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Fractional Hedonic Games With a Limited Number of Coalitions

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Abstract. Recently, fractional hedonic games have received considerable attention. Such a game can be represented by directed weighted graph where the weight of edge (i, j) denotes the value player i has for player j. The utility of player i is the average value that player i assigns to the members of i's coalition. In this paper, we study a variant of this game where there is a specific upper bound k on the number of coalitions that can be formed. We first consider how to find a coalition partition that maximizes the social welfare, i.e., the sum of the utilities. Computing social welfare maximizing partitions for these games of all agents on undirected unweighted graphs is known to be NP-hard. Here, we study the parameterized complexity in terms of k. For all fixed $k \geq 2$, we show that it remains NP-hard to find a social welfare maximizing k-partition for undirected unweighted graphs. For undirected unweighted trees, we present an algorithm finding a social welfare maximizing k-partition in time $O(n^k)$. Moreover, we consider Nash stable outcomes. We show that for all $k \geq 2$, if a fractional hedonic game on a directed unweighted graph with bounded maximum out-degree admits a Nash stable k-partition, then the stable partition is almost balanced. However, we prove that determining whether a fractional hedonic game admits a Nash stable k-partition is NP-complete for all $k \geq 2$.

1 Introduction

Community detection in social networks, or network partitioning, is an important topic in social network analysis. A social network is classically represented by a directed weighted graph over the agents, where a weighted link models the relationship between two agents in the social network. Intuitively, communities in a social network corresponds to groups of the vertices that are internally more densely connected than with the rest of the vertices in the network. Partitioning a social network into disjoint communities, or revealing the hidden community structure, can offer insights regarding the organization of a social network and can significantly simplify the network representation. Furthermore, in online marketing, such as placing online ads or deploying viral marketing strategies, identifying communities in the social network often leads to more accurate targeting and achieves better marketing results.

A key challenge of community detection is to formally define what is a community. Various attempts from various perspectives have emerged in the literature (see [15, 20] for two recent surveys on this topic). Mainstream attempts focus on optimizing a given metric that quantitatively measure the quality of a community structure. However, optimization of a centralized and global metric dictates the global network decomposition from a centralized viewpoint and ignores the natural forces and dynamics underlying the formation of communities.

The field of game theory focuses on interactions between intelligent individuals. Thus, it is natural to apply game theory to capture the dynamics behind the formation of communities in social networks. In recent work, there has been a considerable amount of research on using game-theoretic techniques to study community detection in social networks. We refer to [17] for a recent survey on this topic. Hedonic games are a notable type of game for studying coalition formation (see [4] for a survey). A hedonic game is specified by a set of players who have preferences over the set of all possible partitions of the players into coalitions. The outcome of a hedonic game consists a partition of the players into disjoint coalitions. Of particular relevance to the present paper is the line of research initiated by Aziz et al. [3] on using fractional hedonic games to study community detection. Fractional hedonic games (FHGs), introduced by Aziz et al. [2], are a subclass of hedonic games that can be represented by directed weighted graphs. In particular, an FHG is represented by a directed weighted graph where the weight of edge (i, j) denotes the value player i has for player j and the utility of a player i is the average value that player i ascribes to the members of i's coalition. Outcomes that satisfy some notion of stability or welfare are considered to be desirable community structures for a given FHG. For example, consider FHGs represented by undirected unweighted graphs, i.e., undirected graphs where each edge has weight 0 or 1. This covers situations in which players only distinguish between friends and non-friends and desire to be in a coalition in which the fraction of friends is maximized. Aziz et al. [3] consider the computational complexity of computing welfare maximizing partitions for FHGs. Three different notions of social welfare are considered: (1) utilitarian welfare (or social welfare): sum of utilities; (2) egalitarian welfare: the minimum utility of any agent; and (3) Nash welfare: product of utilities. They show that maximizing utilitarian, egalitarian, or Nash welfare is NP-hard even for the FHGs represented by undirected unweighted graphs. On the positive side, they present approximation algorithms which search for maximal matchings. These algorithms are therefore limited as the maximum number of players in a coalition is two, and it is usually unrealistic in practice to form many tiny coalitions. In this paper, we focus on utilitarian welfare.

Our results. We study a variant of FHGs where there is a specific upper bound k on the number of coalitions that can be formed. To motivate the study, note that in many real-world scenarios, there are physical and organizational restrictions that limit the number of possible coalitions. Consider a setting in which each coalition requires a leader, and where only a small number of agents are qualified to act as a leader. Thus, any feasible partition cannot contain more coalitions than the number of qualified leaders.

A central concern of coalition formation games is to define what constitutes an acceptable or desired outcome. Within our setting, we consider two key objectives. Our first objective is to find a partition maximizing the social welfare, i.e., the sum of the utilities of all players. As mentioned before, computing social welfare maximizing partitions (with no restriction number of coalitions) is proved to be NP-hard by Aziz et al. [3], even for FHGs represented by undirected unweighted graphs. Here, we study the parameterized complexity of the problem in terms of k. We refer to a partition with exactly k coalitions as a k-partition. For all $k \geq 2$, we establish the NP-hardness of finding a social welfare maximizing k-partition on undirected unweighted graphs. For undirected unweighted trees, we prove a structural property of social welfare maximizing k-partitions. In particular, on undirected unweighted trees, we show that any coalition in a social welfare maximizing k-partition is connected. By leveraging this property, for n-node undirected unweighted trees, we present a simple algorithm finding a social welfare maximizing k-partition in $O(n^k)$ time.

A social welfare maximizing partition may not satisfy every player and hence there may exist a player who could increase their utility by deviating to another coalition. Our second objective is to consider Nash stable partitions, where no player can improve their utility by unilaterally changing their coalition. We prove that for all $k \geq 2$, if a Nash stable k-partition exists in an FHG represented by n-node directed unweighted graph with bounded maximum out-degree, then each coalition in such a k-partition is of size $\Omega(n)$. We then study the computational complexity of finding a Nash stable k-partition. Unfortunately, for all $k \geq 2$, we prove that it is NP-complete to determine whether an FHG played on a directed weighted graph with edge weights in $\{0, -1\}$ admits a Nash stable k-partition.

Related works. Aziz et al. [3] studied the FHGs from a social welfare perspective, Subsequently, Flammini et al. [14] investigated how to form welfare maximizing coalitions in FHGs in an online setting. Chen et al. [12] proposed several agent-based (simulation-based) methods for finding social welfare maximizing partitions, and provided numerical results. Bilò et al. [8] initiated the study of Nash stable partitions in FHGs from a non-cooperative point of view. They showed that a Nash stable partition is not guaranteed to exist in FHGs played on undirected graphs with negative weights. However, they proved that such a partition always exists when weights are non-negative. Furthermore, they give bounds on the (Nash) price of anarchy and stability. In addition, they established the NP-hardness of computing a Nash stable partition with maximum social welfare. Further results on the price of stability for FHGs played on undirected unweighted graphs have been presented in [18]. Other stability concepts in FHGs have also been studied [1, 10, 11].

The restriction on the number of coalitions, which is the focus of the present paper, has been mostly overlooked. Skibski et al. [24] studied k-coalitional cooperative games in the transferable utility setting, and developed an extension of the Shapley value for this game. Sless et al. [25] initiated the study of additively separable hedonic games (ASHGs) in which exactly k coalitions must be formed. Estivill-Castro et al. [13] studied modified fractional hedonic games (MFHGs) where k equal-size coalitions must be formed (a balanced k-partition). ASHGs [2] and MFHGs [23] are two related classes of hedonic games that can also be represented by graphs. Aziz et al. [1] explained why efficient or stable outcomes of FHGs provide better partitions than their counterparts for ASHGs and MFHGs, respectively. Sless et al. [25] considered social welfare maximizing partitions and k-coalitions-core stable partitions, an adaptation of the notion of core stability to their setting. They presented an efficient algorithm for finding social welfare maximizing k-partitions in ASHGs played on undirected graphs when the number of negative-weight edges is limited, and prove that for all $k \geq 1$, it is NP-hard to determine whether a given k-partition is k-coalitions-core and whether there exists a k-coalitions-core stable k-partition in ASHGs. Estivill-Castro et al. [13] considered Nash stability. They proved that for all $k \geq 2$, finding a balanced Nash stable k-partition is NP-hard in general undirected unweighted graphs, but polynomially solvable in undirected unweighted trees. Further results on Nash stable 2-partitions for MFHGs have been presented in [5, 6].

Paper organization. The remainder of the paper is organized as follows. Section 2 presents formal definitions. Sections 3 and 4 present our results for social welfare maximizing k-partitions and Nash stable k-partitions, respectively. Section 5 offers some concluding remarks. Due to space restrictions, some of the proofs are omitted. Complete proofs will be provided in the full version.

2 Preliminaries

For all positive integers n, let $[n] = \{1, 2, \ldots, n\}$. In an FHG, we are given a set $N = [n]$ of players. The objective of the game is to partition the players into disjoint coalitions $\mathcal{P} = \{P_1, P_2, \ldots\}$. Let $\Pi(N)$ denote the set of all partitions of players N, and for all integers $k \geq 1$, let $\Pi_k(N)$ denote the set of partitions in $\Pi(N)$ with exactly k coalitions. We refer to each partition in $\Pi_k(N)$ as a k-partition.

Each player i has a value function $v_i : N \to \mathbb{R}$ that denotes how much player i values each of the players in N. We assume that $v_i(i) = 0$. Hence, every FHG can be represented by a tuple of valuation functions $v = (v_1, \ldots, v_n)$. We often associate an FHG with a weighted directed graph. Given a tuple of valuation functions v, let $G = (N, E, v)$ denote the weighted directed graph where the weight of the tuple (i, j) in $N \times N$ is $v_i(j)$ and E contains all tuples of non-zero weight. Let $\mathcal{G}(G)$ denote the fractional hedonic game associated with G. When there is no ambiguity, we simply refer to the FHG $\mathcal{G}(G)$ as G.

For convenience, for each player i, we extend the input domain of the value function v_i to $N \cup \{P \mid P \subseteq N \wedge i \in P\} \cup \Pi(N)$. For any coalition $P \subseteq N$ that contains player i, the utility $v_i(P)$ of agent i is defined as $\frac{\sum_{j \in P} v_i(j)}{|P|}$. For any partition $\mathcal{P}$ in $\Pi(N)$, let $\mathcal{P}(u)$ denote the coalition that contains player u, and the utility $v_i(\mathcal{P})$ of player v_i in the partition $\mathcal{P}$ is defined as $v_i(\mathcal{P}(i))$.

Given an FHG $\mathcal{G}(G)$ and a partition $\mathcal{P}$, the social welfare $SW_G(\mathcal{P})$ of the partition $\mathcal{P}$ is defined as the sum of the utilities of all players, i.e., $SW_G(\mathcal{P}) = \sum_{i\in N} v_i(\mathcal{P})$. We often drop the subscript G when there is no ambiguity.

Given a partition $\mathcal{P}$ in $\Pi(N)$, we say a player i is Nash stable for $\mathcal{P}$ if there is no other coalition $P' \neq \mathcal{P}(i)$ in $\mathcal{P}$ such that $v_i(P' \cup \{i\}) > v_i(\mathcal{P})$. We say a partition $\mathcal{P}$ in $\Pi(N)$ is Nash stable if all players are Nash stable for $\mathcal{P}$.

We use the following notations from graph theory. Let $G = (N, E)$ be an unweighted graph. Given a subset U of N, we denote by $E_G(U)$ the set of edges of G having both endpoints in U. Moreover, for two disjoint sets N_1 and N_2, we denote by $E_G(N_1, N_2)$ the set of edges having exactly one endpoint in N_1 and exactly one endpoint in N_2. We drop the subscript G when there is no ambiguity. For any graph $G = (N, E)$ and any subset S of N, we let $G[S]$ denote the subgraph of graph G induced by S.

We now state the two problems studied in this paper.

- The social welfare maximizing k-partition problem: Given a fractional hedonic game $\mathcal{G}(G)$ and an integer k, find a k-partition $\mathcal{P}$ in $\Pi_k(N)$ that maximizes the social welfare $SW(\mathcal{P})$.
- The Nash stable k-partition problem: Given a fractional hedonic game $\mathcal{G}(G)$ and an integer k, determine whether there is a k-partition $\mathcal{P}$ in $\Pi_k(N)$ that is Nash stable.

3 Social Welfare Maximizing *k*-Partition

In this section, we focus on FHGs played on undirected unweighted graphs. We remark that for an FHG played on an undirected unweighted graph $G = (N, E)$, the social welfare of any partition $\mathcal{P}$ in $\Pi(N)$ is $SW_G(\mathcal{P}) = \sum_{P\in\mathcal{P}} \frac{2|E_G(P)|}{|P|}$. In the following, for FHGs played on undirected unweighted graphs, Section 3.1 establishes the NP-hardness of the social welfare maximizing k-partition problem for every fixed $k \geq 2$. For FHGs played on undirected unweighted trees, Section 3.2 presents an efficient algorithm that solves the social welfare maximizing k-partition problem for all $k \geq 2$.

3.1 NP-Hardness Results

In this section, we prove Theorem 1 below.

Theorem 1. *For FHGs played on unweighted undirected graphs, the social welfare maximizing k-partition problem is NP-hard for every fixed $k \geq 2$.*

We separate our hardness proof into two parts: $k \geq 3$ and $k = 2$.

When $k \geq 3$, we reduce from the k-colorable problem, which was proved to be NP-complete by Leven and Galil [22] for all $k \geq 3$. The k-colorable problem is to determine whether a given undirected graph can be partitioned into k independent sets. By considering complementary graphs, the NP-completeness of the k-colorable problem implies the NP-completeness of determining whether an undirected graph can be partitioned into k cliques. It is straightforward to verify that the social welfare of a k-partition $\mathcal{P}$ for an undirected unweighted graph is at most at most $2(n-k)$. Moreover, if a k-partition $\mathcal{P}$ has social welfare $2(n-k)$, then the k-partition $\mathcal{P}$ partitions G into k cliques. Therefore, if there is an efficient algorithm that solves the social welfare maximizing k-partition problem, then there is an efficient algorithm that solves the NP-complete problem of determining whether an undirected graph can be partitioned into k cliques. Thus, we deduce that the social welfare maximizing k-partition problem is NP-hard for $k \geq 3$.

It remains to prove the case when $k = 2$. We reduce from the max cut problem, which was proved to be NP-complete by Karp [19]. Recall that for the max cut problem, we are given an instance of an unweighted undirected graph G and a positive integer r, and we wish to determine whether there exists a cut (S_1, S_2) of G satisfying $|E(S_1, S_2)| \geq r$. We remark that our reduction is similar to a reduction given by Bonsma et al. [9]. Bonsma et al. use a reduction from the max cut problem to prove that it is NP-hard to find a cut (S_1, S_2) in an undirected graph G such that $\frac{|E_G(S_1,S_2)|}{|S_1||S_2|}$ is minimized.

Reduction. Let $I = (G, r)$ denote an instance of the max cut problem, where $G = (V, E)$ denotes an undirected graph and r denotes a positive integer. We construct an undirected unweighted graph $G^* = (V^*, E^*)$ as an instance of the social welfare maximizing 2-partition problem as follows. For convenience, let n denote $|V|$ and let m denote $|E|$.

We begin by constructing an undirected graph $G' = (V', E')$, and then let $G^* = (V', K' \setminus E')$ be the complementary graph of G', where K' consists of all 2-element subsets of V'. For each v in V, we have two sets I_v and I'_v of vertices, each of size $M = 4m + 2$. Thus, G' has $2nM$ vertices and $V' = \bigcup_{v \in V} I_v \cup I'_v$. For each v in V, we introduce edges connecting each vertex in I_v to each vertex in I'_v. Pick one distinguished vertex from each I_v to form a set A of n vertices, and pick one distinguished vertex from each I'_v to form a set A' of n vertices. We proceed to insert edges in A and A' to create two copies of G. The resulting graph is G'.

To show that our reduction is correct, we first prove two useful properties of social welfare maximizing k-partitions. Then, we prove Lemma 1, which establishes the correctness of our reduction.

Proposition 1. *Let $H = (V, E)$ be an undirected graph, and let $\overline{H}$ be the complementary graph of H. Let $\mathcal{P}$ be a 2-partition in $\Pi(V)$. Then $SW_H(\mathcal{P}) = |V| - 2 - SW_{\overline{H}}(\mathcal{P})$.*

Proof. Let n denote $|V|$. Note that $SW_H(\mathcal{P}) = \sum_{i \in [2]} \frac{2E_H(P_i)}{|P_i|}$. Since $\overline{H}$ is the complementary graph of H, for all subsets P of V, we have $E_H(P) + E_{\overline{H}}(P) = |P|(|P| - 1)/2$. Therefore, we obtain

$$SW_H(\mathcal{P}) = \sum_{i \in [2]} \left(|P_i| - 1 - \frac{2E_{\overline{H}}(P_i)}{|P_i|} \right) = n - 2 - \sum_{i \in [2]} v_{\overline{H}}(P_i) = n - 2 - SW_{\overline{H}}(\mathcal{P}).$$

□

Proposition 2. *Let $G = (V, E)$ be an undirected graph. Let $G' = (V', E')$ be the corresponding graph obtained by our reduction, where $V' = \bigcup_{v \in V}(I_v \cup I'_v)$. Let $\mathcal{P}$ in $\Pi_2(V')$ be a social welfare minimizing 2-partition in G'. Then, for each node v in V and each coalition P in $\mathcal{P}$, either $P \cap (I_v \cup I'_v) = I_v$ or $P \cap (I_v \cup I'_v) = I'_v$.*

Proof. Assume for the sake of contradiction that there is a vertex v in V such that the 2-partition $\mathcal{P}$ does not partition $I_v \cup I'_v$ into the sets I_v, I'_v. Let $\mathcal{P} = (P_1, P_2)$. Therefore, for each i in [2], both $P_i \cap I_v$ and $P_i \cap I'_v$ are non-empty. Let $a = |P_1 \cap I_v| \geq 1$ and $a' = |P_1 \cap I'_v| \geq 1$. Note that $|E(P_1 \cap I_v, P_1 \cap I'_v)| = a \cdot a'$, and $|E(P_2 \cap I_v, P_2 \cap I'_v)| = (M - a)(M - a')$. Without loss of generality, assume that $|E(P_1 \cap I_v, P_1 \cap I'_v)| \geq |E(P_2 \cap I_v, P_2 \cap I'_v)|$, i.e., $aa' \geq (M - a)(M - a')$. That is, $a + a' \geq M$, and hence $aa' \geq a(M - a) \geq M - 1$, as a and a' are positive integers. Thus, we have $|E(P_1 \cap I_v, P_1 \cap I'_v)| = aa' \geq M - 1$.

Therefore, we obtain

$$SW_{G'}(\mathcal{P}) = \sum_{i \in [2]} \frac{2E(P_i)}{|P_i|} \geq \frac{2E(P_1)}{|P_1|} \geq \frac{2|E(P_1 \cap I_v, P_1 \cap I'_v)|}{|P_1|} \geq \frac{M-1}{|P_1|} \geq \frac{M-1}{2nM}.$$

Let $P' = \left(\bigcup_{v \in V} I_v, \bigcup_{v \in V} I'_v\right)$. Thus, $SW_{G'}(\mathcal{P}) \leq SW_{G'}(P') = \frac{2\left|E_{G'}\left(\bigcup_{v \in V} I_v\right)\right|}{nM} + \frac{2\left|E_{G'}\left(\bigcup_{v \in V} I'_v\right)\right|}{nM} = \frac{2m}{nM}$. That is, $\frac{M-1}{2nM} \leq \frac{2m}{nM}$. Rearranging, we obtain $M - 1 \leq 4m$, contradicting $M = 4m + 2$. □

Lemma 1. *Let $G = (V, E)$ be an undirected graph and let (G, r) be an instance of the max cut problem. Let G^* be the corresponding instance of the social welfare maximizing 2-partition problem. Let $n = |V|$, $m = |E|$, and $M = 4m + 2$. Then G has a cut of cardinality r if and only if G^* has a 2-partition with social welfare at least $2nM - 2 - \frac{4m-4r}{nM}$.*

Proof. Let G' be the complementary graph of G^*. By Proposition 1, it suffices to prove that G has a cut of cardinality r if and only if G' has a 2-partition with social welfare at most $\frac{4m-4r}{nM}$. We consider two cases.

Case 1: G has a cut (S_1, S_2) of cardinality r. For each i in $\{1, 2\}$, let $P_i = \{I_v \mid v \in S_i\}$ and $P'_i = \{I'_v \mid v \in S_i\}$. Consider the 2-partition $\mathcal{P}' = (P_1 \cup P'_2, P'_1 \cup P_2)$. It is straightforward to verify that $|P_1 \cup P'_2| = |P'_1 \cup P_2| = nM$. Moreover, notice that $|E_{G'}(P_1 \cup P'_2)| = |E_{G'}(P_1)| + |E_{G'}(P'_2)| = |E_G(S_1)| + |E_G(S_2)| = |E| - |E_G(S_1, S_2)| = m - r$. Similarly, we have $|E_{G'}(P'_1 \cup P_2)| = m - r$. Thus,

$$SW(\mathcal{P}') = \frac{2|E_{G'}(P_1 \cup P'_2)|}{|P_1 \cup P'_2|} + \frac{2|E_{G'}(P'_1 \cup P_2)|}{|P'_1 \cup P_2|} = \frac{4m - 4r}{nM}. \tag{1}$$

Therefore, G' has a 2-partition with social welfare at most $\frac{4m-4r}{nM}$.

Case 2: G' has a 2-partition with social welfare at most $\frac{4m-4r}{nM}$. Consider a social welfare minimizing 2-partition $\mathcal{P}^* = (P^*_1, P^*_2)$ in G'. That is, we have $SW_{G'}(P^*) \leq \frac{4m-4r}{nM}$. By Proposition 2, we deduce that for each v in V, $\mathcal{P}^*$ partitions $I_v \cup I'_v$ into the two sets I_v and I'_v. Therefore, either $I_v \subseteq P^*_1$ and $I'_v \subseteq P^*_2$, or $I_v \subseteq P^*_2$ and $I'_v \subseteq P^*_1$. For each i in $\{1, 2\}$, let $S_i = \{v \in V \mid I_v \subseteq P^*_i\}$

and $S'_i = \{v \in V \mid I'_v \subseteq P^*_i\}$. Clearly, $S_1 = S'_2$, $S_2 = S'_1$, and $(S_1, S'_1) = (S'_2, S_2)$ is a partition in $\Pi_2(V)$. Using a calculation similar to that used to derive Eq.(1), we have $SW_{G'}(\mathcal{P}^*) = \frac{4(m-|E_G(S_1,S'_1)|)}{nM}$. Since $SW_{G'}(P^*) \leq \frac{4m-4r}{nM}$, we deduce that $|E_G(S_1, S'_1)| \geq r$. Therefore, G has a cut of cardinality at least r. □

3.2 Finding Social Welfare Maximizing k-Partitions for Trees

In this section, we first prove Lemma 2, which presents a useful structural property of social welfare maximizing k-partitions on undirected unweighted trees. Then, based on this property, we present a simple $O(n^k)$-time algorithm for the social welfare maximizing k-partitions problem on unweighted undirected trees.

Lemma 2. *Let $G = (N, E)$ be a tree and let k be a positive integer. Let $\mathcal{P}^*$ in $\Pi_k(N)$ be a social welfare maximizing k-partition with k coalitions $P^*_1, \ldots, P^*_k$. Then, for all coalitions P^*_i in $\mathcal{P}^*$, $G[P^*_i]$ is connected.*

Proof. Note that a connected subgraph of a tree is a tree, and a disconnected subgraph of a tree is a forest. Assume for the sake of contradiction that there is a coalition P^*_i in $\mathcal{P}^*$ such that $G[P^*_i]$ is not connected, i.e., a forest. Suppose that $G[P^*_i]$ has p connected components, where $p \geq 2$. Let $(T_1, T_2, \ldots, T_p)$ be the partition of P^*_i such that $G[T_1], \ldots, G[T_p]$ are all connected components in $G[P^*_i]$. Let $t_w = |T_w|$ for each w in $[p]$ and $p_i = |P^*_i|$. That is, $p_i = \sum_{w\in[p]} t_w$. Without loss of generality, assume that $t_1 \leq t_2 \leq \cdots \leq t_p$.

Since G is a tree, we deduce that there is another coalition P^*_j in $\mathcal{P}^*$ with $j \neq i$ such that there is an edge between P^*_j and T_1. Now we are ready to construct a k-partition $\mathcal{P}'$ in $\Pi_k(N)$ with $SW(\mathcal{P}') > SW(P^*)$, which contradicts the assumption that $\mathcal{P}^*$ is a social welfare maximizing k-partition.

We construct such a k-partition $\mathcal{P}'$ with coalitions $P'_1, \ldots, P'_k$ as follows. For each t in $[n] \setminus \{i, j\}$, let $P'_t = P^*_t$. Furthermore, let $P'_i = P^*_i \setminus T_1$ and $P'_j = P^*_j \cup T_1$. Now we prove that $SW(\mathcal{P}') > SW(\mathcal{P}^*)$. For all coalitions S of N, let $d(S) = E(S)/|S|$. Thus, $SW(\mathcal{P}') = \sum_{w\in[k]} 2d(P'_w)$ and $SW(\mathcal{P}^*) = \sum_{w\in[k]} 2d(P^*_w)$. Hence it suffices to prove that $d(P'_i) + d(P'_j) > d(P^*_i) + d(P^*_j)$. To prove this, it is enough to prove that $d(P'_i) \geq d(P^*_i)$ and $d(P'_j) > d(P^*_j)$.

First, we prove that $d(P'_i) \geq d(P^*_i)$. Recall that the subgraph $G[P^*_i]$ contains p trees $G[T_1], \ldots, G[T_p]$, while the subgraph $G[P'_i]$ contains trees $G[T_2], \ldots, G[T_p]$. Thus,

$$d(P'_i) = \frac{\sum_{w=2}^{p}(t_w - 1)}{\sum_{w=2}^{p} t_w} = 1 - \frac{p-1}{p_i - t_1}, d(P^*_i) = \frac{\sum_{w=1}^{p}(t_w - 1)}{\sum_{w=1}^{p} t_w} = 1 - \frac{p}{p_i}.$$

Now, to show that $d(P'_i) \geq d(P^*_i)$, it suffices to show that $\frac{p-1}{p_i-t_1} \leq \frac{p}{p_i}$, i.e., $p \cdot t_1 \leq p_i$. The last inequality follows by $t_1 \leq t_2 \leq \cdots \leq t_p$ and $p_i = \sum_{w=1}^{p} t_w$.

Second, we prove that $d(P'_j) > d(P^*_j)$. Let $p_j = |P^*_j|$ and $e_j = |E(P^*_j)|$. That is, $d(P^*_j) = \frac{e_j}{p_j}$. Since there is at least one edge connecting P^*_j and T_1, we have

$$d(P'_j) = \frac{|E(P^*_j \cup T_1)|}{p_j + t_1} \geq \frac{|E(P^*_j)| + 1 + |E(T_1)|}{p_j + t_1} = \frac{e_j + t_1}{p_j + t_1}.$$

Therefore, to prove $d(P'_j) > d(P^*_j)$, it suffices to prove $\frac{e_j+t_1}{p_j+t_1} > \frac{e_j}{p_j}$, i.e., $p_j \cdot t_1 > e_j \cdot t_1$. This follows by the observation that any subgraph of a tree is either a tree or a forest, that is, $e_j \leq p_j - 1 < p_j$. □

Theorem 2. *For all positive integers k, the social welfare maximizing k-partition problem can be solved in $O(n^k)$ time on undirected unweighted trees.*

Proof. By Lemma 2, we deduce that any social welfare maximizing k-partition on a tree is a partition into k subtrees. Notice that for trees, there is a one-to-one correspondence between removing $k-1$ edges and partitioning into k subtrees. Thus, in $O(n^k)$ time, one can simply enumerate each possible partition of k subtrees in G and identify the optimal one in in $O(n^k)$ time. □

4 Nash Stable k-Partition

In this section, we consider Nash stable k-partitions for all $k \geq 2$. Throughout this section, we assume that $k \geq 2$ unless stated otherwise. As an independent result, Theorem 3 below shows that a Nash stable k-partition of an unweighted directed graph with bounded out-degree is almost balanced. Then, we prove that it is NP-complete to determine whether a directed weighted graph with edges weights -1 admits a Nash stable k-partition. We remark that a directed graph is strongly connected if there is a path in each direction between each pair of vertices of the graph.

Theorem 3. *Let $k \geq 2$ and $\Delta \geq 2$ be two integers. Let $G = (N, E)$ denote a directed unweighted strongly connected graph with out-degree bounded by Δ and $|N| \geq k \cdot \Delta^{k+1}$. Assume that $\mathcal{G}(G)$ admits a Nash stable k-partition $\mathcal{P}$ in $\Pi_k(N)$. Then all coalitions in $\mathcal{P}$ have size at least $\frac{n}{k \cdot \Delta^{k-1}}$.*

Proof. Let $P_1, \ldots, P_k$ denote the k coalitions in $\mathcal{P}$ with $0 < |P_1| \leq |P_2| \cdots \leq |P_k|$. It suffices to prove that $|P_1| \geq \frac{n}{k \cdot \Delta^{k-1}}$. For any t in $\{0, 1, \ldots, k-1\}$, let $Q(t)$ denote the predicate $|P_{k-t}| \geq \frac{n}{k \cdot \Delta^t}$. We use induction on t to prove that $Q(t)$ holds for any t in $\{0, 1, \ldots, k-1\}$. Clearly, $Q(k-1)$ implies that $|P_1| \geq \frac{n}{k \cdot \Delta^{k-1}}$.

For the base case, notice that $0 < |P_1| \leq |P_2| \cdots \leq |P_k|$, and hence $|P_k| \geq \frac{1}{k} \sum_{i \in [k]} |P_i| = \frac{n}{k}$. Therefore, $Q(0)$ holds. For the induction step, let i in $[k-1]$ be given and suppose that $Q(t)$ holds for each t in $\{0, \ldots, i-1\}$. Then, we shall prove that $Q(i)$ holds, i.e., $|P_{k-i}| \geq \frac{n}{k\Delta^i}$. Since $(P_1, \ldots, P_k)$ belongs to $\Pi_k(N)$, we deduce that $\left(\bigcup_{j \in [k-i]} P_j, \bigcup_{j \in [k] \setminus [k-i]} P_j\right)$ is a 2-partition in $\Pi_2(N)$. Furthermore, since G is strongly connected, there is a directed edge (b, a) from $\bigcup_{j \in [k] \setminus [k-i]} P_j$ to $\bigcup_{j \in [k-i]} P_j$. Let $P_{i'}, P_{j'}$ with $i' \leq k-i, j' \geq k-i+1$ denote the two coalitions that contain players a and b, respectively. Therefore, $|P_{i'}| \leq |P_{k-i}|$ and $k - j' \leq i - 1$. By the induction hypothesis, we deduce that $Q(k-j')$ holds, i.e., $|P_{j'}| \geq \frac{n}{k\Delta^{k-j'}} \geq \frac{n}{k\Delta^{i-1}}$. Below we prove that $|P_{i'}| \geq \frac{1}{\Delta}|P_{j'}|$. Since $|P_{i'}| \leq |P_{k-i}|$ and $|P_{j'}| \geq \frac{n}{k\Delta^{i-1}}$, we deduce that $|P_{k-i}| \geq |P_{i'}| \geq \frac{1}{\Delta}|P_{j'}| \geq \frac{1}{\Delta} \cdot \frac{n}{k\Delta^{i-1}} = \frac{n}{k\Delta^i}$, as required.

It remains to prove that $|P_{i'}| \geq \frac{1}{\Delta}|P_{j'}|$. Since $\mathcal{P}$ is Nash stable, we deduce that each player is Nash stable. Since player b is Nash stable, we have

$$v_b(P_{i'} \cup \{b\}) \leq v_b(P_{j'}). \tag{2}$$

Since (b, a) is a directed edge, we deduce that $v_b(a) = 1$ and hence $v_b(P_{i'} \cup \{b\}) = \frac{\sum_{w \in P_{i'} \cup \{b\}} v_b(w)}{|P_{i'}|+1} \geq \frac{v_b(a)}{|P_{i'}|+1} = \frac{1}{|P_{i'}|+1}$. Furthermore, since the out-degree of player b is bounded by Δ and a is an out-neighbor of b outside $P_{j'}$, we deduce that b has at most $\Delta - 1$ out-neighbors in $P_{j'}$, that is, $\sum_{w \in P_{j'}} v_b(w) \leq \Delta - 1$. Thus, $v_b(P_w) \leq \frac{\Delta-1}{|P_{j'}|}$. Using inequality (2), we have $\frac{1}{|P_{i'}|+1} \leq \frac{\Delta-1}{|P_{j'}|}$. By rearranging, we have $|P_{j'}| \leq \Delta|P_{i'}| - |P_{i'}| + \Delta - 1$. Furthermore, we deduce from $Q(k - j')$ and $n \geq k\Delta^{k+1}$ that

$$|P_{j'}| \geq \frac{n}{k \cdot \Delta^{k-j'}} \geq \frac{k\Delta^{k+1}}{k \cdot \Delta^{k-j'}} = \Delta^{j'+1}.$$

Notice that $j' \geq k - i + 1 \geq 2$ as $i \leq k - 1$. Therefore, we have $\Delta^3 \leq \Delta^{j'+1} \leq \Delta|P_{i'}| - |P_{i'}| + \Delta - 1$. Now, we prove that $|P_{i'}| \geq \Delta$. Assume for the sake of contradiction that $|P_{i'}| \leq \Delta$. Hence, we have $\Delta|P_{i'}| - |P_{i'}| + \Delta - 1 \leq \Delta^2 + \Delta - 1$. Since $\Delta \geq 2$, we deduce that $\Delta^2 + \Delta - 1 < 2\Delta^2 \leq \Delta^3$, which contradicts $\Delta^3 \leq \Delta|P_{i'}| - |P_{i'}| + \Delta - 1$. Hence $|P_{i'}| \geq \Delta$.

Thus, $|P_{j'}| \leq \Delta|P_{i'}| - |P_{i'}| + \Delta - 1 \leq \Delta|P_{i'}| - \Delta + \Delta - 1 < \Delta|P_{i'}|$, i.e., $|P_{i'}| > \frac{1}{\Delta}|P_{j'}|$. □

4.1 Hardness

In this section, for each $k \geq 2$, we establish the NP-completeness of determining whether a directed weighted graph with edges weights -1 admits a Nash stable k-partition. We give an NP-completeness proof first for $k = 2$ and then for $k \geq 3$.

First, it is convenient for us to consider Nash stable partitions in FHGs played on undirected unweighted graphs with each player aiming to minimize the utility, rather than considering Nash stability in weighted directed graphs with negative edge weights. Formally, we state the following observation.

Observation 4 *Let $H = (N, E, v)$ denote a directed weighted graph with edge weight -1. Let $H' = (N, E)$ denote the directed unweighted graph that contains the same vertices and edges as H. Let $\mathcal{P}$ denote a k-partition in $\Pi_k(N)$ for all $k \geq 1$. Then, the k-partition $\mathcal{P}$ is Nash stable in $\mathcal{G}(H)$ if and only if $\mathcal{P}$ is Nash stable in $\mathcal{G}(H')$ with each player seeking to minimize, rather than maximize, their utility.*

We now prove the following proposition, which will be used in our NP-completeness proofs for both $k = 2$ and $k \geq 3$.

Proposition 3. *Let N denote the set of utility-minimizing players, and let $k \geq 2$ be an integer. Let $\mathcal{G}(G)$ denote a fractional hedonic game, where $G = (N, E)$ is an unweighted directed graph. Let x in N denote a player such that x has exactly $k - 1$ out-neighbors $y_1, \ldots, y_{k-1}$ in G. Then, for all Nash stable k-partitions $\mathcal{P}$ in $\Pi_k(N)$, we have $\mathcal{P}(x) \neq \mathcal{P}(y_i)$ for each i in $[k - 1]$.*

Proof. Assume for the sake of contradiction that there is an index i in $[k-1]$ such that $\mathcal{P}(x) = \mathcal{P}(y_i)$. That is, $v_x(\mathcal{P}(x)) = v_x(\mathcal{P}(y_i)) > 0$ since y_i is in $\mathcal{P}(y_i)$ and y_i is an out-neighbor of x. Since x has exactly $k-1$ out-neighbors, there is a coalition P in the k-partition $\mathcal{P}$ such that P does not contain x and any out-neighbors of x. Therefore, $v_x(P \cup \{x\}) = 0 < v_x(\mathcal{P}(x))$. That is, the utility-minimizing player x is not Nash stable for $\mathcal{P}$, a contradiction. □

Nash Stable 2-Partition For $k = 2$, we reduce from the balanced unfriendly 2-partition problem. A 2-partition of an undirected graph is called unfriendly if each vertex has at least as many neighbors outside its part as within. Bazgan et al. [7] prove that the decision problem for balanced unfriendly 2-partitions is NP-complete. Our reduction borrows ideas from a known NP-completeness reduction based on the same problem. Kun et al. [21] present an elegant reduction from the balanced unfriendly 2-partition problem to show that determining whether a directed graph has a stable coloring with two colors is NP-complete. They use a gadget that forces any stable 2-coloring to be balanced. We adapt this gadget to our setting to ensure that any Nash stable 2-partition is balanced. Due to space limitations, the proof of Lemma 3 is deferred to the full version of the paper.

Lemma 3. *For FHGs with utility-minimizing players and played on directed unweighted graphs, the Nash stable* 2*-partition problem is NP-complete.*

Nash Stable *k*-Partitions

Lemma 4. *For FHGs with utility-minimizing players and played on directed unweighted graphs, the Nash stable k-partition problem is NP-complete for all $k \geq 3$.*

Proof. Clearly, this problem is in NP. For hardness, we reduce from the NP-complete problem stated in Lemma 3. Let $G = (V, E)$ denote an instance of the problem stated in Lemma 3, where G is a directed unweighted graph with an isolated 2-cycle of vertices p_1, p_2. Let n denote $|V|$, and suppose that $n \geq 3$. We construct a directed unweighted graph $G' = (V', E')$ as follows.

The graph G' has all vertices in V, and all edges in E. Note that p_1, p_2 are the two vertices of the isolated 2-cycle in G. Add $k-2$ vertices, $p_3, \ldots, p_k$, and add edges letting $p_1, p_2, \ldots, p_k$ form a clique, i.e., add directed edges (p_i, p_j) for each $i \neq j$ in $[k]$ unless $\{i, j\} = \{1, 2\}$. Let $M = n^2 - 2n + 2$. For each j in $\{3, \ldots, k\}$, add M dummy vertices $d_{j,q}$ for each q in $[M]$, and add edges connecting these dummy vertices $d_{j,q}$ to all vertices p_i for each i in $[k] \setminus \{j\}$. That is, for each j in $\{3, \ldots, k\}$, the vertices in the set $\{p_j\} \cup \{d_{j,q} \mid q \in [M]\}$ have the same $k-1$ out-neighbors. Add edges connecting all vertices in $V \setminus \{p_1, p_2\}$ to dummy vertices $d_{j,q}$ for all j in $\{3, \ldots, k\}$ and all q in $[M]$. In total, G' has $n + k - 2 + (k-2)M$ vertices. Claims 1 and 2 below imply that the lemma holds

Claim 1: If G admits a Nash stable 2-partition $\mathcal{P} = (P_1, P_2)$ for utility-minimizing players, then G' admits a Nash stable k-partition for utility-minimizing players.

Let $\mathcal{P}'$ denote the k-partition $(P'_i)_{i\in[k]}$, where $P'_1 = P_1, P'_2 = P_2$, and $P'_i = \{p_i\} \cup \{d_{i,q} \mid q \in [M]\}$ for each i in $\{3,\dots,n\}$. Clearly, for each i in $\{3,\dots,n\}$, all players in $P'_i = \{p_i\} \cup \{d_{i,q} \mid \ q \in [M]\}$ have utility 0, and hence are Nash stable. To prove that $\mathcal{P}'$ is Nash stable, it remains to prove that all of the players in $P'_1 \cup P'_2$ are Nash stable. Assume for the sake of contradiction that there exists an integer i in [2] such that the player p in P'_i is not Nash stable. Thus, there is a coalition P'_j such that the utility player p is decreased after p deviates to P'_j. Since $P'_1 = P_1, P'_2 = P_2$, and the 2-partition (P_1, P_2) is Nash stable, we deduce that $j \neq 3-i$. Therefore, j is in $\{3,\dots,k\}$. Notice that for each q in $[M]$, the player $d_{j,q}$ is an out-neighbor of p in P'_j. Therefore, $v_p(P'_j \cup \{p\}) = M/(M+2) = 1 - 2/(M+2)$. Moreover, p has at most $|P'_i| - 1$ out-neighbors in P'_i and has utility at most $1 - 1/|P'_i| \leq 1 - 1/(n-1)$ as $|P'_i| = |P_i| = |V \setminus P_{3-i}| \leq n-1$. Note that $M = n^2 - 2n + 2 = (n-1)^2 + 1 > 2n - 2$ as $n \geq 3$. Therefore, we have $2/(M+2) < 1/n$, i.e., $v_p(P'_j \cup \{p\}) = 1 - 2/(M+2) > 1 - 1/n \geq v_p(P'_i)$. Thus, player p's utility does not decrease by letting p deviate to P'_j, a contradiction. This completes the proof of Claim 1.

Claim 2: If G' admits a Nash stable k-partition $\mathcal{P}' = (P'_i)_{i\in[k]}$ in $\Pi_k(V')$, then G admits a Nash stable 2-partition.

Notice that for each i in $[k]$, the player p_i has $k-1$ out-neighbors in $\{p_j \mid j \in [k] \setminus \{i\}\}$. Thus, we deduce by Proposition 3 that $\mathcal{P}'(p_i) \neq \mathcal{P}'(p_j)$ for all j in $[k] \setminus \{i\}$. Therefore, the k vertices $p_1, \dots, p_k$ belong to distinct coalitions in the k-partition. Without loss of generality, suppose that P'_i contains p_i for all i in $[k]$. That is, $\mathcal{P}'(p_i) = P'_i$ for all i in $[k]$. It now suffices to prove that (P'_1, P'_2) is a 2-partition in $\Pi_2(V)$. After we prove this, the desired statement that G admits the Nash stable 2-partition (P'_1, P'_2) directly follows, since the Nash stability of (P'_1, P'_2) follows by the Nash stability of $\mathcal{P}' = (P'_i)_{i\in[k]}$.

To prove that (P'_1, P'_2) is in $\Pi_2(V)$, it suffices to prove that $P'_1 \cup P'_2 = V$. That is, it is enough to prove that $V' \setminus V \subseteq \bigcup_{i\in\{3,\dots,k\}} P'_i$ and $V \cap \bigcup_{i\in\{3,\dots,k\}} P'_i = \emptyset$. We first prove that $V' \setminus V \subseteq \bigcup_{i\in\{3,\dots,k\}} P'_i$. Since p_i belongs to P'_i for each i in $[k] \setminus [2]$, it remains to consider the dummy vertices. For each j in $[k] \setminus [2]$ and q in $[M]$, the dummy vertex $d_{j,q}$ has $k-1$ out-neighbors in $\{p_i \mid i \in [k] \setminus \{j\}\}$, and hence we deduce by Proposition 3 that $\mathcal{P}'(d_{j,q}) \neq \mathcal{P}'(p_i) = P'_i$ for all i in $[k] \setminus \{j\}$. That is, we obtain that $\mathcal{P}'(d_{j,q}) = P'_j$ for all i in $[k] \setminus \{j\}$. Therefore, for all j in $[k] \setminus [2]$, we deduce that $\{p_j\} \cup \{d_{i,q} \mid \ q \in [M]\} \subseteq P'_j$. That is, except for the $n-2$ vertices in $V \setminus \{p_1, p_2\}$, the coalitions that contain the remaining vertices in V' have been fixed. Therefore, $|P'_1| \leq n-1, |P'_2| \leq n-1$, and $|P'_j| \leq 1 + M + n - 2 = M + n - 1$ for all j in $[k] \setminus [2]$. Then we prove that $V \setminus \{p_1, p_2\} \cap \bigcup_{i\in\{3,\dots,k\}} P'_i = \emptyset$. Assume by contradiction that there is a vertex u in $V \setminus \{p_1, p_2\}$ such that u is not in $P'_1 \cup P'_2$. Let j be an index in $[k] \setminus [2]$ such that $P'_j = \mathcal{P}'(u)$. Note that P'_j has at least M out-neighbors of u, $d_{j,q}$ for each q in $[M]$. That is, $v_u(P'_j) \geq \frac{M}{|P'_j|} \geq \frac{M}{M+n-1} = 1 - \frac{n-1}{M+n-1}$, where the last inequality follows by $|P'_j| \leq M + n - 1$. Moreover, $v_u(P'_1 \cup \{i\}) \leq \frac{|P'_1 \cup \{i\}| - 1}{|P'_1 \cup \{i\}|} \leq \frac{n-1}{n} = 1 - \frac{1}{n}$, where the last inequality follows by $|P_1| \leq n-1$. Since $M = n^2 - 2n + 2$, it follows that $\frac{1}{n} = \frac{n-1}{n(n-1)} > \frac{n-1}{n^2-n+1} = \frac{n-1}{M+n-1}$, i.e., $v_u(P'_1 \cup \{i\}) < v_u(P'_j)$. That is, player

u deceases its utility by deviating to P'_1, contradicting the Nash stability of u. Therefore, we conclude that $V' \setminus V \subseteq \bigcup_{i\in\{3,\ldots,k\}} P'_i$ and $V \cap \bigcup_{i\in\{3,\ldots,k\}} P'_i = \emptyset$, i.e., $(P'_1, P'_2) \in \Pi_2(V)$. This completes the proof of Claim 2. $\square$

The theorem below summarizes the main result of this section.

Theorem 5. *For FHGs played on directed weighted graphs where all edges have weight -1, the Nash stable k-partition problem is NP-complete for every fixed $k \geq 2$.*

5 Concluding Remarks

Following the direction of using game-theoretic methods to study community detection, we initiated the study of the fractional hedonic games by restricting the number of coalitions. We considered this scenario from two aspects: social welfare maximization and Nash stability. We applied parameterized complexity theory to understand the computational barriers.

For future work, given our NP-hardness results, we propose to design approximation algorithms or heuristic algorithms. As a starting point, one could study the approximation algorithms for finding social welfare maximizing k-partition in FHGs played on undirected unweighted graphs. Note that for this problem, it is easy to see that the classical algorithm finding a densest subgraph by Goldberg [16] provides a k-approximation algorithm. It is interesting to study whether there is an efficient $O(\log k)$-approximation algorithm. Furthermore, it is interesting to study price of anarchy and price of stability for Nash stable partitions in our model. In particular, study the performance in terms of k. Finally, we conjecture that it is NP-hard to find a Nash stable k-partition on undirected unweighted graphs for all $k \geq 2$, but this remains an open problem.

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