Equation of the Related Riemann-Hilbert-Privalov Problem with Zeros and Poles of the Coefficient in the Half-plane*

Tatyana G. Voytik (0000-0002-1126-9860)[1], Gennadij S. Poletaev (0000-0002-1989-808X)[2], and Angela A. Stehun (0000-0003-3140-2689)[2(*)]

[1] Odessa National Maritime University, Odessa, Ukraine
beauty5@i.ua

[2] Odessa State Academy of Buildings and Architecture, Odessa, Ukraine
poletayev_gs@ukr.net, angela.stehun@gmail.com

Abstract. Various mathematics and mechanics questions are connected with the Riemann (Riemann-Hilbert-Privalov) problem from the theory of analytic functions. This justifies the relevance of the equation associated with the specified Riemann problem. Until now, the specificity of the location of the singular points of the rational coefficient of the equation has not been considered. The paper aims to obtain general formulas for solving abstract equations in the corresponding ring of rational functions, considering this specificity. The authors propose a way to solve an equation and a related problem. For the first time, the sought general formulas were established and proved to be simpler than those obtained earlier. The research indicates the signs when the use of these formulas is justified. The authors apply the general provisions of (1) the theory of rings and functional analysis, (2) the theory of linear operators, (3) equations in rings with factorization pairs, (4) the problem of factorization by subrings, and (5) the properties of the used ring of rational functions. The authors' approach is free from the theory of integrals (Cauchy and Fourier types), the requirements of the Hölder condition, and the index. In a similar situation, it applies to (1) integral equations of the Wiener-Hopf type and other equations of the convolution type, as well as to related functional equations; (2) to matrix equations that can be used in mechanics with projectors onto the corresponding subrings and unknown triangular matrices.

Keywords: Riemann problem · Equation · Ring · Projector · Factorization pair

1 Introduction

1.1 Relevance, Problem Statement

The equations and problems considered in the paper are related to the corresponding famous Riemann problem (Riemann-Hilbert, Riemann-Hilbert-Privalov) from the theory of analytic functions, but for a narrower class of functions. This problem is important in the theory of differential and integro-differential equations, integral equations of convolution type, the corresponding differential equations of mathematical physics, in the theory of elasticity, torsion problems, and others (Akopyan & Dashtoyan, 2013; Cherskij, P. Kerekesha & D. Kerekesha, 2010; Gahov, 1963; Gahov & Cherskij, 1978; Krejn, 1958; Mhitaryan, 1968; Mushelishvili, 1968; Popov, Kerekesha & Kruglov, 1976; Voytik, Poletaev & Yatsenko, 2017). Therefore, the authors develop (1) general provisions on the solvability of equations associated with it or related problems; (2) ideas about the possible results of the theory; (3) a way of considering the specifics. The research analyzes the solvability, criteria, and formulas for solutions of an equation in the corresponding ring of rational functions:

$$A(z)X^{+}(z) + Y_{-}(z) = B(z); \quad z \in \mathbb{C} \cup \{\infty\}. \tag{1}$$

This equation for $z = x \in \{-\infty, \infty\}$ expresses the boundary conditions of the following, called related, problem:

For given rational coefficient functions $A(x)$, $B(x)$, $-\infty < x < \infty$ *find a pair of rational functions* $X^{+}(z) \in \mathfrak{R}_{r}^{+}$, $Y_{-}(z) \in \mathfrak{R}_{r-}$, $z \in \mathbb{C}$. *All poles of the first one (if they exist) are located only inside the lower one; the second one is located only inside the upper one of the half-plane bounded by the closed real axis and satisfying the linear equation on this axis:*

$$A(x)X^{+}(x) + Y_{-}(x) = B(x); \quad x \in \{-\infty; \infty\}.\text{”} \tag{2}$$

The closed real axis here is the contour [1]. The authors consider the case when all zeros and poles of the coefficient $A(z)$ (if they exist) are located simultaneously in the lower or upper half-plane of the extended complex plane.

1.2 Analysis of Research and Publications

The study of the Riemann-Hilbert problem's solvability by exact methods goes back to the works of I. I. Privalov, F. D. Gahov, Yu. I. Cherskij, M. G. Krejn, and others (Gahov & Cherskij, 1978; Krejn, 1958). The connection between the theory of integral equations of Wiener-Hopf type and this problem was first noted by I. M. Rapoport (1948, 1949). Using the provisions of M. G. Krejn (1958, p. 114) concerning the N. I. Mushelishvili book (1968), the authors conclude that this problem was solved under the assumption that the additional Hölder condition for functions on a contour is fulfilled. The apparatus of the Cauchy integral theorem, an index, was often used. This can lead to the need to overcome serious analytical difficulties, sometimes unjustified. Information of history elements to the Riemann problem, integral equations of convolution type, is in the works of F. D. Gahov, Yu. I. Cherskij, M. G. Krejn, and others (Gahov, 1963; Gahov & Cherskij, 1978; Krejn, 1958). Here arises the question about easing restrictions. In M. G. Krejn work (1958), new ideas and an indication of possible approaches to research based on the theory of Banach algebras, under different assumptions and without the requirement that the functions be Hölder, appeared for the first time. Related to the Riemann-

Hilbert problem, integral equations of convolution type, the study of abstract equations with an arbitrary right-hand side in associative rings with a special pair of subrings, and their realizations in concrete rings, the results of individual works by G. S. Poletaev (1973–2020) and in co-authorship (Poletaev, 1988; Voytik et al., 2017). The works by V. N. Akopyan & L. L. Dashtoyan (2013); A. McNabb & A. Schumitzky (1972); and G. Ya. Popov, P. V. Kerekesha & V. E. Kruglov (1976) confirmed the relevance of a range of issues close to the Riemann-Hilbert problem. Along with others, the case is important when in this type of problems and related, the coefficients are rational functions (Gahov, 1963; Mushelishvili, 1968; Popov et al., 1976). However, it does not consider the specifics of the case when the zeros and poles of the coefficient of a related problem and the equation under consideration are simultaneously located in one of the half-planes of the extended complex plane. Therefore, the search for solutions to equations (1) in the considered setting is relevant.

1.3 Research Objective

The research aims to establish the form of general formulas for solutions of abstract equations (1), considering the specifics of assumptions in cases where the coefficient $A(z)$ is a function of the considered ring of rational ones. Moreover, a function, all zeros, and poles of which (if they exist) are finite, immaterial, and are concentrated simultaneously either inside the upper half-plane or the lower half-plane of the extended complex plane. The aim can be achieved using the corresponding statements and theorems, decision formulas established in the works of the abovelisted authors (Krejn, 1958; McNabb & Schumitzky, 1972; Poletaev, 1988; Voytik et al., 2017).

2 Materials and Methods

2.1 Designations, Definitions, and General Provisions

The current research uses designations and statements from the works of the abovementioned authors. The symbol R denotes an arbitrary commutative and associative ring with unit e. Let p^+, p^- - be the commuting projectors $R \to R$. Therefore:

$$p^0 := p^+ p^- (= p^- p^+), \quad p_{\mp} := p^{\mp} - p^0, \quad p_* := p_+ + p_-.$$

For any subset $B \subseteq R$ let us denote $B^{\mp,\,0} := p^{\mp,\,0} B$; $B_{\mp} := p_{\mp} B$; $B^* = B^+ + B^-$; $B_* = B_+ + B_-$. For any element $x \in R$, we assume:

$$x^{\mp,\,0} := p^{\mp,\,0} x, \quad x_{\mp} := p_{\mp} x, \quad x_* := p_* x.$$

Definition 1. A pair of subrings (R^+, R^-) of a ring R with unit e will be called a factorization pair [*FP*], if it is generated in force in R commuting projectors p^+, p^-:$R^{\mp} = p^{\mp}(R)$, and the following axioms (McNabb & Schumitzky, 1972):

$$e \in R^0 \; \left(= R^{\mp} \cap R^{\pm}\right); \qquad (*)$$

$p^0(= p^{\mp}p^{\pm})$ *- is the ring homomorphism* R^+*and* R^- *in* R^0; (**)

$$R^+R^-, \quad R^-R^+ \subseteq R^+ + R^- \quad (:= R^*). \tag{***}$$

Definition 2. Any ring R with unit $a_1^{'}$, considered together with a fixed *FP* of the subrings $a_2^{'}$ R, in other words, subrings with axiomatically specified properties (*), (**), (***), will be called a "ring with a factorization pair" or, in short - a ring with *FP*.

2.2 Factorization of an Element of a ring by its Factorization Pair

Let us say that the element $a \in R$ admits factorization in the commutative ring R by the factorization pair $a_2' = t_2^- s_2^0 r_2^+$ (– by the *FP* $a_2' = t_2^- s_2^0 r_2^+$), if there are elements $r^{\ +} \in R^+$, $s^0 \in R^0$, $t^- \in R^-$ such that: $a = r^+ s^0 t^-$. This factorization is called, namely:

1. Correct factorization факторизацией [*CF*], if $r^{\ +} \in R^+, s^0 \in R^0, t^- \in R^-$ - are invertible in their subrings, respectively;
2. Normalized factorization [*NF*], if $t^0 = r^0 = e$;
3. Normalized regular factorization [*NRF*] if it is *CF* and $t^0 = r^0 = e$. The correct factorization of an element from R^- by the *FP* $t_1^-{}' s_1^0{}' r_1^+{}' x_+ r_2^+{}' s_2^0{}' t_2^-{}' = b_+ + q^-$ can be normalized (McNabb & Schumitzky, 1972; Poletaev, 1988). There is only one normalized correct factorization.

2.3 Ring $\mathfrak{R}_r$ with Factorization Pair ($\mathfrak{R}_r^+, \mathfrak{R}_r^-$)

Further, we will use definitions, designations, and provisions from work by T. G. Voytik, G. S. Poletaev, & S. A. Yatsenko (2017). By $\mathfrak{R}_r$, we denote the collection of all rational functions of a complex variable $z \in \mathbb{C}$, all whose poles (if they exist) are finite and immaterial. The limits of functions from $\mathfrak{R}_r$ at infinity exist and are finite. Let $\mathfrak{R}_r^+$ ($\mathfrak{R}_r^-$) – be subsets of functions from $\mathfrak{R}_r$, all poles of which (if they exist) are located inside the lower (upper) half-plane $H_-(H_+)$, respectively (Krejn, 1958, pp.14–15). It can be shown that $\mathfrak{R}_r^{\pm,0} = p^{\pm,0}(\mathfrak{R}_r)$, where $p^0 := p^+p^-(= p^-p^+)$, $\mathfrak{R}_r^0 = \mathfrak{R}_r^+ \ \cap \mathfrak{R}_r^-$,
$\mathfrak{R}_r$ - ring with *FP* ($\mathfrak{R}_r^+, \mathfrak{R}_r^-$). For any function $A(z) \in \mathfrak{R}_r$ the decomposition is true: $A(z) := A^0 + A_+(z) + A_-(z)$.

Let us establish formulas for solutions for (1) considering corresponding to the research objective.

3 Results

3.1 Main Result

Results of individual works by G. S. Poletaev (1973–2020) and in co-authorship (Poletaev, 1988; Voytik et al., 2017) suggest that the following is true:

Theorem 1. Let the function $\boldsymbol{A(z)} \in \boldsymbol{\mathfrak{R}_r}$ have no real zeros and $\boldsymbol{\lim_{z\to\infty} A(z) = const \neq 0}$. If, in this case, the element inverse in $\boldsymbol{\mathfrak{R}_r}$ is the function $\boldsymbol{A^{-1}(z)}$ admits a normalized regular factorization: $\boldsymbol{A^{-1}(z) = G^{+}(z)\ S^0(z)\ T^{-}(z)}$, $\boldsymbol{z} \in \mathbb{C} \cup \{\infty\}$ concerning the factorization pair $(\boldsymbol{\mathfrak{R}_r^+, \mathfrak{R}_r^-})$, then the abstract equation (1) and the problem concerning $\boldsymbol{X^+(z)} \in \boldsymbol{\mathfrak{R}_r^+}, \boldsymbol{Y_-(z)} \in \boldsymbol{\mathfrak{R}_r^-}$, for any right-hand side of $B(z) \in \boldsymbol{\mathfrak{R}_r}$, в $\boldsymbol{\mathfrak{R}_r}$ are uniquely solvable. The formulas find their only solution:

$$X^+(z) = G^+(z)\ S^0\ [T^-(z)B^+(z)\ ]^+,$$
$$Y_-(z) = B_-(z) + (T^-(z))^{-1}[T^-(z)B^+(z)]_-\ , \quad z \in \mathbb{C} \cup \{\infty\}, \qquad (3)$$

where:

$S^0 := S^0(z) = const \in \mathbb{C}$.

Using *theorem 1*, let us establish a statement corresponding to the research objective.

Theorem 2. Let the function $A(z) \in \mathfrak{R}_r$ с $A^0 := p^0 A(z) = \lim_{z\to\infty} A(z) = = const \neq 0$ has no real zeros and all its zeros and poles (for existence) are located: 1) only inside the lower, or 2) only inside the upper, from the half-planes H_-, H_+ of the extended complex plane, bounded by a closed real axis. Then for it there exists in $\boldsymbol{\mathfrak{R}_r}$ the inverse - function $A^{-1}(z) \in \mathfrak{R}_r$ and there is a normalized correct factorization of $A^{-1}(z) = G^+(z)S^0(z)T^-(z), z \in \mathbb{C} \cup \{\infty\}$ by the factorization pair $(a_1, a_2 \in invR)$, for which: in the case 1):

$$G^{+}(z) = [A^{-1}(z)A^0]^+ = A^{-1}(z)A^0,$$
$$A^0 := [A(z)]^0, \quad S^0(z) = [A^0]^{-1} := [A^{-1}(z)]^0,$$
$$T^-(z) = 1, \quad z \in \mathbb{C} \cup \{\infty\};$$

and in case 2) it will be:

$$G^+(z) = 1\ , \ S^0(z) = [A^0]^{-1},$$
$$T^-(z) = [A^0 A^{-1}(z)]^- = A^0 A^{-1}(z), \quad z \in \mathbb{C} \cup \{\infty\}.$$

For arbitrary $B(z) \in \mathfrak{R}_r$ there is a unique solution to equation (1) and the problem. This solution in case 1) can be found according to following the formulas:

$$X^+(z) = A^{-1}(z)B^+(z), \ \ Y_-(z) = B_-(z), \ \ z \in \mathbb{C} \cup \{\infty\}; \qquad (4)$$

and in case 2) - according to the formulas:

$$X^+(z) = [A^{-1}(z)B^+(z)]^+,$$
$$Y_-(z) = B_-(z) + A(z)[A^{-1}(z)B^+(z)]_-, \ \ z \in \mathbb{C} \cup \{\infty\}. \qquad (5)$$

Proof. Under the conditions of *theorem 2*, for the coefficient $A(z) \in \mathfrak{R}_r$ there exists an inverse in $\mathfrak{R}_r$, and by virtue of the definitions of the subrings $\mathfrak{R}_r{}^+$, $\mathfrak{R}_r{}^-$ and the assumptions, in case 1) $A(z), A^{-1}(z) \in \mathfrak{R}_r^+$, and in case 2) $A(z),\ A^{-1}(z) \in \mathfrak{R}_r^-$. Therefore, there is a normalized correct factorization of $A^{-1}(z) = G^+(z)S^0(z)T^-(z)$, $z \in \mathbb{C} \cup \{\infty\}$ over the factorization pair $(a_1, a_2 \in invR)$, which will be: in the case 1) $G^+(z) = [A^{-1}(z)A^0)]^+ == A^{-1}(z)A^0$, $A^0 := [A(z)]^0$, $S^0(z) = [A^0]^{-1} = [A^{-1}(z)]^0$, $T^-(z) = 1$, $z \in \mathbb{C} \cup \{\infty\}$; and in case 2) $G^+(z) = 1, S^0(z) = [A^0]^{-1}$, $T^-(z) = [A^0 A^{-1}(z)]^- = A^0 A^{-1}(z)$, $z \in \mathbb{C} \cup \{\infty\}$. Therefore, by *theorem 1*, for an arbitrary right-hand side $B(z) \in \mathfrak{R}_r$ there is a unique solution in $\mathfrak{R}_r$ to equation (1) and *the problem.* It can be found by formulas (3), which, in case 1), give:

$$X^+(z) = A^{-1}(z)A^0\ [A^0]^{-1}\ [1 \cdot B^+(z)]^+\ = A^{-1}(z)\, B^+(z),$$
$$Y_-(z) = B_-(z) + (1)^{-1}[1 \cdot B^+(z)]_- = B_-(z).$$

The formulas (4) are valid. In case 2), from the same formulas (3), it follows that:

$$X^+(z) = 1 \cdot [A^0]^{-1}[A^0 A^{-1}(z)\, B^+(z)]^+ = [A^{-1}(z)\, B^+(z)]^+,$$
$$Y_-(z) = B_-(z) + [A^0]^{-1}A(z)[A^0 A^{-1}(z) B^+(z)]_- =$$
$$= B_-(z) + A(z)[A^{-1}(z)B^+(z)]_-\ ,$$

that is, formulas (5) are valid. The theorem is proved.

Consequence. Under the conditions of *theorem 2*, equation (1) and *the problem* with the right-hand side $B(z) = 1,\ \ B(z) \in \mathfrak{R}_r,\ z \in \mathbb{C} \cup \{\infty\}$ have in $\mathfrak{R}_r$ the only solution $X_e^+(z),\ \ Y_{-e}(z)$. In case 1) from *theorem 2*, it can be found by the formulas:

$$X_e^+(z) = A^{-1}(z),\ \ Y_{-e}(z) = 0,\ \ z \in \mathbb{C} \cup \{\infty\}, \tag{6}$$

and in case 2) - according to the following formulas:

$$X_e^+(z) = [A^{-1}(z)]^0 = [A^0]^{-1},\ \ Y_{-e}(z) = A(z)[A^{-1}(z)]_-,\ z \in \mathbb{C} \cup \{\infty\}. \tag{7}$$

The transformation of the second of equalities (7) can be represent in the as:

$$X_e^+(z) = [A^0(z)]^{-1},\ \ Y_{-e}(z) = 1 - A(z)[A^0]^{-1},\ z \in \mathbb{C} \cup \{\infty\}. \tag{8}$$

3.2 Example

Let us find a solution to an equation of the form (1) and *problem* in $\mathfrak{R}_r$ for:

$$A(z) = \frac{11z^2 - 33iz - 22}{3z^2 - 24iz - 45};\quad B(z) = \frac{5z + 5i}{z^2 + 22iz - 120}.$$

In this case,

$$A^0 = \frac{11}{3};\quad A(z) = \frac{11(z-i)(z-2i)}{3(z-3i)(z-5i)};\quad A^{-1}(z) = \frac{3(z-3i)(z-5i)}{11(z-i)(z-2i)};$$
$$B(z) = \frac{5(z+i)}{(z+10i)(z+12i)} = B^+(z) = B_+(z),$$
$$B^-(z) = 0 = B_-(z).$$

In this example, all zeros and poles of the function $A(z)$ are located in the upper half-plane. Thus, $A(z),\ \ A^{-1}(z) \in \mathfrak{R}_r^-$ and, by virtue of *theorem 2*, formulas (5) are applicable. Using these formulas, let us find a unique solution to equation (1) and *the problem* in the form:

$$X^+(z) = \left[\frac{3(z-3i)(z-5i)}{11(z-i)(z-2i)} \cdot \left[\frac{5(z+i)}{(z+10i)(z+12i)}\right]^+\right]^+ =$$

$$=\left[\frac{15(z-3i)(z-5i)(z+i)}{11(z-i)(z-2i)(z+10i)(z+12i)}\right]^{+}=$$

$$=\frac{15}{11}\cdot\left[\frac{2805}{364(z+12i)}-\frac{1755}{264(z+10i)}\right]=\frac{225}{242\cdot 364}\times$$

$$\times\frac{565z-1448i}{(z+10i)(z+12i)}=\frac{225}{88088}\cdot\frac{565z-1448i}{z^{2}+22iz-120},$$

$$Y_{-}(z)=\left[\frac{5(z+i)}{(z+10i)(z+12i)}\right]_{-}+\frac{11}{3}\frac{(z-i)(z-2i)}{(z-3i)(z-5i)}\times$$

$$\times\left\{\frac{3(z-3i)(z-5i)}{11(z-i)(z-2i)}\cdot\left[\frac{5(z+i)}{(z+10i)(z+12i)}\right]^{+}\right\}_{-}=$$

$$=5\cdot\frac{(z-i)(z-2i)}{(z-3i)(z-5i)}\cdot\left[\frac{-16}{143(z-i)}+\frac{3}{56(z-2i)}\right]=$$

$$=\frac{5}{8008}\cdot\frac{(z-i)(z-2i)(-467z+1363i)}{(z-3i)(z-5i)(z-i)(z-2i)}=$$

$$=\frac{5}{8008}\cdot\frac{(-467z+1363i)}{z^{2}-8iz-15}.$$

By substitution in (1), one can make sure that the solution is actually found. The desired solution is as follows:

$$X^{+}(z)=\frac{225}{88088}\cdot\frac{565z-1448i}{z^{2}+22iz-120},\ Y_{-}(z)=-\frac{5}{8008}\cdot\frac{467z-1363i}{z^{2}-8iz-15}.$$

4 Discussion

In the range of questions related to integral equations of the convolution type and the Riemann problem, research methods based on the integral theory (Cauchy type) are distinguished by a significant analytical barrier. It is quite challenging to bring research to the level of observable theorems. In different classes of assumptions, it is often necessary to rebuild the theory from scratch. Examples of new approaches go back, particularly to the research of M. G. Krejn (1958). Simultaneously, it is sometimes possible to simplify and obtain the final results and exact formulas for solutions to equation (1). This is how the current research is carried out. Using a new approach, the authors obtained a general solvability theorem that considers the specifics of the assumptions and contains convenient solution formulas. The resulting theorem is consistent with the stated objective. The authors note that the special nature of the used *FP* ring can be both a strength and a weakness of the research. The proposed approach is simpler and more efficient to use. Other authors did not have similar results.

The authors have reflected the research results in many publications. With the received materials (abstract and specific), they participated in scientific conferences of the following structures:

1. Kherson National Technical University (KNTU) in Kherson (Ukraine);
2. Taras Shevchenko National University of Kyiv (KNU);
3. National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute" (NTUU KPI) in Kyiv (Ukraine);
4. Bauman Moscow State Technical University (BMSTU) in Moscow (Russia);
5. Odessa State Academy of Civil Engineering and Architecture in Odessa (Ukraine) (2015–2020), and others.

Experts showed a significant interest to the obtained results.

5 Conclusion

The paper considers the specifics of the assumptions about the location of the singular points of the rational coefficient of equation (1). The authors obtained final results that are easy to use, and the main results of the research are consistent with the objective. For the first time, under the conditions of *theorem 2*, signs are established that consider the specifics of assumptions about the coefficients of equation (1) and general formulas (4)-(8) of solutions in $\mathfrak{R}_r$. They are more convenient than the ones obtained by the authors earlier. The results are applicable when studying the existence, uniqueness, or non-uniqueness of solutions, their construction for specific equations of the form (1), and corresponding related problems. Their practical significance lies in the possibility of facilitating their use for finding solutions of specific equations of the considered form and demonstrating a new general approach. The new research methodology for this problem is also of practical interest. Other researchers did not use the proposed method in the considered ring. The authors want to study the cases further when factorization concerning the factorization pair for $A^{-1}(z)$ is not correct. The expansion of classes of equations, examples of rings with factorization pairs, the expansion of the scope of the approach based on the approximation of functions, and applied aspects are also promising. The obtained results indicate the fruitfulness of the approach from the theory of equations in rings with factorization pairs for the considered issues and problems. According to assumptions, they deepen the fundamental understanding of issues related to the Riemann problem and simplify the process of solving specific examples. The formulation of the problem of solvability of the equation, considering the specifics of the location of the singular points of the rational coefficient, is given for the first time and the "circular" approach to the research. It demonstrates the possibility of a unified approach based on the provisions on equations in rings with FP and provides an opportunity to see the interpretation and results of research.

References

Akopyan, V. N., & Dashtoyan, L. L. (2013). *Closed solutions of some mixed problems for an orthotropic plane with a cut. Current problems of deformable solid body mechanisms, differential, and integral equations. Abstracts of reports of the international scientific conference.* Odessa, Ukraine: Odessa I. I. Mechnikov National University.

Cherskij, Yu. I., Kerekesha, P. V., & Kerekesha, D. P. (2010). *Method of conjugation of analytical functions with applications.* Odessa, Ukraine: Astroprint.

Gahov, F. D. (1963). *Two-point boundary value problem*. Moscow, USSR: Gosudarstvennoe izdanie fiziko-matematicheskoj literatury.

Gahov, F. D., & Cherskij, Yu. I. (1978). *Equations of the convolution type*. Moscow, USSR: Nauka.

Krejn, M. G. (1958). Integral equations on a half-line with kernels depending on the difference of arguments. *Uspekhi Matematicheskikh Nauk*, *13*(5(83)), 3-120.

McNabb, A., & Schumitzky, A. (1972). Factorization of operators I: Algebraic theory and examples. *Journal of Functional Analysis*, *9*(3), 262-295.

Mhitaryan, S. M. (1968). On some plane contact problems in the exercises' theory, considering adhesion forces and related integral and differential equations. *Bulletin of the Academy of Sciences of the Armenian SSR. Series: Mechanics*, *21*(5-6), 3-20.

Mushelishvili, N. I. (1968). *Singular integral equations.* Moscow, USSR: Nauka.

Poletaev, G. S. (1988). *On equations and systems of the same type in rings with factorization pairs.* Kyiv, USSR: Mathematics Institute, Academy of Sciences of the Ukrainian SSR.

Popov, G. Ya., Kerekesha, P. V., & Kruglov, V. E. (1976). *The factorization method and its numerical implementation.* Odessa, Ukraine: Odessa I. I. Mechnikov National University.

Rapoport, I. M. (1948) On a class of singular integral equations. *Proceedings of the USSR Academy of Sciences*, *59*(8), 1403-1406.

Rapoport, I. M. (1949). On some "paired" integral and integro-differential equations. *Proceedings of the Institute of Mathematics and Mechanics of the Academy of Sciences of the USSR*, *12*, 102-118.

Voytik, T. G., Poletaev, G. S., & Yatsenko, S. A. (2017). Projector approach to the general linear equation with variables from the subring of the rational functions and a factorable coefficient. *Journal of Physics: Conference Series*, *918*, 012032. DOI: 10.1088/1742-6596/918/1/012032.