Mamdani-Assilian rules: with or without continuity?

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Abstract

Fuzzy rules and fuzzy inference systems have become the central point of fuzzy modeling since the early beginnings of fuzzy systems. Hence, distinct desirable properties of the rules, their models, and the whole systems are studied. The non-conflictness of rules and/or the preservation of modus ponens seem to be considered the most crucial one(s). However, under the standard setting, such properties are semantically equivalent to the continuity of the modeled dependency. A natural question arises whether such a requirement is consistent with semantics of fuzzy rules. While the answer is positive in the case of implicative rules, in the case of the more often used Mamdani-Assilian rules, we may consider another perspective. This article foreshadows another perspective that could lead to the investigation of a desirable property of the Mamdani-Assilian model that is different from continuity.

Keywords

Fuzzy rules, Mamdani-Assilian rules, inference mechanism, modus ponens, functionality

1. Preliminaries

1.1. Basic concepts

Since we assume that all readers are familiar with the basic concepts of fuzzy sets, we only briefly recall them. A fuzzy set A is defined as a mapping from non-empty universe X to the unit interval, i.e., $A : X \to [0, 1]$. Let us also recall the denotation of the set of all fuzzy sets on a given universe: $\mathcal{F}(X) = \{A \mid A : X \to [0, 1]\}$. A fuzzy relation is a fuzzy set on a Cartesian product of universes, e.g., a binary fuzzy relation R can be an element of $\mathcal{F}(X \times Y)$. Let us recall the definition of two fundamental properties of a fuzzy set.

Definition 1. Fuzzy set $A \in \mathcal{F}(X)$ is called

- *normal* if there exists $x \in X$ such that $A(x) = 1$;
- *bounded* if there exists $x \in X$ such that $A(x) = 0$.

The operations on fuzzy sets can be approached from distinct perspectives; however, probably the most often accepted setting stems from a residuated lattice $\langle [0, 1], \wedge, \vee, \otimes, \to, 0, 1\rangle$; therefore, we adopt it too. Thus, any operations on fuzzy sets appearing in this article come from the above-mentioned algebraic structure, where $\otimes$ is a left-continuous t-norm [1] and $\to$ is its adjoint fuzzy implication [2].

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1.2. Fuzzy rules

If we omit the difference between the syntactical (linguistic) level and the semantic level, fuzzy rules can be viewed as conditional sentences:

$$\text{IF } x \text{ is } A_i \text{ THEN } y \text{ is } B_i \ , \quad i = 1, \dots, n \tag{1}$$

where $A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(Y)$ are antecedent and consequent fuzzy sets, respectively.

If a fuzzy relational inference is considered, *fuzzy rule base* (1) is modeled by a single fuzzy relation on the Cartesian product of both universes $X \times Y$. These fuzzy relations are either formed as the conjunction of implications (so-called *implicative model*):

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \ , \tag{2}$$

or as the disjunction of conjunctions (so-called *Mamdani-Assilian model*):

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) \otimes B_i(y)) \ . \tag{3}$$

Fuzzy relational models of a fuzzy rule base constitute one of the major blocks in the block structure of fuzzy relational inference systems. Another crucial block is the *inference mechanism*. It deals with a fuzzy set $A \in \mathcal{F}(X)$ as an input, and with a direct use of a fuzzy relational model of the given fuzzy rule base, maps it to the output fuzzy set $B \in \mathcal{F}(Y)$. Mathematically, it is defined as an image of a fuzzy set under a fuzzy relation—a concept derived from fuzzy relational compositions/products [3, 4].

The more common one is known under the name *Compositional rule of inference* (CRI) [5], and for an input $A \in \mathcal{F}(X)$ and a relational model of a fuzzy rule base $R \in \mathcal{F}(X \times Y)$ it is defined as follows:

$$(A \circ R)(y) = \bigvee_{x \in X} (A(x) \otimes R(x, y)) \ . \tag{4}$$

Another alternative stems from the composition called *Bandler-Kohout subproduct* [6, 7], and carries the same name. It is defined by the following formula:

$$(A \lhd R)(y) = \bigwedge_{x \in X} (A(x) \to R(x, y)) \ , \tag{5}$$

and it is worth mentioning that it has been firstly proposed as an alternative to CRI already in [8] and later on it has been shown to be an equally appropriate inference mechanism [9].

Note that if we consider a crisp input $x' \in X$ represented by a singleton, i.e., by $A' \in \mathcal{F}(X)$ such that $A'(x') = 1$ and $A'(x) = 0$ elsewhere, both inferences degenerate to a simple and practical substitution that makes the choice between inference mechanisms redundant: $(A' \circ R)(y) = (A \lhd R)(y) = R(x', y)$.

There are numerous research studies focusing on the preservation of desirable properties of fuzzy rules or whole inference systems [10, 11, 12, 13]. Vast majority of them include the preservation of modus ponens or consistency of fuzzy rules. In the first case, we consider the input A to be equal to one of the antecedents A_i, and investigate whether the inferred output B is equal to B_i, which leads to the solvability of fuzzy relational equations [14, 15, 16]. In the latter case, most works consider concepts that define the conflict as the existence of two rules with equal or very similar antecedents but dissimilar consequents. Let us recall, e.g., the *coherence*.

Definition 2. [17] Fuzzy relation $\hat{R} \in \mathcal{F}(X \times Y)$ is called *coherent* if for any $x \in X$ there exists $y \in Y$ such that $\hat{R}(x, y) = 1$.

Clearly, the coherence of a fuzzy relation can be defined for an arbitrary fuzzy relation, not only restrictively as a property of the implicative model $\hat{R}$ of fuzzy rules; however, we avoid doing so by purpose as the definition was intended for $\hat{R}$. Using it for other fuzzy relations, e.g., for $\check{R}$, would be meaningless. An analogous approach for the Mamdani-Assilian model stemming from the fact that conflicting rules in this model do not lower membership degrees, but generate non-convex results, was investigated in [18].

2. What are the desirable properties?

2.1. Preservation of modus ponens

The preservation of modus ponens leads to the solvability of fuzzy relational equations. We recall the most fundamental results that can be found in the literature cited above.

Theorem 1. *Let us consider the following systems of fuzzy relational equations*

$$A_i \circ (\lhd) R = B_i \ , \quad i = 1, \ldots, n \ . \tag{6}$$

Then the system is solvable if and only if $\hat{R}$ ($\check{R}$) is its solution.

Theorem 1 states that $\hat{R}$ has the primary position for the CRI inference while $\check{R}$ keeps the same position for the Bandler-Kohout subproduct inference. Although we may find conditions under which the opposite combinations also preserve modus ponens [19, 20], they usually bring other disadvantages as long as we do not accept additional restrictions, e.g., on the choice of the algebra, see [21]. Therefore, we avoid going into a deeper discussion on assumptions for these combinations, and we simply consider $\hat{R}$ to be the predetermined model for the inference $\circ$ and vice-versa, and analogously we assume that $\check{R}$ and $\lhd$ constitute such a pair too. Note that the latter pair does not constitute a logical inference; however, Mamdani-Assilian rules have their meaningful logical models that have been successfully studied by logical tools [22, 23].

2.2. Preservation of modus ponens as a sort of functionality or continuity

In this section, we show that both the preservation of modus ponens and consistency are closely related to each other and also to the set-theoretic definition of a function, and, consequently, also to (the Lipschitz-type of) the continuity.

If there are inconsistent rules in (1), then there does not exist any fuzzy relation that would solve the related system of fuzzy relational equations. Thus, modus ponens cannot be preserved.

The equivalence of the solvability with (a sort of Lipschitz-like) continuity of the related fuzzy mapping has been demonstrated in [24].

In principle, for two identical inputs (antecedents), considering two different outputs (consequents) is in contradiction with the definition of a function in set theory. And if we consider it in a bit weaker form, i.e., two close inputs cannot lead to two outputs located far from each other, we come to the Lipschitz continuity. And taking into account that the consistency means the nonexistence of conflicting rules, i.e, rules that at the same time impose different outputs for the same inputs, the connection of the preservation of modus ponens and the continuity is straightforward.

The same view is mirrored in the definition of the coherence. Indeed, $a \to b = 1$ if and only if $a \leq b$ for any residual implication. Hence, the requirement of the existence of $y \in Y$ such that $\hat{R}(x, y) = 1$ actually means that for all rules and for arbitrary $x \in X$, there has to be a y such that $A_i(x) \leq B_i(y)$. Therefore, for any input, there is an element in the output universe that belongs to any consequent in a degree higher than or equal to the degree enforced by the respective antecedent. Suppose that there are two rules with identical and normal antecedents A_1 and A_2 but completely different consequents. Then, obviously, for an input x such that $A_1(x) = A_2(x) = 1$, such y cannot be found.

2.3. Another view

If we consider the argumentation mentioned above, it corresponds to the semantics expressed in the conditional form of (1) that is mirrored in the implicative model $\hat{R}$ given by (2). However, the Mamdani-Assilian model $\check{R}$ actually expresses rather the semantics "x is A_i AND y is B_i", with the disjunctive aggregation by the connective OR, see [11, 25]. And then, one can doubt what is wrong with having two identical or close inputs and two different outputs. If rules specify "options" or possibilities (see [11]) a decision-maker has, then there seems to be nothing wrong with rules with similar or even equal antecedents but contradictory consequents.

Consider, for example, the following rules for going around an obstacle:

$$\begin{aligned}&\mathsf{IF}\ obstacle\ \text{is}\ front\ \mathsf{THEN}\ direction\ \text{is}\ left,\\&\mathsf{IF}\ obstacle\ \text{is}\ front\ \mathsf{THEN}\ direction\ \text{is}\ right.\end{aligned}$$

Clearly, if we consider the implicative interpretation of these rules, that is, if they are supposed to hold simultaneously (connected by the AND connective), we observe that it is impossible to fulfill their requirement to change the direction of our vehicle to the left and to the right at the same time, and we obtain inconsistency. However, if we consider the disjunctive interpretation of these rules:

$$\begin{aligned}&obstacle\ \text{is}\ front\ \mathsf{AND}\ direction\ \text{is}\ left,\\&\mathsf{OR}\\&obstacle\ \text{is}\ front\ \mathsf{AND}\ direction\ \text{is}\ right,\end{aligned}$$

we see that we are given two options for going around, and it is up to us which one we choose. However, what is important, these rules implicitly exclude the direction *forward* causing the crash with the obstacle. Note also that, provided that there are three possible directions (*left*, *right* and *forward*), if there were also the third rule "$obstacle$ is $front$ AND $direction$ is $forward$", these three rules together would bear no information, since all possible directions would be equally represented.

From the logical inference point of view, it is not necessary to expect the equality in (6), but an inclusion $A_i \circ \hat{R} \subseteq B_i$ would be sufficient. However, this inclusion is ensured automatically, but the problem is that even an empty fuzzy set on the output of the system, which carries no information, fulfils the inclusion.

Taking this into account and adding the coherence as an additional property to the assumptions, we may obtain the following proposition ensuring that we do not get a meaningless or even empty fuzzy set on the output.

Proposition 1. *Let A be normal and let $\hat{R}$ be coherent. Then $(A \circ \hat{R}) \in \mathcal{F}(Y)$ is normal.*

Proof: Let $x' \in X$ be a point of normality of A. Then,

$$(A \circ \hat{R})(y) = \bigvee_{x \in X} \Big(A(x) \otimes \hat{R}(x, y) \Big) \geq A(x') \otimes \hat{R}(x', y) = \hat{R}(x', y)$$

and, as $\hat{R}$ is coherent, for the given x' there has to exist some y' such that $\hat{R}(x', y') = 1$. □

Consequently, if some antecedent A_i is the input, we automatically obtain the desirable inclusion $A_i \circ \hat{R} \subseteq B_i$, and, jointly with the assumption on the normality of A_i and coherence of $\hat{R}$, we have ensured that the output will be meaningful. Thus, the inclusion $A_i \circ \hat{R} \subseteq B_i$ will not be satisfied by trivial outputs.

The remarks above as well as Proposition 1 are still closely related to the functionality/continuity. However, as we know, the situation between $\circ$ and $\hat{R}$ on the one side and $\lhd$ and $\check{R}$ on the other side is dual [26]. We obtain, for example, an automatic preservation of the following inclusion: $A_i \lhd \check{R} \supseteq B_i$, and, with respect to the inference $\lhd$ that is based on an implication, the meaningless output would be a fuzzy set equal to 1 on the whole universe Y. This naturally brings us to defining a concept for $\check{R}$ that would be dual to the coherence for $\hat{R}$.

Definition 3. Fuzzy relation $\check{R} \in \mathcal{F}(X \times Y)$ is called *concise* if for any $x \in X$ there exists $y \in Y$ such that $\check{R}(x, y) = 0$.

As the concept of coherence is related to the concept of normality, the concept of a concise fuzzy relation $\check{R}$ is closely related to boundedness. This immediately leads to the following proposition.

Proposition 2. *Let A be normal and let $\check{R}$ be concise. Then $(A \lhd \check{R}) \in \mathcal{F}(Y)$ is bounded.*

Proof: Let $x' \in X$ be a point of normality of A. Then

$$(A \lhd \check{R})(y) = \bigwedge_{x \in X} \big(A(x) \rightarrow \check{R}(x, y) \big) \leq A(x') \rightarrow \check{R}(x', y) = \check{R}(x', y)$$

and, as $\check{R}$ is concise, for the given x' there has to exist some y' such that $\check{R}(x', y') = 0$. □

And again, if some antecedent A_i is the input, we automatically obtain the desirable inclusion $A_i \lhd \check{R} \supseteq B_i$, and, jointly with the assumption on the normality of A_i and conciseness of $\check{R}$, we have ensured that the output will be meaningful. Thus, the inclusion will not be satisfied by trivial outputs.

Propositions 1 and 2 state an analogous knowledge that could be expressed as follows: "if the input is significant and the fuzzy rule base model is *correct* (coherent or concise), then the output also brings a significant information". We only have to carefully distinguish between two different models and inferences, which influence what is a *'significant information'*.

2.4. Is functionality always desirable?

Let us move a bit forward in our thoughts on the intuitively expected properties of fuzzy rules, especially the Mamdani-Assilian ones. Consider their crisp variant. Thus, let all antecedents and consequents be classical sets $A_i \subset X, B_i \subset Y$, e.g., intervals. Let the antecedents meet the *finitary* condition, that is, let for each antecedent A_i there is a point x_i such that $x_i \in A_i$ but $x_i \notin A_j$ for any $j \neq i$. Then it is easy to prove that $A_i \circ \hat{R} = B_i$ and $A_i \lhd \check{R} = B_i$ for all i.

Now, let us consider that the rule base (still in the crisp case) describes the driving example of avoiding the obstacle introduced above. There, we have two rules such that $A_i = A_j$ and $B_i \neq B_j$, and they are even disjoint. In the case of implicative rules, this amounts to a clear conflict, incoherence, or inconsistency. Indeed, if the rules were viewed as special axioms of some theory, such a theory would be contradictory. However, if we do not view the rules as special axioms in the conditional form but consider the Mamdani-Assilian form that determines possibilities, we do not observe any contradiction. Then, for the rule base containing two rules with $A_i = A_j$ and disjoint $B_i \neq B_j$, we obtain $A_i \lhd \check{R} = B_i \cup B_j$.

This result is not contradicting anything, nor our intuition. If we have two rules, one of them giving us an option to avoid an obstacle located in front of us by going to the left, the other one giving an option to go to the right, and the observation is that there is an obstacle in front of us, we should deduce the conclusion that we may go either to the left or to the right.

The principal problem is that this view rather fits decision-making situations, not control situations with expected functional dependency, where a defuzzification is employed. Consequently, most of the defuzzifications such as COG or COA would "average" the output which would lead to a frontal collision with the obstacle. However, the problem does not lie in the rules nor in the union of disjoint intervals on the output but in an inappropriately chosen combination of the tool (Mamdani-Assilian interpretation of rules) and the modeled functional dependency. Let us shortly come back to the solvability of systems of fuzzy relational equations. In [27] and then independently in [28], so called *finitary* (originally *boundary*) condition has been defined.

Definition 4. Let $I = \{1, \ldots, n\}$ be the index set and let A_i be normal for $i \in I$. Then A_i are said to meet the *finitary condition* if for any $i \in I$ there exists an $x \in X$ such that $A_i(x) = 1$ and $A_j(x) = 0$ for any $j \neq i$.

The finitary condition has been proved to be sufficient for the solvability of both systems. For distinct proofs and formulations of the problem, we refer to [27, 28, 29]. Let us recall the version devoted to the Bandler-Kohout subproduct.

Theorem 2. [28] *Let A_i meet the finitary condition. Then,*

$$A_i \lhd \check{R} = B_i \ , \quad i = 1, \ldots, n \ . \tag{7}$$

Theorem 2 might be viewed as violating the functionality or continuity idea as it imposes no assumptions on the closeness of the consequents B_i; however, it is simply due to the fact that finitarity has to be understood as "sufficient disjointness" of the input (fuzzy) nodes. And if input nodes are far from each other, the respective output nodes can be arbitrary, and none of the above-mentioned properties is harmed.

As discussed above, preservation of modus ponens is a natural expectation; however, only when a functional relationship is assumed, which is often not the case in decision-making situations, where two or more (disjoint) choices are possible and natural. For such cases, Mamdani-Assilian rules seem to perfectly fit the goal with their semantics. However, it does not mean that we should not expect any reasonable behavior of the system or no reasonable properties should be preserved. Analogously to the case of crisp inputs, where the conciseness of the fuzzy relation $\check{R}$ played the "good property" role, we should be willing to obtain not too general outputs of the system. The following sequence of propositions will provide us with a knowledge showing that Mamdani-Assilian systems can give us natural and reasonable outputs no matter the fact that we are harming the modus ponens.

Proposition 3. *Let $I = \{1, \ldots, n\}$ be the index set and let $i, j \in I$ be such that $A_i = A_j$. Then,*

$$A_i \lhd \check{R} \supseteq B_i \cup B_j \ . \tag{8}$$

Proof: Using the isotonicity of $\to$ in the second argument and the property $a \to (a \otimes b) \geq b$, we get

$$\begin{aligned}(A_i \lhd \check{R})(y) &= \bigwedge_{x \in X} \left(A_i(x) \to \bigvee_{k \in I} (A_k(x) \otimes B_k(y)) \right) \\ &\geq \bigwedge_{x \in X} ((A_i(x) \to (A_i(x) \otimes B_i(y))) \vee (A_i(x) \to (A_j(x) \otimes B_j(y)))) \\ &\geq B_i(y) \vee B_j(y) \ .\end{aligned}$$

□

Note that Proposition 3 does not assume anything, no finitarity or normality is needed. The result is intuitive, but we still do not know whether not 'too much' would be produced by the inference system and what is needed to prevent that the output is trivial, that is, the universal fuzzy set. Therefore, let us add the finitarity.

Proposition 4. *Let $I = \{1, \ldots, n\}$ be the index set and let $i, j \in I$ be such that $A_i = A_j$. Let the set $\{A_k \mid k \in I \setminus \{j\}\}$ meet the finitary condition. Then,*

$$A_i \lhd \check{R} = B_i \cup B_j \ . \tag{9}$$

PROOF: Let $x' \in X$ be such that $A_i(x') = 1$ and $A_k(x') = 0$ for any $k \neq i, j$. Then we get

$$(A_i \lhd \check{R})(y) = \bigwedge_{x \in X} \left(A_i(x) \to \bigvee_{k \in I} (A_k(x) \otimes B_k(y)) \right) \leq \left(A_i(x') \to \bigvee_{k \in I} (A_k(x') \otimes B_k(y)) \right)$$
$$= 1 \to \left((1 \otimes B_i(y)) \vee (1 \otimes B_j(y)) \vee \bigvee_{k \in I \smallsetminus \{i,j\}} (A_k(x') \otimes B_k(y)) \right)$$
$$= B_i(y) \vee B_j(y) \vee \bigvee_{k \in I \smallsetminus \{i,j\}} (0 \otimes B_k(y)) = B_i(y) \vee B_j(y) \; .$$

□

Proposition 4 provides us with a valuable result, i.e., that the output of the system is the desirable union of both consequents of the fully fired rules. Thus, the system does not build a confusing fog of information around the necessary one we want to be given.

3. Conclusions

We have recalled the basic components of fuzzy inference systems and the most frequently discussed desirable properties, namely, the consistency of rules that is also mirrored in the preservation of modus ponens. We showed that under the standard setting this property leads to the functionality or continuity of the model. However, taking into account the semantic meaning of the Mamdani-Assilian rules, this requirement does not seem that natural. An alternative approach to investigating the “correct” behavior of Mamdani-Assilian rules is foreshadowed.

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