, \ldots, T_t$ are top cells, the right cells are $R_1, \ldots, R_r$ and the bottom cells are $B_1, \ldots, B_b$. Any cell contained in the closed interior of R is called an *interior* cell.

The cells C_{TL}, C_{TR}, C_{BL} and C_{BR} each contain a corner of the rectangle and have a single vertex contained in the interior of the rectangle. These four cells, called *corner cells*, are not plus adjacent to any interior cell. As in Figure 6, we assume that the cell L_1 is plus adjacent to L_2 and C_{TL} and does not share a vertex with any other left, right,top or bottom cell. We make analogous assumptions for each of the left, right, top and bottom cells.

Finally, we also assume that the cells are *nicely padded* in the following way: Let $\mathcal{L}_{top}$ be the infinite line containing the top side of R and define $\mathcal{L}_{bottom}, \mathcal{L}_{left}$ and $\mathcal{L}_{right}$ analogously. If Z_1 is any cell intersecting $\mathcal{L}_{top}$ and star adjacent to a cell intersecting R and Z_2 is any cell intersecting $\mathcal{L}_{bottom}$ and star adjacent to a cell intersecting R, then Z_1 and Z_2 are not star adjacent. A similar assumption holds for cells intersecting $\mathcal{L}_{left}$ and $\mathcal{L}_{right}$.

Assuming that R is nicely covered and padded as described above, we have the following definition of left right crossings.

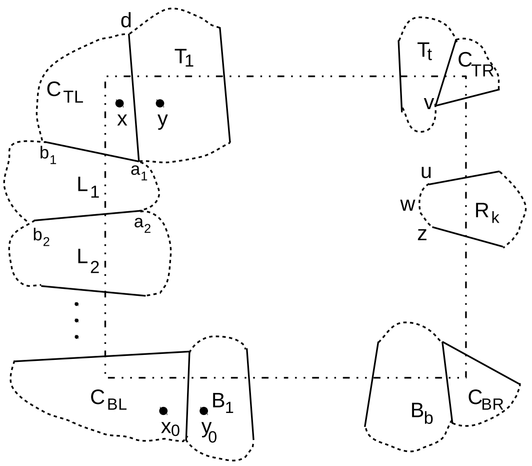


Figure 6: The sides of the rectangle are nicely covered by the cells of G.

Definition 4. *A plus connected cell path* $L = (J_1, \ldots, J_m) \subset \{S_k\}, m \geq 3$ *is said to be a* plus connected left right crossing *of a rectangle* R *if* J_1 *is a left cell, the cell* J_m *is a right cell and every* $J_i, 2 \leq i \leq m-1$ *is an interior cell.*

By the nicely padded assumption, L must contain at least one interior cell. Every interior cell is now assigned one of the following two states: occupied or vacant. If every interior cell in a left right crossing L of R is occupied, we say that L is an *occupied plus connected left right crossing* of the rectangle R. We denote $LR^+(R, O)$ and $LR^+(R, V)$ to be the events that the rectangle R contains an occupied and vacant plus connected left right crossing, respectively.

We have a similar definition of plus connected top down crossings and denote $TD^+(R, O)$ and $TD^+(R, V)$ to be the events that the rectangle R contains an occupied and vacant plus connected top down crossing, respectively. Replacing plus adjacent with star adjacent, we obtain an analogous definition for star connected left right and top down crossings. We have the following result.

Theorem 6. *Suppose properties* $(a1) - (a5)$ *hold and* R *is nicely covered and nicely padded by cells as in Figure 6. Also suppose that every vertex of a cell intersecting* R *is present in the interior of a dual cell and that there is at least one plus connected left right crossing and one plus connected top bottom crossing. We have the following.*
(i) *One of the events* $LR^+(R, O)$ *or* $TD^*(R, V)$ *always occurs but not both.*
(ii) *One of the events* $LR^*(R, O)$ *or* $TD^+(R, V)$ *always occurs but not both.*

The above result describes the mutual exclusivity of occupied and vacant left right and top down crossings in any rectangle.

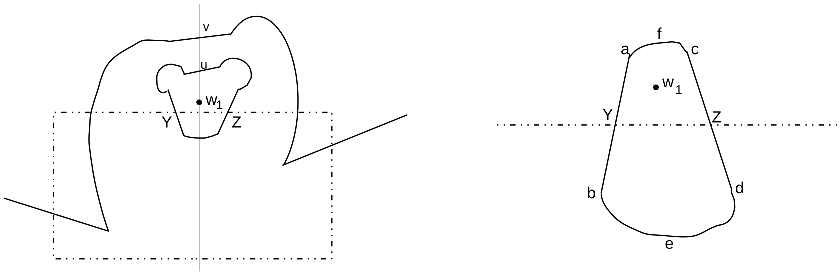
Proof of Theorem 6

We prove the following three statements.
(I) The events $LR^*(R, O)$ and $TD^+(R, V)$ cannot both occur simultaneously.

(II) If $LR^*(R,O)$ does not occur, then $TD^+(R,V)$ must necessarily occur.
(III) If $LR^+(R,O)$ does not occur, then $TD^*(R,V)$ occurs.
Using $(I)-(III)$ and the fact that a top down crossing of R is a left right crossing of the rectangle R' obtained by rotating R by ninety degrees around the centre, we then get Theorem 6.

Proof of (I): Suppose there exists a star connected occupied left right crossing L_s of R and let $\Gamma = (e_1, \ldots, e_t)$ be a path in L_s crossing R from left to right so that e_1 intersects the left edge of R, the edge e_t intersects the right edge of R and each $e_i, 2 \leq i \leq t-1$ belongs to an interior occupied cell of L_s. The path Γ divides the rectangles into two halves.

Suppose $TD^+(R,V)$ also occurs and let $L = (J_1, \ldots, J_m)$ be a vacant plus connected top bottom crossing where J_1 is a top cell, J_m is a bottom cell and every other J_i is an interior cell. Let w_i be the vertex of G_d present in the cell J_i so that $P := (w_1, w_2, \ldots, w_m)$ is a path in G_d. We claim that the vertex w_1 necessarily above Γ. This is because if w_1 were to be present below Γ, then the cell $J_1 \in G$ containing w_1 in its interior also lies below Γ. Let L_{w_1} be the infinite line parallel to the left side of R and containing the point w_1. Since w_1 is contained in the interior of J_1, the some edge of J_1 contains a point $u \in L_{w_1}$ lying above w_1. Since Γ lies above J_1, there exists $v \in \Gamma$ lying above u (see Figure 7 (a) for illustration). This would imply that some edge of Γ crosses the top side of R, a contradiction.



(a) (b)

Figure 7: (a) If J_1 lies below Γ, then the path Γ would cross the top side of R. (b) The top cell $T = afcdeba$ containing the dual vertex w_1 in its interior. The dual edge (w_1, w_2) with w_2 present in the interior of R, must cross the top edge of R in the segment YZ.

From the above paragraph we therefore get that the dual vertex w_1 is necessarily above Γ and an analogous analysis implies that w_m is below Γ. Next, we argue that the "first" edge $(w_1, w_2) \in P$ is either present in the interior of R or crosses the top edge of R. For illustration we consider a magnified top cell $T = afcdeba$ in Figure 7(b) containing edges ab and cd that intersect the top edge of R (the dotted-dashed line). Because of the interior edge property $(a2)$, the dual edge (w_1, w_2) must cross the top edge of R in the segment YZ.

Summarizing, the edge (w_1, w_2) is either present in the interior of R or crosses the top edge of R. Similarly the final edge (w_{m-1}, w_m) is either contained in the interior of R or crosses the bottom edge of R. Every other vertex $w_i, 2 \leq i \leq m-1$ is present in the interior of Γ. Therefore the path P necessarily crosses Γ in the sense that there are edges $f \in \Gamma$ and $e = (w_i, w_{i+1}) \in P$ such f intersects e. The end-vertices of the dual edge e belong to cells J_i and J_{i+1} and the

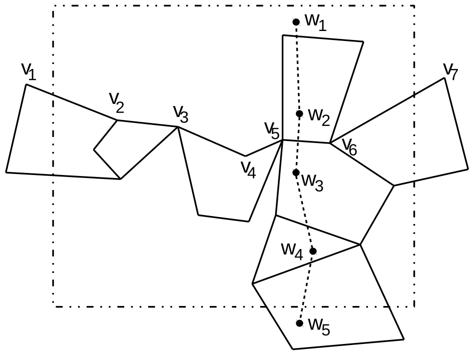


Figure 8: The edge $f = (v_5, v_6)$ belonging to the path $\Gamma = (v_1, v_2, \ldots, v_7)$ and the dual edge $e = (w_2, w_3)$ belonging to the path $P = (w_1, \ldots, w_5)$ intersect.

edge f is common to J_i and J_{i+1}. At least one of the two cells J_i or J_{i+1} lies in the interior of R and so the edge f is necessarily contained within the rectangle R. Thus f is an edge of some occupied cell in the left right crossing L_s and so one of the cells J_i or J_{i+1} must lie in the interior of R and also be occupied, a contradiction.

We illustrate the above argument in Figure 8, where the edge $f = (v_5, v_6)$ belonging to the path $\Gamma = (v_1, v_2, v_3, v_4, v_5, v_6)$ and the dual edge $e = (w_2, w_3)$ in the path $P = (w_1, w_2, w_3, w_4, w_5)$ intersect. $\square$

Proof of (II): The collection $\mathcal{I}_{tot}$ of all left cells $L_1, \ldots, L_l$ and the corner cells C_{TL} and C_{BL} in Figure 6 is a plus connected component. To each cell in $\mathcal{I}_{tot}$ we now assign a *label* ω. If some occupied cell Q in the interior of R is connected to a left cell by a star connected occupied path, we assign the label ω to Q as well. The collection of all cells with the label ω is a star connected component which we denote as $\mathcal{F}_{tot}$. If R is any cell star adjacent to some cell in $\mathcal{F}_{tot}$ and not present in $\mathcal{F}_{tot}$, we assign the label δ to R.

By assumption, any vertex of a cell $Q \in \mathcal{F}_{tot}$ is contained in the interior of some dual cell. Therefore by Theorem 4, there exists a plus connected cell cycle $\Delta_{vac} = (Z_1, \ldots, Z_t)$ surrounding $\mathcal{F}_{tot}$ in such a way that the outermost boundary cycle ∂_Z of Δ_{vac} contains all the cells of $\mathcal{F}_{tot}$ in its interior. The cell cycle Δ_{vac} contains a plus connected cell sub-path $\Delta_Z = (Z_{u_1}, \ldots, Z_{u_2})$ that lies to the right of $\mathcal{L}_{left}$, with Z_{u_1} intersecting $\mathcal{L}_{top}$ and Z_{u_2} intersecting $\mathcal{L}_{bottom}$.

In Figure 9, we illustrate the part of the cycle Δ_{vac} that intersects the rectangle R with the cell labelled i denoting Z_i. As in Figure 9, the cycle Δ_{vac} may intersect the top side of R multiple times but there exists a “last" cell after which the cycle never intersects the top side of R. Formally, u_1 be the largest index i such that the cell Z_i intersects the top side of R and Z_{u_2} is the “first" cell after Z_{u_1} that intersects the bottom side of R. In Figure 9, $u_1 = 6$ and $u_2 = 10$.

By definition the cell Z_{u_1} intersects the line $\mathcal{L}_{top}$, the cell Z_{u_2} intersects the line $\mathcal{L}_{bottom}$ and that every other $Z_j, u_1 < j < u_2$ neither intersects $\mathcal{L}_{top}$ nor intersects $\mathcal{L}_{bottom}$ but lies between

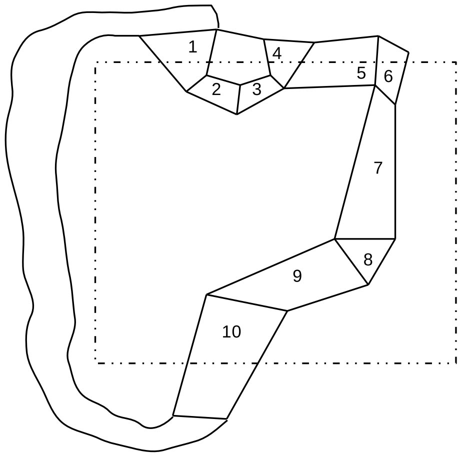


Figure 9: The part of the vacant cycle Δ_{vac} that intersects the rectangle R.

these two lines. By the nicely padded assumption we have that $u_2 > u_1$. No cell $Z_j, u_1 < j < u_2$ can be a right cell because then there would exist an occupied cell contained in the interior of R which is star adjacent to Z_j and is connected to some left cell L_x by an occupied star connected cell path P. The concatenation (L_x, P, Z_j) would then form an occupied star connected left right crossing of R, a contradiction. Thus each cell $Z_j, u_1 < j < u_2$ is necessarily an interior cell.

By the nicely covered assumption, this necessarily means that Z_{u_1} must be a top cell and not a corner cell. This is because, no corner cell is plus adjacent to an interior cell. Similarly Z_{u_2} must be a bottom cell and not a corner cell and so $(Z_{u_1}, Z_{u_1+1}, \ldots, Z_{u_2})$ forms a vacant plus connected top down crossing of R. $\square$

Proof of (III): The proof is analogous to the proof of (II) with minor modifications. Here $\Delta_{vac} = (Z_1, \ldots, Z_t)$ is star connected and if the corner cell C_{TR} or the bottom cell C_{BR} in Figure 6 appear in Δ_{vac}, we simply remove the corresponding entry from Δ_{vac}. The resulting sequence of vacant cells is still star connected and we proceed as before to get the desired vacant star connected top bottom crossing of R. $\square$

Bond Percolation

In this section, we consider bond percolation in the lattice G and the mutual exclusivity of left right and top down crossings of in a rectangle. We consider unoriented bond percolation and an analogous analysis holds for the oriented case as well. As before we assume that the percolation lattice $G = \bigcup_{k=0}^{N} S_k$ and the dual lattice $G_d = \bigcup_{l=0}^{M} W_l$ satisfy properties $(a1) - (a5)$ and the origin is the present in the interior of the cell S_0.

Moreover, we also assume that for a fixed rectangle R, we assume that the sides of R nicely covered and nicely padded by cells of G as described prior to Definition 4.

Assuming that R is nicely covered as described above, we have the following definition of left right crossings.

Definition 5. *A path* $P = (v_1, \ldots, v_m) \subset G, m \geq 4$ *is said to be a* left right crossing *of a rectangle* R *if:*
$(d1)$ *The edge* (v_1, v_2) *intersects the left side of* R *and* v_1 *lies in the exterior of* R.
$(d2)$ *The edge* (v_{m-1}, v_m) *intersects the right side of* R *and* v_m *lies in the exterior of* R.
$(d3)$ *Every other edge* $(v_i, v_{i+1}), 2 \leq i \leq m-2$ *is contained in the interior of* R.

By definition any left right crossing must contain at least one edge in the interior of R. Every edge in the closed interior of R is now assigned one of the following two states: open or closed. Moreover, if edge $e \in G$ is open and if f is the unique dual edge intersecting G_d, then we assign f to be open as well. If every interior edge in a left right crossing P of R is open, we say that P is an *open left right crossing* of the rectangle R. An analogous definition holds for top down crossings. We denote LR to be the event that the rectangle R contains an open left right crossing of G.

For the dual crossing we have a slightly different definition. We say that $P_g := (g_1, \ldots, g_t)$ is a dual top bottom crossing of R if the dual vertex g_1 lies in a top cell, the dual vertex g_t lies in a bottom cell and each edge $(g_i, g_{i+1}), 1 \leq i \leq t-1$ intersects an interior edge of G; i.e., an edge of G with both end-vertices present in the interior of R.

We now see that every edge in a dual top bottom crossing has a state. By definition it suffices to see that the first edge (g_1, g_2) and the last edge (g_{t-1}, g_t) both have states. First consider the edge (g_1, g_2) and let $(v_1, v_2) \in G$ intersect (g_1, g_2). By the nicely covered assumption in Figure 6, we see that the edge (v_1, v_2) belongs to one of the interior paths represented by the dotted lines and so necessarily lies in the interior of R and consequently has a state. This implies that the dual edge (g_1, g_2) has the same state as (v_1, v_2). Similarly, the last edge (g_{t-1}, g_t) also has a state. If every edge in P_g is closed, we say that P_g is a closed dual top bottom crossing and we let TD_d be the event that R contains a closed dual top bottom crossing consisting of edges of G_d.

We have the following result.

Theorem 7. *Suppose properties* $(a1) - (a5)$ *hold. Further suppose that the rectangle* R *is nicely covered and nicely padded by cells in* G *as in Figure 6. One of the events* LR *or* TD_d *always occurs, but not both.*

As before we need to prove three statements:
(I) Both LR and TD_d cannot occur simultaneously.
(II) If LR does not occur, then TD_d occurs.
(III) If TD_d does not occur, then LR occurs.
The proof of (I) is analogous to the proof of (I) in Theorem 6. If Γ is any open left right crossing and Δ is any top bottom dual crossing, we obtain that one vertex of the dual crossing lies above Γ and one vertex lies below Γ and so these two paths must necessarily meet. The dual edge intersecting any edge $e \in G$ has a state and in fact the same state as e and this leads to a contradiction. $\square$

The proof of (III) is analogous to (II) and we prove (II) below.
Proof of (II): Let $\{e_i\}_{1 \leq i \leq t}$ be the set of edges of G intersecting the left side of R arranged in

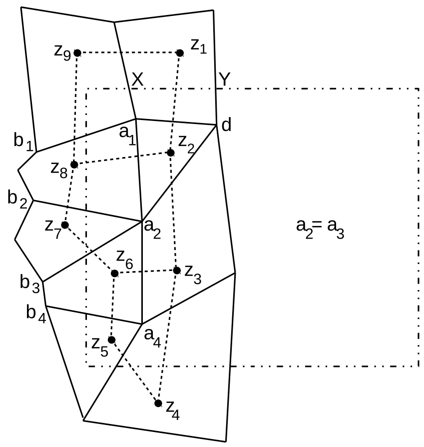


Figure 10: The outermost boundary cycle $\partial_E = (z_1, \ldots, z_9, z_1)$ denoted by the dotted lines, intersects the top and bottom sides of the rectangle R.

the decreasing order of the $y-$coordinate of the intersection point and for $1 \leq i \leq t$, let a_i and b_i be the end-vertices of e_i present in the interior and exterior of R, respectively. For example, in Figure 6, the edge $e_1 = a_1 b_1, e_2 = a_2 b_2$ and so on.

Let $\mathcal{I}_{tot}$ be the set of all open edges lying in the interior of R and connected to some vertex in $\{a_i\}_{1 \leq i \leq t}$ by an open path and for $1 \leq i \leq t-1$, let A_i be the path between a_i and a_{i+1} lying in the interior of the rectangle R. The union $\mathcal{E}_{tot} = \mathcal{I}_{tot} \cup \{A_i\}_{1 \leq i \leq t-1}$ is then a connected component and each vertex $v \in \mathcal{E}_{tot}$ is present in the interior of some dual cell $W(v) \subset G_d$. The union of the dual cells $\{W(v)\}_{v \in \mathcal{E}_{tot}}$ forms a plus connected dual component whose outermost boundary ∂_W is a single cycle in G_d containing all edges of $\mathcal{E}_{tot}$ in its interior.

We now see that ∂_W contains at least one dual vertex present in the interior of a top cell. From Figure 6, the dual edge joining x and y intersects the edge (a_1, d) and so belongs to the dual cell $W(a_1)$ containing a_1 in its interior. The dual vertex y lying in the interior of the top cell T_1 therefore belongs to the dual cell $W(a_1)$, by property $(a5)$. The dual edge (x, y) is not present in any other dual cell $W(v), v \in \mathcal{E}_{tot}$ and so belongs to the final cycle ∂_W as well.

By an analogous argument, all the dual vertices present in the interior of the left cells $L_1, \ldots, L_l$ and the corner cells C_{BL} and C_{TL} form a sub-path P_W of ∂_W and the dual edge $(x_0, y_0) \in \partial_W$ as well, where x_0 and y_0 are the dual vertices is present in the interior of the bottom left corner cell C_{BL} and the first bottom cell B_1, respectively (See Figure 6). The subpath $\Delta_W := \partial_W \setminus P_W$ has end-vertices y and y_0. For illustration, in Figure 10, the outermost boundary cycle ∂_E formed by the merging of the dual cells $\{W(a_i)\}_{1 \leq i \leq t}$ is shown by the dotted line $(z_1, z_2, \ldots, z_9, z_1)$. Here $z_1 = y, z_9 = x, z_5 = x_0$ and $z_4 = y_0$. The sub-path $P_W = (z_5, z_6, z_7, z_8, z_9)$ and $\Delta_W = (z_1, z_2, z_3, z_4)$.

The path Δ_W might contain many dual vertices present in the interior of some top cell and so we pick the "last" such vertex and call it y_1. Similarly we pick the first dual vertex present in the interior of a bottom cell. Formally, we extract a sub-path $\Gamma_W := (y_1, \ldots, y_p) \subset \Delta_W$ such

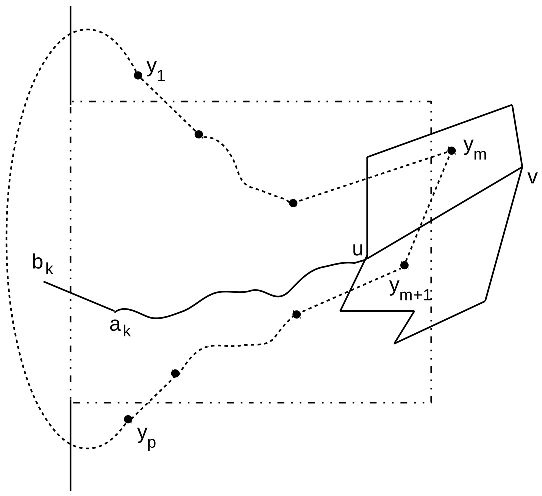


Figure 11: The edge $(y_m, y_{m+1}) \in \Gamma_W$ cuts an edge intersecting the right side of R.

that the vertex y_1 lies in the interior of a top cell, the vertex y_p lies in the interior of a bottom cell and every other y_i lies in the interior of a right, corner or interior cell. To prove that each edge of Γ_W has a state it suffices to see that there does not exist m such that y_m belongs to a corner or a right cell and y_{m+1} belongs to a right cell.

As in Figure 11, we assume that y_m and y_{m+1} both belong to right cells and an analogous argument holds for the corner cells since no corner cell is plus adjacent with an interior cell. From Figure 11 we see that the edge $(y_m, y_{m+1}) \in \Gamma_W$ intersects some edge (u, v) that cuts the right side of R as in Figure 11. The vertex $u \in G$ is therefore contained in the interior of the dual cell containing (y_m, y_{m+1}) (property $(a5)$) and so by definition of the component $\mathcal{E}_{tot}$, there exists an open path P from u to some vertex a_k of the edge e_k that intersects the left side of R. The concatenation $(e_k, P, (u, v))$ would then form an open left right crossing of R, a contradiction.

Finally by the nicely padded assumption there must exist at least two edges in Γ_W and so the subpath $(y_1, \ldots, y_p) \subset \partial_W$ is a dual top bottom crossing of R. It remains to see that each such edge is closed. Suppose not and the edge $(y_i, y_{i+1}) \in \Gamma_W$ is open. By the interior edge property $(a2)$, there exists exactly one edge $(v_1, v_2) \in G$ that intersects (y_i, y_{i+1}). By construction, one of the vertices, say v_1 belongs to the component $\mathcal{E}_{tot}$ and so there is an open path from v_1 to some end-vertex a_k of the edge e_k intersecting the left side of R.

Since (y_i, y_{i+1}) is open, the edge (v_1, v_2) is open as well and so v_2 also belongs to $\mathcal{E}_{tot}$. But if $R(v_1)$ and $R(v_2)$ denote the dual cells containing v_1 and v_2, respectively, then from property $(a4)$ the cells $R(v_1)$ and $R(v_2)$ are plus adjacent and share the edge (y_i, y_{i+1}). This implies that (y_i, y_{i+1}) is present in the interior of the cycle formed by merging $R(v_1)$ and $R(v_2)$ and consequently (y_i, y_{i+1}) must be present in the interior of the outermost boundary cycle ∂_W as well, a contradiction. $\square$

Acknowledgments

I thank Professors Rahul Roy, Thomas Mountford, Federico Camia, C. R. Subramanian and the referee for crucial comments that led to an improvement of the paper. I also thank IMSc for my fellowships.

References

[1] B. Bollobás, Modern Graph Theory, Springer, 1998.
[2] B. Bollobás and O. Riordan, Percolation, Academic Press, 2006.
[3] G. Grimmett, Percolation, Springer Verlag, 1999.
[4] H. Kesten, The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$, Communications in Mathematical Physics, 74 (1980) 41–59.
[5] M. Penrose, Random Geometric Graphs, Oxford, 2003.
[6] A. Timár, Boundary-Connectivity via Graph Theory, Proceedings of the American Mathematical Society, 141 (2013), 475–480.