Modeling of Random Variables on Fuzzy Intervals of Their Values

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Abstract
An approach to modeling random variables on fuzzy intervals of their values is proposed. The approach includes two stages. At the first stage, a triangular or trapezoidal fuzzy number is built on the basis of a fuzzy linguistic evaluation of the boundaries of the values of a random variable, the fuzzy coefficients of which determine the boundaries of the interval of values of a fuzzy variable. Such numbers are constructed using the Gaussian membership function. At the second stage, Monte Carlo simulation is carried out using Gaussian membership functions and beta distribution. In addition, declaring a random parameter by a fuzzy number makes it possible not only to determine the interval of its possible values when modeling a random variable, but also to use this parameter in fuzzy arithmetic.

Keywords [1]
random variable, fuzzy set, membership function, linguistic assessment, fuzzy interval, modeling, beta distribution, Monte Carlo method.

1. Introduction

Modern real systems and processes to one degree or another have development in time, therefore, they are stochastic. This means that the characteristics that describe their functioning are probabilistic and are random variables. The values of these quantities, as a rule, are in a certain interval, which sometimes has clearly defined boundaries, and more often - the boundaries are indefinite, vague. For example, such boundaries are inherent in parameters that are predictive in nature. Moreover, the more distant in time the forecasting horizon, the less its accurate, i.e. accuracy of estimates of the boundaries of possible values of such parameters. Therefore, in such conditions, the use of fuzzy intervals is preferable. Declaring model parameters in the form of a fuzzy interval is a convenient form for formalizing imprecise values. It is psychologically easy to give a fuzzy interval estimation, and the carrier of a fuzzy interval is guaranteed to contain the value of the parameter under consideration. Recently, fuzzy modeling has become one of the most active and promising areas of applied research in various fields [1-4]. In fuzzy modeling, to represent fuzzy sets, fuzzy values are most often used, which are the basis for constructing mathematical models using linguistic variables. Fuzzy Monte Carlo Simulation (FMCS) [5-9] is widely used in stochastic fuzzy models for modeling random variables. The main point of the FMCS considered in these works is the representation of parameters and variables only by triangular fuzzy numbers. However, in practice, the intervals of possible values of a random variable are often known. In this case, such parameters are given by trapezoidal fuzzy numbers. In article [10], a mechanism for fuzzy modeling of random variables by the Monte Carlo method based on the Gaussian membership function is proposed. This article is a development of these studies. It discloses a method for modeling random variables, the value intervals of of which are given in a fuzzy linguistic form. In this case, both the Gaussian membership function and the beta distribution are used.

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2. Representation of fuzzy linguistic evalutions by fuzzy values

As noted, under conditions of uncertainty, it is easier to specify the range of values of a random variable by a fuzzy linguistic evaluation. Fuzzy linguistic evaluation is understood as a numerical score, which is expressed using the modalities " approximately / near".

The intervals of possible values of random variables are determined by their physical content. If the random variable is, for example, the *diameter of the machined part*, then its deviation from the specified value under the influence of various factors may be insignificant. That is, the values of this random variable are near the norm. If we consider the random variable *project implementation time*, then in this case there is a large uncertainty and it is most plausible to assert that the possible values of the project implementation time are approximately in a certain interval. With this in mind, we will use two types of fuzzy linguistic evaluations " *the value is near c* " or " *the value is approximately in the range from c to d* " and represent them as fuzzy values. From a linguistic point of view, a fuzzy value is an imprecise, indefinite numerical value of a certain parameter of a model, which is the result of its evaluation in the absence of complete and accurate information. Fuzzy variables include fuzzy numbers and fuzzy intervals. To declare such values when solving practical problems, several approaches can be used [11,13]. In the case of fuzzy modeling, an approach using the standard and combined (double) Gaussian membership function (MF) [14] (Fig. 1) was applied.

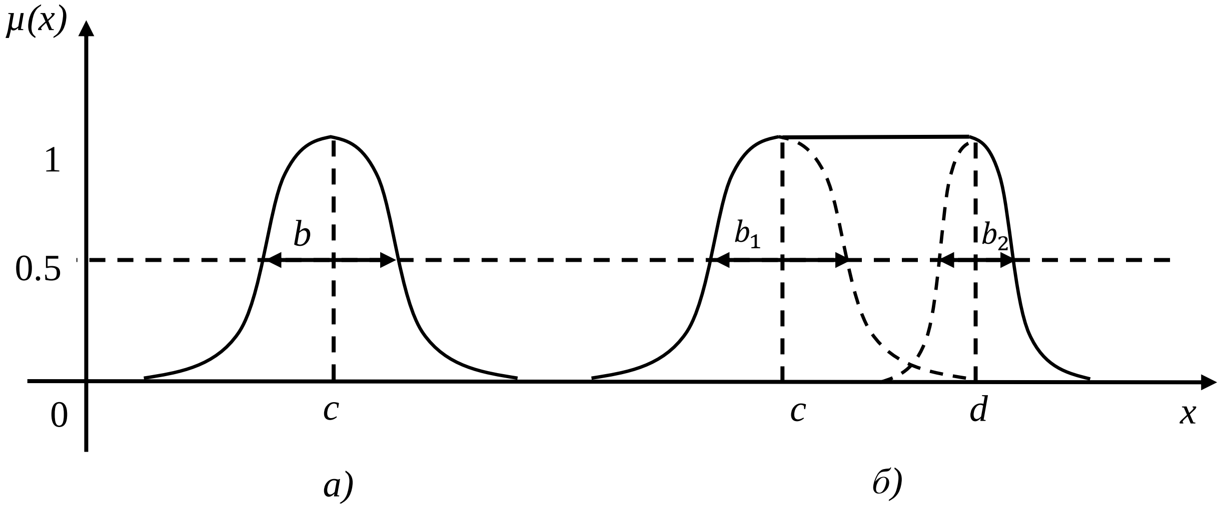

Figure 1: *Gaussian membership functions: a) - standard; b) - combined (double)*

The standard Gaussian function is used to define fuzzy sets $\tilde{C} \triangleq$ "the number is near c". We will use the Gaussian function of the form:

$$\mu_{\tilde{C}}(x) = \exp(-a(x-c)^2), \tag{1}$$

where $\alpha = -\frac{4ln0.5}{b^2(c)}$, and $b(c)$ is the distance between the transition points.

The combined function describes fuzzy set $\tilde{A} \triangleq$"the number is approximately in the range from c to d". This function has the form:

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_{\tilde{C}}(x), & x < c \\ 1, & c \leq x \leq d \\ \mu_{\tilde{D}}(x), & x > d \end{cases} \tag{2}$$

where $\mu_{\tilde{C}}(x)$ is the membership function of the fuzzy set $\tilde{C} \triangleq$"the number is near c", and $\mu_{\tilde{D}}(x)$ is the membership function of the fuzzy set $\tilde{D} \triangleq$ "the number is near d". These functions are built in a similar way.

The Gaussian function has an unbounded support, since it tends to zero asymptotically on the left and right. However, in practice, the carrier of this function can be considered limited by points $x = c \pm 3\sigma$, at which its value is approximately equal to 0.01. Therefore, it can be assumed that the value of the function equal to 0.01 corresponds to the complete non-belonging of the element to the fuzzy set $\tilde{A}$. If we go from σ to $b(c)$ then the boundaries of this interval will be equal to $c \pm \frac{k \cdot b(c)}{2}$, where $k \approx 2.5$ is the scaling factor [10]. These boundaries are the coefficients α and β of the triangular fuzzy number $M_1 = (c, \alpha, \beta)$.

The fuzzy interval, described by this function, is constructed in a similar way. In this case, the indistinctness coefficients will be equal to $\alpha = c - \frac{k \cdot b(c)}{2}$ and $\beta = d + \frac{k \cdot b(d)}{2}$. As a result, we get trapezoidal fuzzy interval $M_2 = (c, d, \alpha, \beta)$.

Thus, depending on the type of fuzzy linguistic evaluation of the interval of possible values of a random variable, its boundaries will be the fuzzy coefficients α and β, accordingly, a fuzzy triangular number $M_1 = (c, \alpha, \beta)$ or a trapezoidal number $M_2 = (c, d, \alpha, \beta)$.

3. Harmonization of linguistic evaluations

When determining the range of values of random parameters, the problem of objectivity (reliability) of their evaluations may arise. This is especially true for parameters that are predictive in nature. In this case, in order to increase the reliability of the evaluations of such parameters, a group examination is carried out. The results of the examination are considered to be reliable if there is good agreement in the evaluations of experts. The issues of harmonization of evaluations of group expertise were considered in many studies [15-18], among which the article [18] can be highlighted. In this paper, a mechanism for harmonization interval evaluations is presented. In this case, the coefficient of variation is used as a measure of the consistency of evaluations. This coefficient is determined separately for the left and right boundaries of the intervals by the formula $V = s/\bar{x}$, where s is the sample standard deviation of the evaluations; $\bar{x}$ - their average value.

Let (c_1, d_1),…, (c_k, d_k) be evaluations of values c and d of interval linguistic evaluation of some random parameter, which are given by k experts. Then the coefficients of variation of the boundaries of the corresponding intervals are determined as follows:

for left borders by the formula

$$V_L = s_L/\bar{x}_L, \tag{3}$$

where

$$s_L = \sqrt{\frac{1}{k-1}\sum_{j=1}^{k}(c_j - \bar{x}_L)^2 r_j}, \ \bar{x}_L = \sum_{j=1}^{k} c_j r_j, \tag{4}$$

for right borders by the formula

$$V_R = s_R/\bar{x}_R, \tag{5}$$

where

$$s_R = \sqrt{\frac{1}{k-1}\sum_{j=1}^{k}(d_j - \bar{x}_R)^2 r_j}, \ \bar{x}_R = \sum_{j=1}^{k} d_j r_j, \tag{6}$$

Here r_{ij} is the weighting coefficient of the j-ith expert, moreover $\sum_{j=1}^{k} r_j = 1$.

The practice of applying the methods of expert evaluations shows that the results of the examination can be considered satisfactory, if $0{,}2 \le V \le 0{,}3$, and good, if $V < 0{,}2$. These conditions can be used as a criterion for the consistency of estimates and the basis for their specification. This approach can also be used for fuzzy point evaluation. In this case, the coefficient of variation of the point value is used.

4. Calculation of the distance between the transition points

When constructing a Gaussian function, the distance between the transition points is determined mainly by an expert. At the same time, the task of measuring such a distance is complicated by the fact that a person, as a rule, has a lack of confidence in the accuracy of his evaluation. Therefore, a more constructive approach is that excludes the conduct of such examinations [14]. This algorithm is based on experimental data, which, according to experts, reflect the transition points for numbers approximately equal to T. Based on this data, formulas were obtained to calculate the distance between the transition points for each number $T \in [1, 99]$. The results are shown in Table 1.

Table 1
The distance between the transition points

Number x	The distance between the transition points $b(x)$
1, 2, 3, 4, 6, 7, 8, 9	$0{,}46\,x$
10, 20, 30, 40, 60, 70, 80, 90	$(0{,}357 - 0{,}00163x)x$
35, 45, 55, 65, 75, 85, 95	$(0{,}213\ -\ 0{,}00067x)x$
5	2,8
15	6,48
25	6,75
50	24
Other two-digit numbers	$\frac{1}{2}(b\left(\left[\frac{x}{10}\right]\cdot 10+5\right)+b(x-\left[\frac{x}{10}\right]\cdot 10))$

For the purpose of a holistic perception of the material, we present the main provisions of this approach. Let a fuzzy linguistic quantity "the number is near the number T" be given, where T is a natural number. If $T \in [1, 99]$, then $b(T)$ could be found according to the Table 1. Otherwise, the following algorithm is used. Let the least significant digit of T has an order of q. We divide the possible values of q into the residue classes modulo 3. As a result, we obtain three classes M_d, $d \in \{0, 1, 2\}$, where $d = q \bmod 3$. In this case the value $b(T)$ also depends on the class M_d, to which the number T belongs. Let r_q be the numeral that is in the q th place of the number T. Then:

1. If $T \in M_0$ (for example, 300, 300000 etc.), then $b(T) = b(x) \cdot 10^{q-2}$, where $x = r_q \cdot 10$ and $b(x)$ is taken from Table 1.
2. If $T \in M_1$ (for example, 101, 202000, 15000 etc.), then two options are possible:
   a) if $r_{q+1} = 0$, then $b(T) = b(x) \cdot 10^{q-1}$, where $x = r_q$;
   b) if $r_{q+1} \neq 0$, then $b(T) = b(x) \cdot 10^{q-1}$ where $x = r_{q+1} \cdot 10 + r_q$.
3. If $T \in M_2$ (for example, 2030, 2140 etc.), then two options are also possible:
   a) if $r_{q+1} = 0$, then $x = r_q \cdot 10$; $b(T) = b(x) \cdot 10^{q-2}$;
   b) if $r_{q+1} \neq 0$, then $x = r_{q+1} \cdot 10 + r_q$; $b(T) = b(x) \cdot 10^{q-1}$;

As a result, the value $b(T)$ will be obtained.

This algorithm can be used in the case when T is expressed as a decimal fraction. In this case, the algorithm is applied to the mantissa of the fraction, and then its order is taken into account.

5. Modeling of random variables

Modeling of a random variable using the Monte Carlo method involves drawing a specific value of the random variable. Consider two ways of drawing: based on the Gaussian membership function (1) and in accordance with a given distribution law.

Draw based on the Gaussian function. Consider two cases.

Case 1. Let a random variable X be given a fuzzy linguistic estimate "value is near c". The membership function $\mu_{\tilde{C}}(x)$ of the fuzzy set $\tilde{C} \triangleq$"the number is near c" and the corresponding fuzzy number $M = (c, \alpha, \beta)$ are constructed, which sets the boundaries α and β of the interval of possible values of the X.

Further, the evaluation "*value near c*" assumes that the random variable X in the interval $[\alpha, \beta]$ is distributed according to the normal law. Therefore, the membership function $\mu_{\tilde{C}}(x)$ can be considered as a density function with mathematical expectation c and variance $b^2(c)$. And since $\max \mu_{\tilde{C}}(x) = 1$, then this function will be used in the drawing of a random variable X.

Let r is a random number. If $r\epsilon[0.01,0.5]$, then the value of X is the root of the equation $\mu_{\tilde{C}}(x) = 2r$ that is in the interval$[\alpha, c]$. If $r\epsilon(0.5,0.99]$, then the root of the equation $\mu_{\tilde{C}}(x) = 2(1-r)$ is taken that belongs to the interval $[c, \beta]$.

Case 2. Let a random variable *X* be given a fuzzy linguistic estimate "the value is approximately in the range from c to *d*". First, the membership functions $\mu_{\tilde{C}}(x)$ and $\mu_{\tilde{D}}(x)$ of the fuzzy sets $\tilde{C} \triangleq$"the number is near c" and $\tilde{D} \triangleq$ "the number is near d" are constructed, as well as the corresponding fuzzy numbers $M_1 = (c, \alpha_1, \beta_1)$ and $M_2 = (d, \alpha_2, \beta_2)$. As a result, we get the following fuzzy trapezoidal number $M = (c, d, \alpha_1, \beta_2)$. In this case, the possible values of *X* will belong to the interval $[\alpha_1, \beta_2]$.

Then the value of the middle of the tolerance interval $b = \frac{c+d}{2}$ is calculated and the membership function $\mu_{\tilde{B}}(x)$ is constructed (Fig. 2).

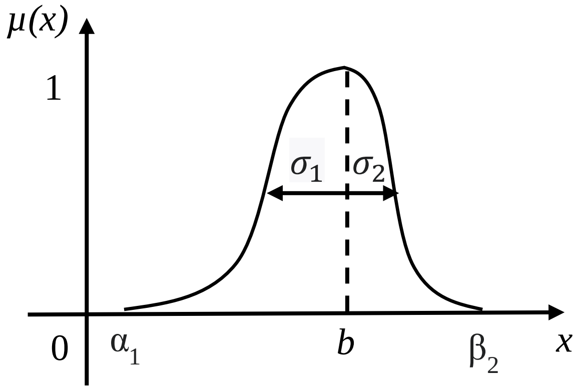

Figure 2: Membership function $\mu_{\tilde{B}}(x)$.

This function is defined by the expression:

$$\mu_{\tilde{B}}(x) = \begin{cases} \mu_{\tilde{B}}^1, x\epsilon[\alpha_1, b] \\ \mu_{\tilde{B}}^2, x\epsilon\{b, \beta_2] \end{cases}, \tag{7}$$

where $\mu_{\tilde{B}}^1 = e^{-\frac{(x-b)^2}{2\sigma_1^2}}$ and $\mu_{\tilde{B}}^2 = e^{-\frac{(x-b)^2}{2\sigma_2^2}}$ are Gaussian functions. In these functions, the parameters σ_1 and σ_2 are found from the equations $e^{-\frac{(\alpha_1-b)^2}{2\sigma_1^2}} = 0.01$ and $e^{-\frac{(\beta_2-b)^2}{2\sigma_2^2}} = 0.01$.

The function $\mu_{\tilde{B}}(x)$ constructed in this way will be used to model the random variable *X* on an interval $[\alpha_1, \beta_2]$. Then, as in case 1, if $r\epsilon[0.01,0.5]$, then the value of *X* is the root of the equation $\mu_{\tilde{B}}^1(x) = 2r$, which is in the interval $[\alpha_1, b]$, and if $r\epsilon(0.5,0.99]$, then the root of the equation $\mu_{\tilde{B}}^2(x) = 2(1-r)$, which belongs to the interval $[b, \beta_2]$.

The drawing in accordance with distribution law. As noted, the result of a fuzzy linguistic evaluation of variable *X* is approximated by the interval of its possible values. To describe random variables, the values of which are limited to a finite interval, beta distribution is mainly used [19]. The beta distribution is parameterized by two positive parameters α and β, which determine its shape. Due to the fact that the beta distribution can have a different shape, practically all the applied probability distributions can be expressed in terms of this distribution.

The standard beta distribution over the interval $x\epsilon[0,1]$ is given by the density function:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{8}$$

where $B(\alpha, \beta) = \int_0^1 y^{\alpha-1} * (1-y)^{\beta-1}dy$ - Euler's beta function.

In this case, the distribution function is expressed through the incomplete beta function:

$$(x) = \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1} * (1-y)^{\beta-1}dy,$$

moreover this function is tabulated [20].

In practice, of greater interest are, as a rule, beta values determined in an arbitrary interval $[a, b]$. Taking into account that the function $F(x)$ is tabulated, the drawing of random variables on the interval $[a, b]$ will be carried out by the Neumann elimination method [21]. This method is based on the following theorem.

Let the random variable X be defined on an interval $[a, b]$ and has an upper bounded density function *f(x)*. Let also $r_1, r_2 \epsilon [0,1]$ be independent implementations of the base random variable ξ, also

$$x = a + (b - a)r_1 \text{ and } y = Mr_2$$

where $M = \max_{a \le x \le b} f(x)$.

Then if $y < f(x)$, then the value *x* is the realization of the random variable *X*. At the same time, the effectiveness of the elimination method is directly proportional to the probability of fulfilling the condition $y < f(x)$, i.e.

$$P\{y < f(x)\} = [M(b - a)]^{-1}. \quad (9)$$

This probability allows for the desired number of realizations of a random variable *X* to determine the number of necessary model runs. The main advantage of this method is its versatility, i.e. applicability for generating random variables having any computable or tabular probability density.To better understand such calculations, consider an example. Let us draw a random value $X \triangleq$"project implementation time". First, the range of its values is determined. Since the random variable has a predictive nature, therefore, in order to obtain more reliable values of the boundaries of the region, a group of, say, three equivalent experts is involved, whose evaluations are given in Table 2.

Table 2

Evaluations of intervals in days

The project implementation time is approximately in the interval	
expert 1	from 87 to 123
expert 2	from 90 to 134
expert 3	from 93 to 145

Using formulas (3) - (6), for the boundaries of the given intervals, we obtain the coefficients of variation, respectively, 0.03 and 0.08. Since the experts' assessments are reasonably well agreed, there is no need to refine them. Therefore, the average values of 90 and 134 are taken as the boundaries of the range of values of the random variable *X*. As a result, we obtain the collective estimate "the time of the project implementation is approximately in the range from 90 to 134".

Then, the membership functions $\mu_{\tilde{C}}(x)$ and $\mu_{\tilde{D}}(x)$ of both fuzzy sets $\tilde{C} \triangleq$"the number is near the number 90" and $\tilde{D} \triangleq$ "the number is near the number 134" are constructed. To construct the function $\mu_{\tilde{C}}(x)$, it is necessary to calculate the distance $b(90)$. This value is found according to Table 1 and is equal to $b(90) \approx 32$. As a result, we get a function $\mu_{\tilde{C}}(x) = e^{-\frac{4ln0.5((x-90)^2}{32^2}}$ and a fuzzy number M_1 = (90,50,130). The value $b(134)$ for the function $\mu_{\tilde{D}}(x)$ is calculated by the above algorithm. The least significant digit of number 134 is in the ones place (q = 1), therefore $r_q = r_1 = 7$, $r_{q+1} = r_2 = 3$ is a digit whose order is one higher than the order of the least significant digit of number 134. When dividing q by 3 in the remainder, we get 1, therefore, the number 134 belongs to the equivalence class M_1, so $d = 1$.

Since $r_{q+1} \neq 0$, then, according to clause 2b of this algorhythm, we have $x = r_{q+1} * 10 + r_q = r_2 * 10 + r_1 = 34$ and $b(134) = b(34)$, and $b(34)$ is calculated by the formula

$$b(37) = \frac{1}{2}\left(b\left(\left[\frac{34}{10}\right] * 10 + 5\right) + b\left(34 - \left[\frac{34}{10}\right] * 10\right)\right) = \frac{1}{2}\big(b(35) + b(4)\big),$$

in which $b(35)$ and $b(35)$ are found in Table 1: $b(35) = 6.63$ and $b(4) = 1.84$. Then $b(134) = \frac{1}{2}(6.63 + 1.84) \approx 4$. As a result, we get $\mu_{\tilde{D}}(x) = e^{-\frac{4ln0.5((x-134)^2}{4^2}}$ the corresponding fuzzy number $M_2 = (134{,}129{,}139)$. Then the interval of possible values of the random variable *X* will be the interval [50,139]. We will model the values of a random variable *X* using two methods: based on the Gaussian function and beta distribution. According to the first method, we determine the value $b = \frac{90+134}{2} = 112$ and build the membership function (7):

$$\mu_{\tilde{B}}(x) = \begin{cases} \mu_{\tilde{B}}^1, x\epsilon[50{,}112] \\ \mu_{\tilde{B}}^2, x\epsilon\{112{,}139] \end{cases},$$

Where $\mu_{\tilde{B}}^1 = e^{-\frac{(x-112)^2}{2*21^2}}$ and $\mu_{\tilde{B}}^2 = e^{-\frac{(x-112)^2}{2*9^2}}$. Then, in the process of modeling, if $r\epsilon[0.01{,}0.5]$, then the value of *X* is the root of the equation $\mu_{\tilde{B}}^1(x) = 2r$, which is in the interval [50,112], and if $r\epsilon(0.5{,}0.99]$, then the root of the equation $\mu_{\tilde{B}}^2(x) = 2(1-r)$, which belongs to the interval [112,139]. For example, if $r = 0.4$, then the value *x* is equal to 98, if $r = 0.7$, then it is equal to 118. Let us now consider modeling a random variable *X* using the beta distribution, for example, with parameters $\alpha = 2, \beta = 3$. In this case $B(2{,}3) = \frac{1}{12}$. Since the value *X* is determined on interval [50, 139], the density function (8) must be scaled. In the general case, for the interval *[a,b]* this function has the form

$$f(t) = \begin{cases} \frac{12}{(b-a)^4} * (t-a)(b-t)^2, if\ a \le t \le b \\ 0, otherwise \end{cases}.$$

For the interval [50, 139] density function has the form

$$f(t) = \begin{cases} \frac{12}{89^4} * (t-50)(139-t)^2, if\ 50 \le t \le 139 \\ 0, otherwise \end{cases}$$

and takes maximum value $M \approx 0.02$.

Let the random numbers $r_{1i} = 0.4$ and $r_{2i} = 0.7$ be obtained by different generators (independence condition). These numbers are scaled to the interval [50, 139] and [0, 0.02]: $x_i = 50 + 89 * 0.4 = 85.6$ and $y_i = 1.78 * 0.7 = 0.014$. Then $f(x_i) = f(85.6) \approx 0.019$ is calculated. Since the condition $y_i \le f(x_i)$ is met, the value $x_i = 85.6$ is taken as a realization of the random variable *X*. Otherwise, this value is discarded. In this case, the efficiency of modeling by the elimination method on the interval [50, 139], according to (9), is directly proportional to the probability 0.56. That is, to obtain, for example, 1000 realizations of a random variable, it is necessary to carry out approximately 1800 runs of the model.

6. Conclusion

An approach to modeling random variables on fuzzy intervals of their values is proposed. This approach includes two stages. At the first stage, on the basis of a fuzzy linguistic evaluation of a random parameter, a fuzzy number is constructed, which declares a fuzzy interval of its possible values. Fuzzy linguistic evaluations can be either point or interval. Depending on the type of evaluation, a triangular or trapezoidal fuzzy number is constructed, the fuzzy coefficients of which determine the range boundaries of values of the fuzzy variable. Such numbers are constructed using the Gaussian membership function. At the second stage, a random variable is modeled on the constructed interval of its values. The drawing is performed by the Monte Carlo method using Gaussian membership functions and beta distribution. In this case, the drawing of a random variable by the beta distribution function is carried out by the Neumann method. Note that the representation of a random parameter as a fuzzy number allows not only to determine the interval of its possible values, but also to use this parameter in calculations in the process of fuzzy modeling.

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