The Signature Investigation of one Class of Universal Boolean Algebras

Ihor Mych, Volodymyr Nikolenko and Olena Vartsaba

Uzhhorod National University, University Str., 14, Uzhhorod, 88000, Ukraine

Abstract

The paper has been introducing new concepts such as universal Boolean algebra, l – basic algebras, free and canonical algebras, functionally complete and incomplete algebras, threshold universal algebra. The class of universal Boolean algebras M_2 is represented in the form of an eleven-dimensional signature cube. This cube has been divided into four nine-dimensional cubes M_2^1, M_2^2, M_2^3, M_2^4. Also the set of functionally incomplete algebras are found and the signature graph of this class is constructed. The class of all functionally complete algebras has been divided into fifteen classes and the signature graphs of each of these classes were constructed. All functionally complete algebras have been represented in the form of a signature graph. The powers of a class canonical algebras and free algebras have been installed and these signature graphs have been constructed. In the second part of paper the class of algebras M_2 divided into four subclasses: internal functionally incomplete algebras, threshold functionally incomplete algebras, threshold functionally complete algebras and internal functionally complete algebras. Signature graphs of classes are constructed, the isomorphism of some subclasses of threshold functionally complete algebras is proved, the power of these classes and the location of algebras on signature graphs are determined. The signature graph of all threshold functionally complete algebras is given. Also, the concept of basic equivalence is introduced in the paper, the factor-grating M_2/σ of the class of algebras M_2 is obtained.

Keywords [1]

Universal Boolean Algebra, Signature Cube, Basis

1. Introduction

The Theory of Boolean functions is one of the most important sections of discrete mathematics. It is the theoretical foundation of modern theories such as machine learning, artificial intelligence, methods of decision support and fuzzy logic.

In the paper, results are summarized obtained in works [1,2]. The study of theories of universal algebras laid the fundamental work of Birghof [3], and they were continued in the works of Maltsev [4], Kurosh [5] and others. The most famous investigations of the theory of Boolean algebras by Sikorsky [6], of the theories of Boolean functions by Glushkov [9], Zhuravlev [12], of modern works [9, 10] and many others.

2. Universal Boolean Algebras

In this research, we will propose to consider the theory of Boolean functions as Universal Boolean Algebras.

II International Scientific Symposium «Intelligent Solutions» IntSol-2021, September 28–30, 2021, Kyiv-Uzhhorod, Ukraine
EMAIL: ihor.mych@uzhnu.edu.ua (A. 1); volodymer.nikolenko@uzhnu.edu.ua (A. 2); olena.vartsaba@uzhnu.edu.ua (A. 3)
ORCID: 0000-0002-3392-1442 (A. 1); 0000-0003-0071-6896 (A. 2); 0000-0001-9158-2365 (A. 3)

Definition 1. Universal algebra U is called the compile pair $\langle A,\Omega\rangle$ at set A (porter of algebras) and Ω (set of operations based on A) [3, 4, 5].

Definition 2. Universal Boolean Algebra $U=\langle A,\Omega\rangle$ is called Universal Algebra, where $A=\{0,1\}$, Ω – set of Boolean operations. In the following Universal Boolean Algebras, we will call Boolean Algebras or Algebras.

Consider the set of Universal Boolean Algebras M to the signature of them enter the operations $Q=\{0, 1, \neg, \wedge, \vee, \oplus, \Rightarrow, \Leftarrow, \Leftrightarrow, \uparrow, |\}$ that is $\mathrm{M}=\{U_i=\langle A,\Omega_i\rangle | \Omega_i \subset Q\}$. Of course, $|\mathrm{M}|=2^{11}=2048$ algebras are formed 11-dimensional cube. In the following, we will call Signature cube or Ω-cube. Each vertex of Ω-cube put on correlation the Boolean vector of length 11. In this Boolean vector true (one) value defines the operations which enter the signature corresponding algebra. Vertexes of Ω-cube are reflected signature (list of operations), Ω-vector or natural number, the decomposition of them modulo 2 defines Ω-vector. Order relation of Ω-cube represented in the following forms: $U_i=\langle A,\Omega_i\rangle \le U_j=\langle A,\Omega_j\rangle$, if $\Omega_i \subset \Omega_j$. Zero values of Ω-cube is trivial algebra, where $\Omega=\varnothing$. Ω-cube has $C_n^1=11$ algebras on the first circle and on each consecutive circle – C_n^k, $n=11$, $k=\overline{1,11}$. An edge of Ω-cube join algebras $U_i=\langle A,\Omega_i\rangle$ and $U_j=\langle A,\Omega_j\rangle$, where $U_j>U_i$ increase (diminish) signature U_j for one more operation in comparison with U_i, if moves into the edge from down to up (up to down).

Definition 3. Algebra $U=\langle A,\Omega\rangle$ is called functionally complete if the set of it functions, that is accord operation from Ω, are formed the functionally complete systems otherwise algebra is called functionally incomplete [11].

Definition 4. Algebra $U=\langle A,\Omega\rangle$ is called $l-$basic if with operations of signature Ω can be construct $l-$ bases.

Definition 5. An operation $f_i \in \Omega$ is called connected if it enters to composition to some base otherwise operation is called free operation.

Definition 6. Algebra $U=\langle A,\Omega\rangle$ is called free algebra if its signature has free operations.

Definition 7. Algebra $U=\langle A,\Omega\rangle$ is called canonical algebra if it doesn't have free operations.

Definition 8. The rank of free algebra is called a number that is equal numerosity of free operations.

Definition 9. Algebra $U=\langle A,\Omega\rangle$ is called saturated if its arbitrary expansion of signature increases the basis of algebra.

Definition 10. The potential of edge that connects two adjacent algebras U_1 and U_2 of $\Omega-$cube with basics η_1 and η_2 ($\eta_1 \ge \eta_2$) is called number $\eta_1-\eta_2$.

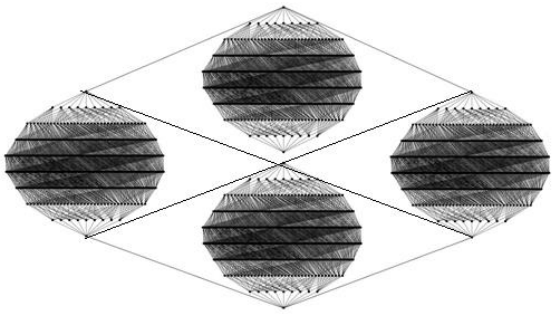
Definition 11. The potential of algebra is called the sum potentials of all edges (upper) going out of the algebra.

3. Signature Cube of the class algebras M_2

Consider the set of algebras $\mathrm{M}_2=\{U=\langle A,\Omega\rangle\}$, where $A=\{0,1\}$ and Ω – set of the Boolean operations arnost of them does not exceed two. The class of algebras M_2 can be represented as an 11-dimensional signature cube in Figure 1.

Construct partition of the set $\mathrm{M}_2=\mathrm{M}_2^1 \cup \mathrm{M}_2^2 \cup \mathrm{M}_2^3 \cup \mathrm{M}_2^4$, where M_2^1 – the set of algebras M_2 its signature is not contained the Pierce's arrow $(\uparrow)$ and the Sheffer's touch $(|)$, and M_2^2, M_2^3, M_2^4 – sets of all algebras M_2, the signature of them is contained the operations $\{\uparrow\}$, $\{|\}$, $\{\uparrow,|\}$.

Consider the set of algebras of the class M_2^1 formed an 11-dimensional signature cube ($\Omega-$cube) with the fixed tenth and eleventh zero value coordinates (Figure 2).



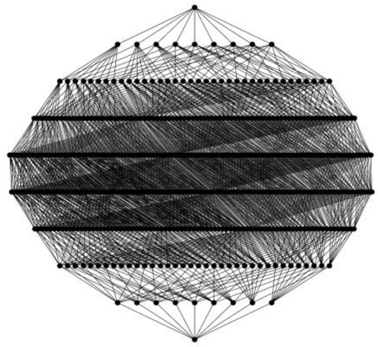


Figure 1: Signature Cube of the class algebras M_2 **Figure 2**: Signature Cube of the class algebras M_2^1

4. Class of Functionally Incomplete Algebras η_0.

Denote the set of functionally incomplete algebras by η_0. This class contains eighty-eight algebras. The signature graph of functionally incomplete algebras is shown in Figure 3.

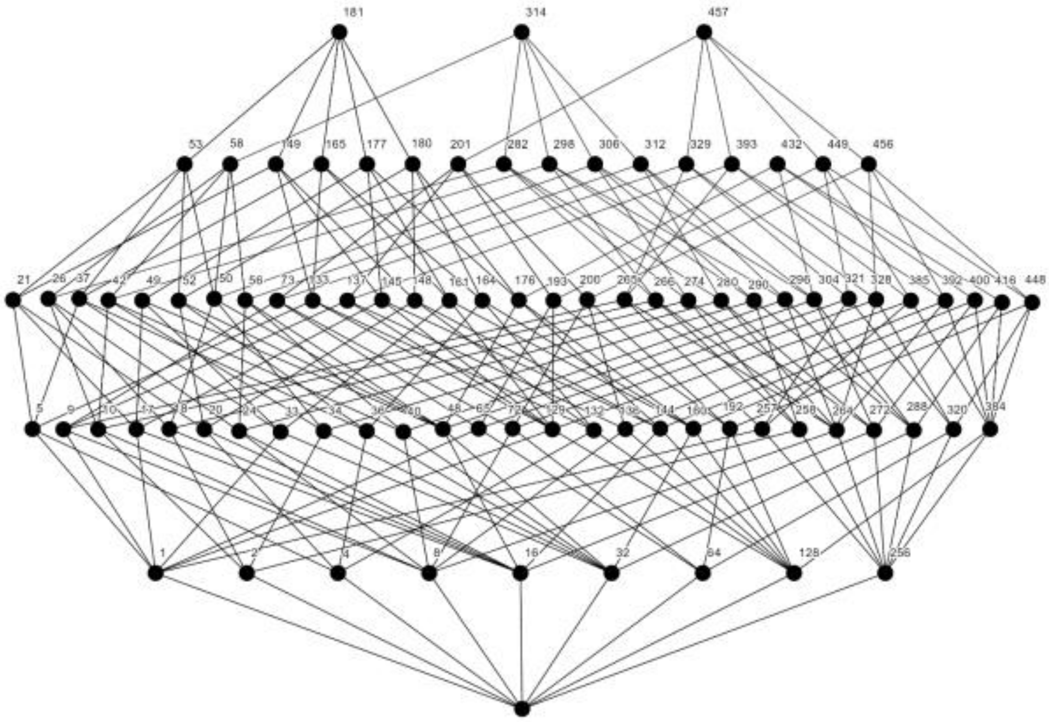


Figure 3: Signature Graph of the Class of Functionally Incomplete Algebras η_0

From definitions 4, 9, 10 we get the following assertion.

Assertion 1.

1. $l-$base of algebras of class η_0 is equal to zero.
2. The potential of all edges of the Ω-cube is equal to zero represented in Figure 3.

To each vertex of $n-$dimensional Ω-cube conduct n edges $(n=9)$. The number of edges conducted to each vertex of the graph represented in Figure 3 is less than nine (except trivial algebra).

Absent edges connect algebras with the functionally complete of algebras. The following assertion takes place.

Assertion 2. The trivial algebra is the sole internal functionally incomplete algebra, another eighty-seven algebras are the threshold.

Definition 9 follows that the class η_0 contains four saturated (pre-full) algebras with signatures $\{0,1,\wedge,\vee\}$, $\{0,1,\neg,\oplus,\Leftrightarrow\}$, $\{0,1,\neg,\oplus,\Leftrightarrow\}$, $\{1,\wedge,\vee,\Rightarrow,\Leftrightarrow\}$. From Definition 6, 8 we get the following assertions.

Assertion 3.

1. All algebras of class η_0 are free.
2. The rank of each algebra $U_i = \langle A,\Omega_i \rangle \in \eta_0$ equals the power signature Ω_i or number of the circle where algebra is situated.

5. Class of Functionally Complete of Algebras.

Construct partition of the set $\mathrm{M}_2^1 = \eta_0 \cup \eta_1 \cup \eta_2 \cup ... \cup \eta_{15}$, where $\eta_l -$ the set of $l-$basic of algebras, $l = 0,1,...,15$. The class of functionally incomplete algebras η_0 is defined in the previous sections. Consider the class of one-basic algebras η_1. The power of this class η_1 is equal to seventy-two. $\Omega-$graph of class η_1 represented in Figure 4.

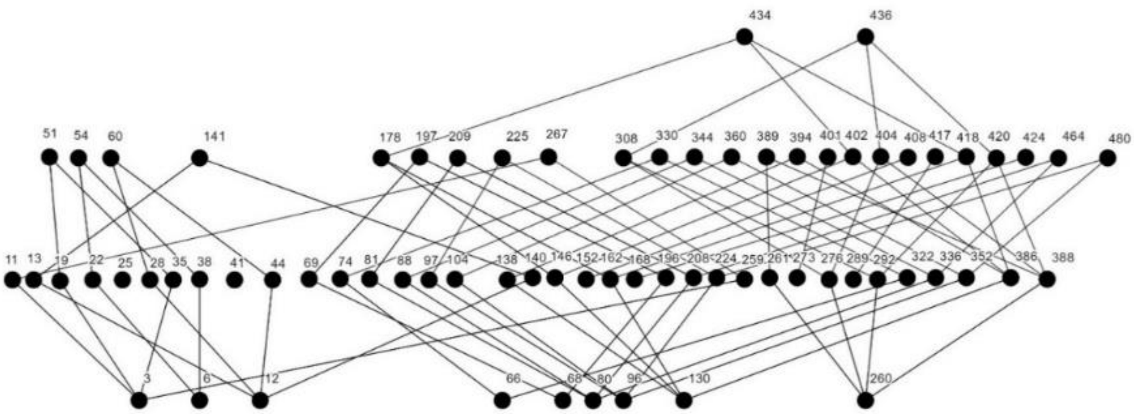


Figure 4: $\Omega-$graph of algebras η_1 of the class M_2^1.

From Figure 4 we can see that algebras are situated in the circle as follows:

- second circle is contained nine canonical unsaturated algebras with two-operation bases;
- third circle is contained six canonical unsaturated algebras with three-operation bases, thirty free algebras which obtained by corresponding algebras second circle in way of expansion for one more operation. Since the basicity of algebras is the same as adjacent algebras of the second circle, therefore, the rank of these algebras is equal to one, and the edges which it connects have a potential value zero;
- fourth circle includes twenty-five free algebras, where there the rank is equal to two; saturated algebras are all algebras of this circle without algebras with numbers 178, 308, 402, 404, 418, 420.
- fifth circle is contained two free saturated algebras 434, 436 where the rank is equal to three.

Based on performed analysis for canonical algebras of class η_1 takes place assertion.

Assertion 4. Class algebras include seventy-two algebras, where

- fifteen canonical algebras: two saturated canonical algebras and twenty-one saturated free algebras;
- fifty-seven free algebras: thirty algebras have the rank is equal to one, twenty-five algebras have the rank is equal to two, two algebras have the rank is equal to three.

The type of algebra we can recognize on the $\Omega-$graph by the features described below.

6. l – Basic Algebras

Class η_2 contains one hundred and five algebras. The third circle is contained the fourteen canonical unsaturated algebras. The fourteen canonical saturated algebras, twelve canonical unsaturated algebras, and thirty-five free unsaturated algebras are on the fourth circle where their rank is equal to one. The thirty free saturated algebras with the rank value two are contained on the fifth circle. $\Omega-$ graphs for the classes $\eta_l, l = 2,3..., 6$, are shown in Figures 5-9.

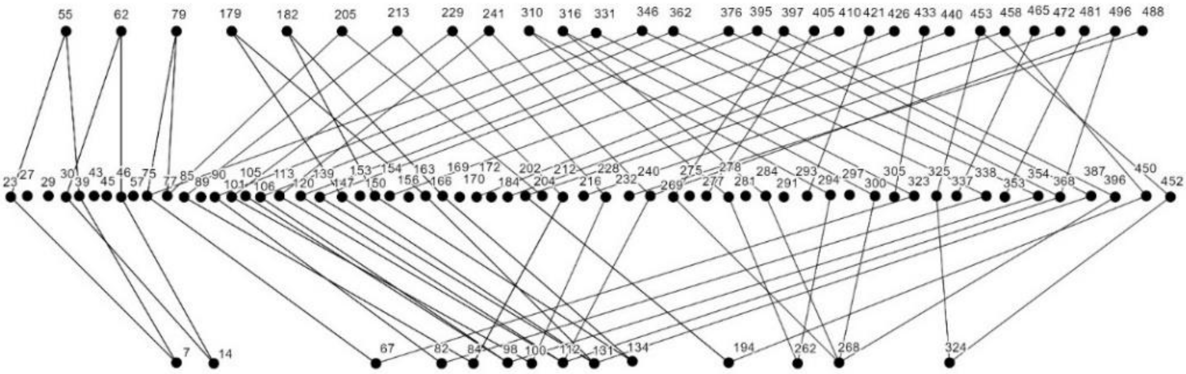


Figure 5: $\Omega-$ graph of algebras η_2 of the class M_2^1

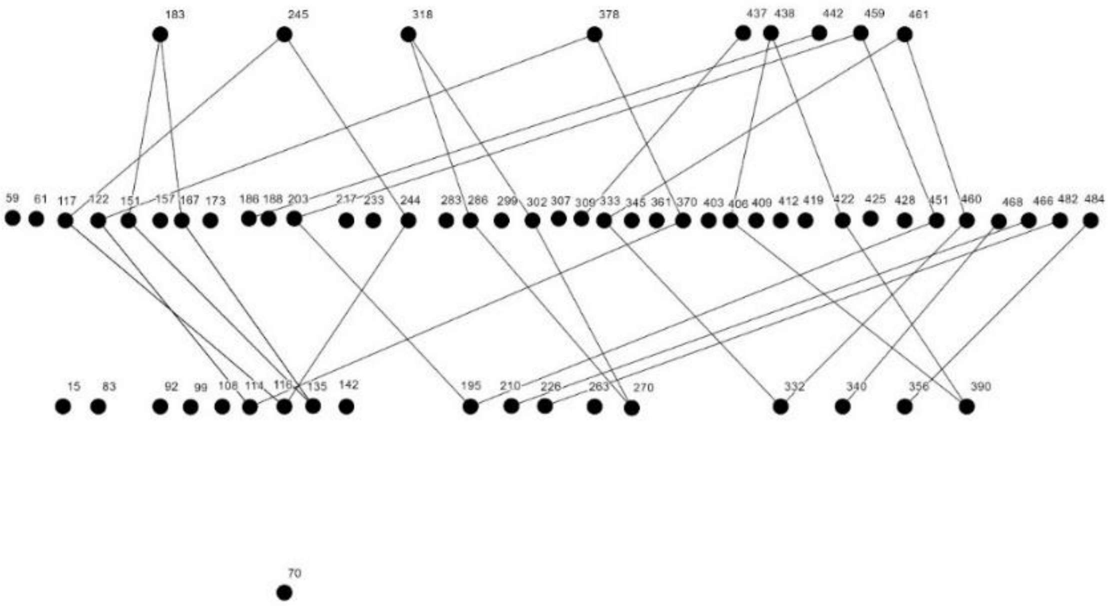


Figure 6: $\Omega-$ graph of algebras η_3 of the class M_2^1

The sets of algebras of classes η_7 - η_{15} are canonical saturated algebras and their graphs are contained only isolated points:

$\eta_7 = \{95, 111, 222, 238, 343, 359, 247, 382, 463, 475, 477, 491, 493, 499, 502, 508\}$;

$\eta_8 = \{191, 253, 319, 379, 415, 431, 443, 445, 471, 478, 487, 494, 505\}$;

$\eta_9 = \{127, 223, 239, 251, 254, 351, 367, 375, 381\}$; $\eta_{10} = \{503, 510\}$;

$\eta_{11} = \{447, 479, 495, 507, 509\}$; $\eta_{12} = \{255, 383\}$; $\eta_{15} = \{511\}$.

From the obtained results we get the following assertions.

Assertion 5. The class of algebras M_2^1 contains four hundred twenty-four functionally complete algebras.

The number of $l-$basic algebras, $l=1,2,\dots,15$ and their location in circle of $\Omega-$cube are given in Table 1.

Table 1
The number of $l-$base algebras $\Omega-$cube

Base	1	2	3	4	5	6	7	8	9	10	11	12	15
Number	72	105	66	57	39	37	16	13	9	2	5	2	1
Circle													
2	9												
3	36	14	1										
4	25	61	18	6									
5	2	30	38	39	15								
6			9	12	24	33	6						
7						4	10	13	9				
8										2	5	2	
9													1

For example, two one-base, thirty-two-basic, thirty-eight three-basic, thirty-nine four-basic, and fifteen five-basic algebras are contained on the fifth circle. The three-basic algebras are situated as follows: one algebra – on the third circle, eighteen algebras – on the fourth circle, thirty-eight – on the fifth circle, and nine algebras – on the sixth circle.

Assertion 6. The two hundred sixty-four canonical algebras exist in the class algebras M_2^1 .

These algebras are formed the signature grating which is represented in Figure 10 and the distribution of algebras by the number of basics is given in Table 2.

All the functionally incomplete algebras are free algebras where the rank is equal to the power of signature.

Base	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Quality	248	337	348	294	219	172	129	82	51	33	18	14	9	2	1	2	1

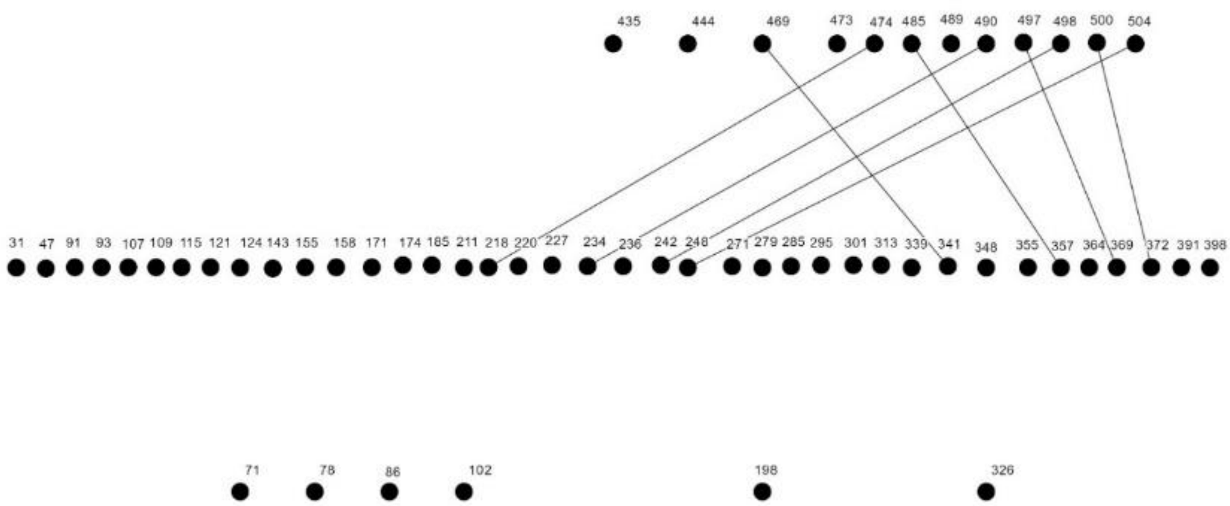


Figure 7: $\Omega-$graph of algebras η_4 of the class M_2^1

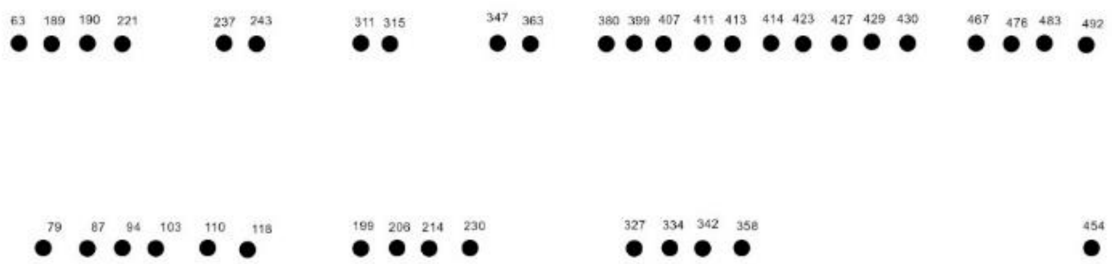


Figure 8: $\Omega-$graph of algebras η_5 of the class M_2^1

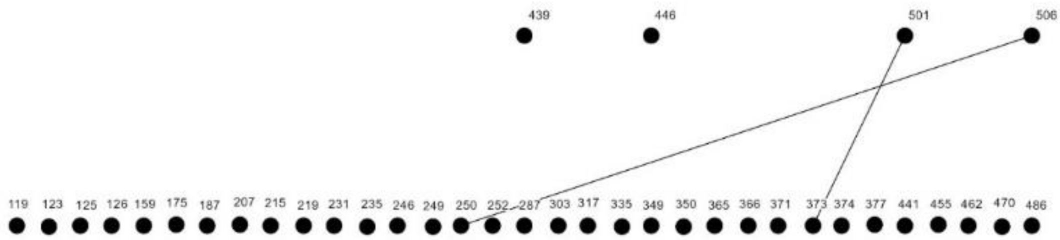


Figure 9: $\Omega-$graph of algebras η_6 of the class M_2^1

Assertion 7. The two hundred fifty-nine free algebras exist in the class algebras M_2^1.
The distribution of free algebras by rank is shown in the following Table 3.

Table 2
The distribution of algebras by the quantity of bases

Base	1	2	3	4	5	6	7	8	9	10	11	12	15
Quantity of algebras	15	39	39	49	39	35	16	13	9	2	5	2	1

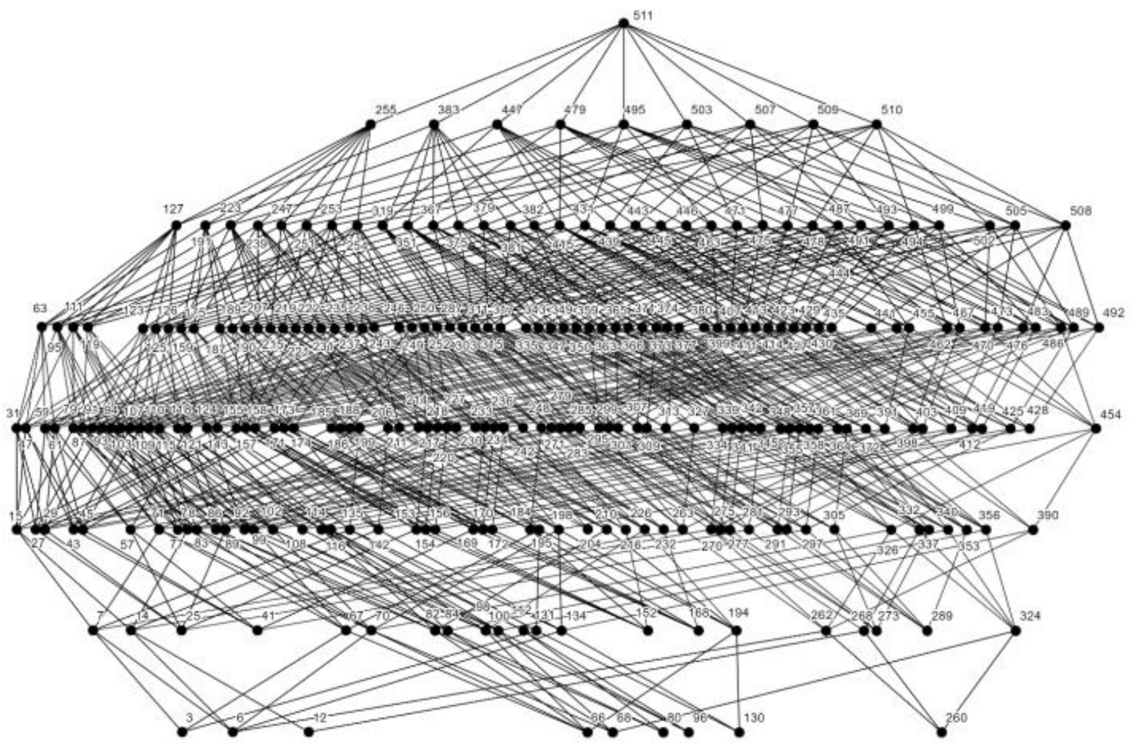


Figure 10: Canonical algebras of the class M_2^1

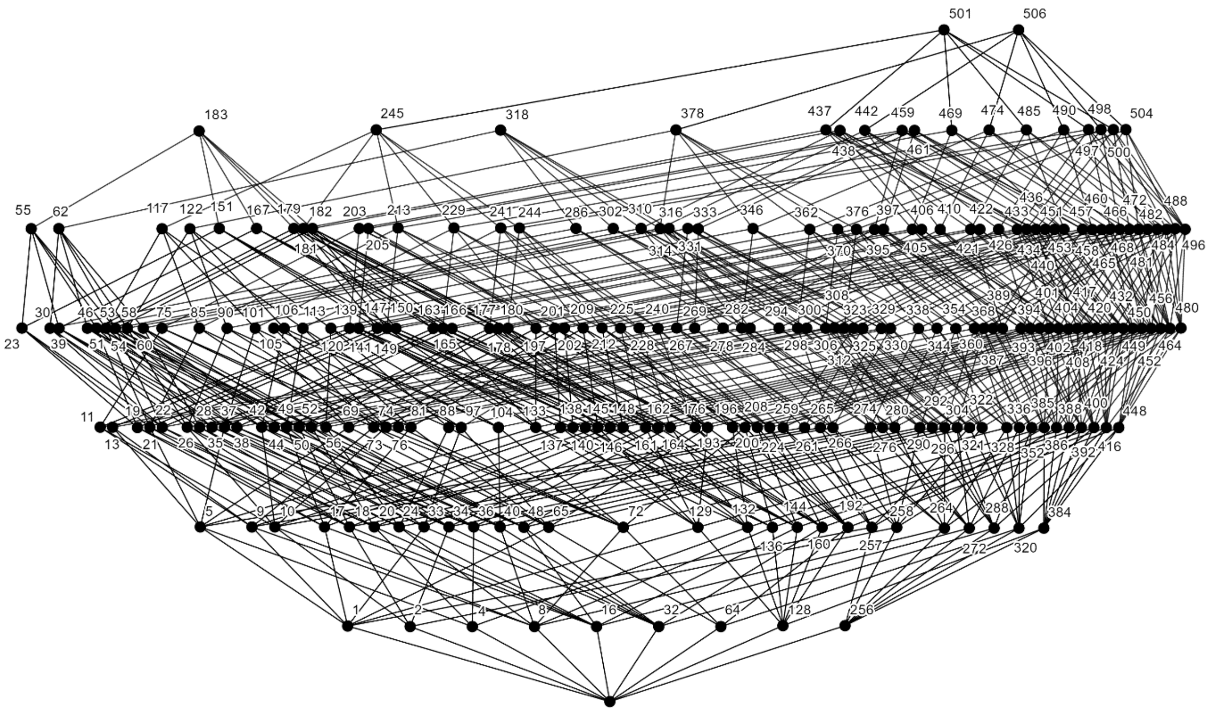


Figure 11: Free algebras of the class M_2^1

Table 3
The distribution of free algebras by the rank

Rank	0	1	2	3	4	5	6
Quantity	88	59	65	27	8	-	2

The free algebras of the class M_2^1 form the signature graph are represented in Figure 11. This graph is obtained from signature graph of sets M_2^1 by the withdrawal canonical algebras.

Consider generalization of results is obtained from the class M_2^1 to the classes M_2^2, M_2^3, M_2^4, that is class M_2. The class of algebras M_2^2 forms 11-dimensional vectors with the fixed tenth one value coordinate and eleventh zero value coordinate. We can construct a 9-dimensional $\Omega-$cube similar the $\Omega-$cube of the class M_2^1. For the each $k-$base algebra of class M_2 exist sole corresponding $k+1-$basic algebra of the class M_2^1 (in the corresponding algebras, the first nine coordinate coincide).

Similar considerations we can develop for algebras of classes M_2^3 and M_2^4. The basicity of algebras of class M_2^3 in relation to M_2^1 increase to two. We can determine the quality $k-$basic algebras of the class M_2 defined by $|\eta_k|+2|\eta_{k-1}|+|\eta_{k-2}|$, where $|\eta_k|-$the number $k-$basic algebras of the class M_2^1.

Assertion 8. The class of algebras M_2 contains 2048 algebras where 88 algebras are functionally incomplete.

7. Threshold Universal Algebras

Definition 12. Algebras $U_1=\langle A,\Omega_1\rangle$, $U_2=\langle A,\Omega_2\rangle$ are called adjacent algebras if exist edge which connects these algebras.

An adjacent algebras U_1 and U_2 differ only one operation in Ω-cube that is $|\Omega_1 \cup \Omega_2 - \Omega_1 \cap \Omega_2| = 1$. An adjacent algebras are located in nearby circle Ω-cube.

Definition 13. Algebra $U = \langle A, \Omega \rangle$ is called threshold if it is contained an adjacent functionally complete and functionally incomplete algebras.

Definition 14. Algebra $U = \langle A, \Omega \rangle$ is called internal if all adjacent algebras are functionally complete or functionally incomplete.

Definition 15. Functionally incomplete algebra is called pre-full if the expansion of signature converts algebra to functionally complete algebra.

Boolean algebras of class M_2 consist four subclasses: $M_1 -$ class of internal functionally incomplete algebras; $M_2 -$ class of threshold functionally incomplete algebras; $M_3 -$ class of threshold functionally complete algebras; $M_4 -$ class of internal functionally complete algebras [1].

In Figure 3 the graph of functionally incomplete algebras is shown.

Assertion 9. All functionally incomplete algebras of class M_2 are threshold.

Assertion 10. Functionally complete algebra is a threshold algebra if and only if there are operation enters into all bases of this algebra.

Construct partition of the set of threshold functionally complete algebras M_3 to subclasses $M_3 = M_3^1 \cup M_3^2 \cup ... \cup M_3^9 \cup M_3^{10} \cup M_3^{11}$, where $M_3^i,\ i = \overline{1,11} -$ the set of all threshold functionally complete algebras which have adjacent edges are connected with algebras of class M_2. The each an adjacent edge responds i^{th} operation of set Q. It means that the signature of algebras of class M_3^i enters i^{th} operation and in the signature of corresponding adjacent algebras this operation doesn't enter. Denote the class of functionally incomplete algebras by M_2^{+i} where signature is contained i^{th} operation and M_2^{-i} – the class of algebras is got from M_2^i by the withdrawal of i^{th} operation of signature that is

$$M_2^i = M_2 - \left(M_2^{+i} \cup M_2^{-i}\right). \qquad (1)$$

Class of algebras M_3^i we can get with M_2^i by the expansion of signature i^{th} operation. If signature all of algebras M_2 expanse i^{th} operation that is algebras M_2^{+i} and M_2^{-i} remain functionally incomplete and another algebras converts to class functionally complete algebras. Since $|M_2| = 88$ and $|M_2^{+i}| = |M_2^{-i}|$ we get $|M_2^i| = 88 - 2|M_2^{+i}|$. Based on (1) is given Table 4.

Table 4

Number of algebras where the signature is contained i^{th} operation

i	1	2	3	4	5	6	7	8	9	10	11
M_2^{+i}	33	33	16	32	32	30	16	16	30	0	0
M_2^i	22	22	56	24	24	28	56	56	28	88	88

Definition 16. Graphs G and H are called isomorphic if it is possible to establish between their vertexes such a bijection f that two vertexes u, v of a graph are adjacent if and only if $f(u)$ and $f(v)$ adjacent vertexes in the graph H.

The problem of recognizing isomorphisms of graphs belongs to the class NP - full. In [19] it was argued that "the search for individual criteria and features of isomorphisms for graphs of a class can be very difficult and not always successful." Modern papers of graph theory [20] are confirmed that the problem of graph isomorphism has not been solved.

Denote by $G_i - \Omega$-graph that is defined the class of threshold functionally complete algebras M_2^i. For each class M_2^i graphs of the signature $G_1 - G_9$ shown in Figures 12 - 16.

Assertion 6. The Ω-graphs are isomorphic: $G_1 \cong G_2$, $G_4 \cong G_5$, $G_6 \cong G_9$, $G_7 \cong G_8$, $G_{10} \cong G_{11}$.

The graphs $G_1 - G_9$ are combined into one graph of signature depicted in Figure 17.

The isomorphism of the signature graphs $G_i, i = \overline{1,9}$ is proved by the bijection of the vertexes given in Table 5.

Definition 17. Algebra $U = \langle A, \Omega \rangle$ is called saturated in a class K if, as a result arbitrary extension of the signature, we obtain an algebra that does not belong K.

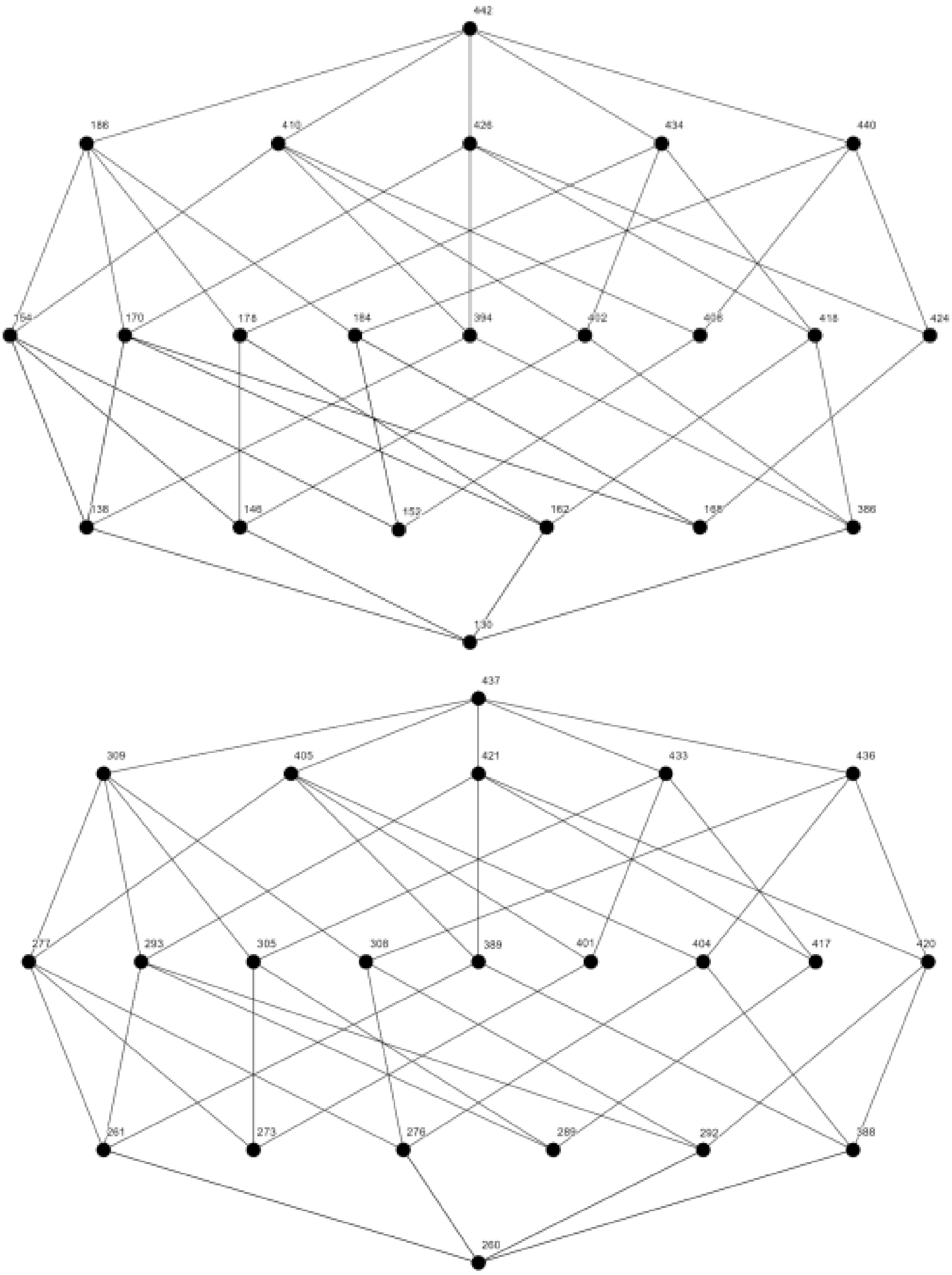


Figure 12: Graphs G_1 and G_2 of the class threshold algebras M_2^1 i M_2^2

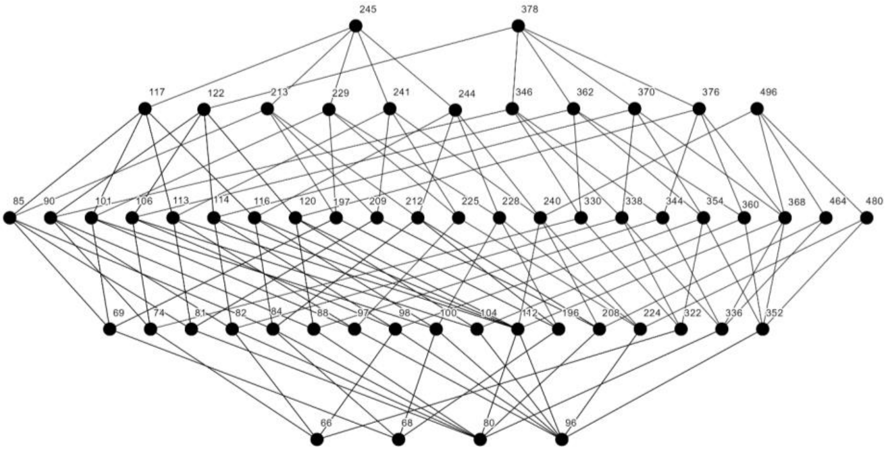


Figure 13: Graphs G_3 of the class threshold algebras M_2^3

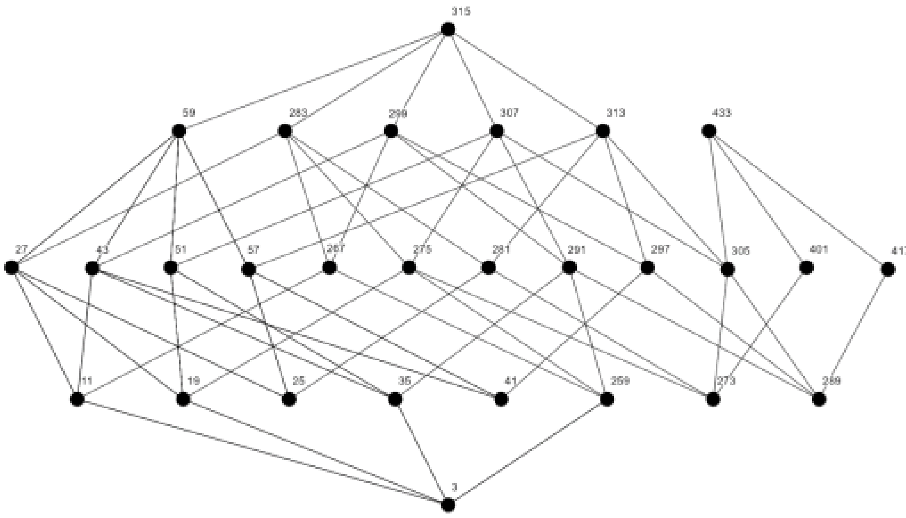


Figure 14: Graphs G_4 and G_5 of the class threshold algebras M_2^4 and M_2^5

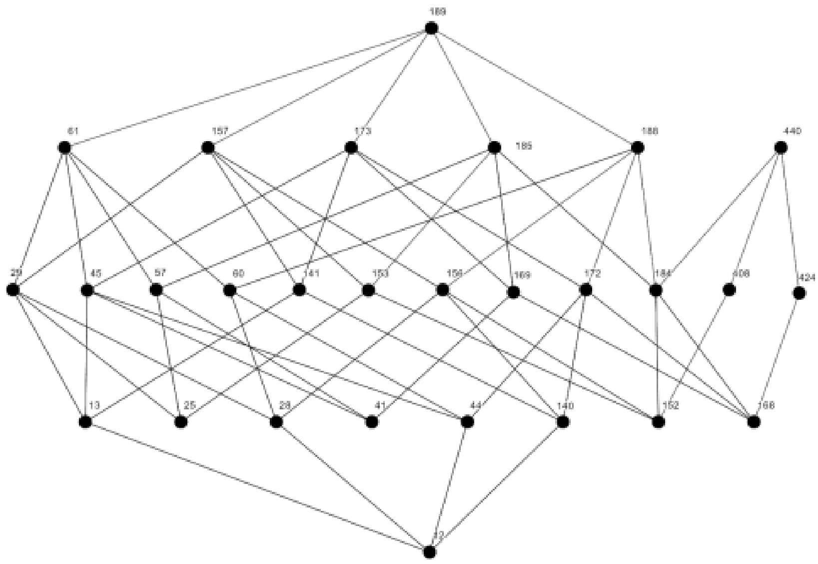


Figure 15: Graphs G_6 and G_9 of the class threshold algebras M_2^6 and M_2^9

Based on Figure 17 the Ω-graph of a class G of threshold algebras M_2 include two hundred and nineteen algebras; fifteen canonical algebras with numbers 3, 6, 12, 66, 68, 80, 96, 130, 260, 25, 41, 152, 168, 273, 289; twelve the threshold algebras with numbers 183, 189,245, 315, 318, 378, 437, 442, 459.

Consider the classes of algebras M_2^{1-10}, $M_2^{1-10,11}$, M_2^{1-11} that have been from M_2^{1-9} the extension of their signatures by the Pierce's arrow $(\uparrow)$ and the Sheffer's touch $(|)$. $M_3^{1-9,10}$, $M_3^{1-9,11}$ – the classes of threshold algebras, and M_3^{10-11} – the class of internal functionally complete algebras. G_2, G_2^{10}, G_2^{11}, G_2^{10-11} – Ω-graphs of the corresponding classes M_2, M_2^{1-10}, $M_2^{1-10,11}$, G_2^{1-11}. These graphs are isomorphic and their unification forms a Ω-graph represented schematically in Figure 18. Based Ω-graph represented in Figure 7 takes place the following theorems.

Theorem 1. Internal functionally incomplete algebras in the class M are absent.

Theorem 2. The set of Boolean algebras M consists of 88 threshold functionally incomplete algebras; 395 threshold functionally complete algebras; 1565 internal functionally complete algebras.

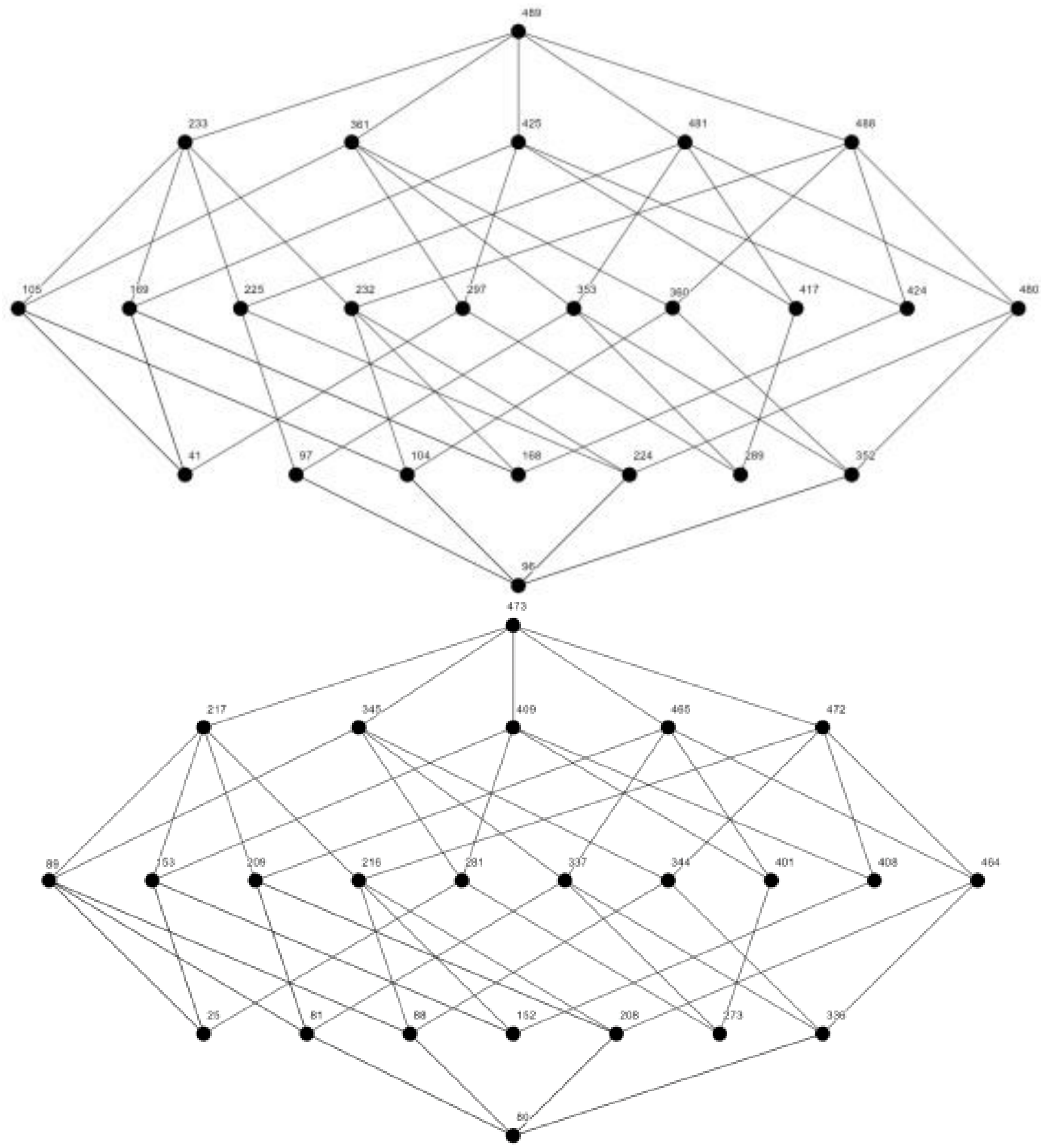


Figure 16: Graphs G_7 and G_8 of the class threshold algebras M_2^7 and M_2^8

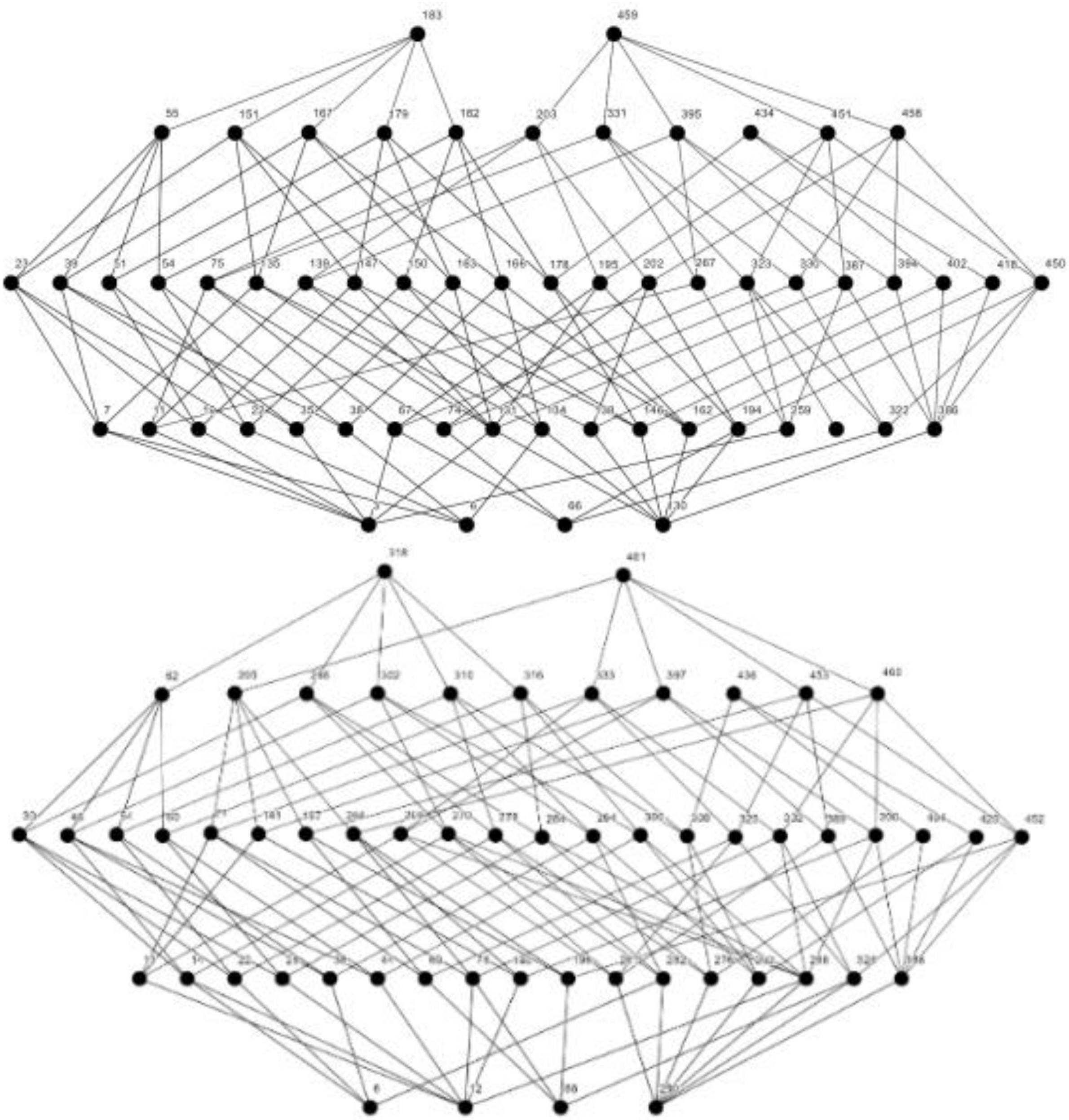


Figure 16: (continue)

8. Basic Equivalence in the Class of Universal Boolean Algebras

We can construct nine two-operation basics $a_1 = \{0, \Rightarrow\}$, $a_2 = \{0, \overline{\Rightarrow}\}$, $a_3 = \{\neg, \wedge\}$, $a_4 = \{\neg, \vee\}$, $a_5 = \{\neg, \Rightarrow\}$, $a_6 = \{\neg, \overline{\Rightarrow}\}$, $a_7 = \{\oplus, \Rightarrow\}$, $a_8 = \{\overleftarrow{\Rightarrow}, \Rightarrow\}$, $a_9 = \{\overleftarrow{\Rightarrow}, \Leftrightarrow\}$ and six three-operation basics $a_{10} = \{0, \wedge, \Leftrightarrow\}$, $a_{11} = \{0, \vee, \Leftrightarrow\}$, $a_{12} = \{1, \wedge, \oplus\}$, $a_{13} = \{1, \vee, \oplus\}$, $a_{14} = \{\wedge, \oplus, \Leftrightarrow\}$, $a_{15} = \{\vee, \oplus, \Leftrightarrow\}$ and two single-operation basics $a_{16} = \{\uparrow\}$, $a_{17} = \{\mid\}$. It is possible to form 2^{17} various combinations of basics from seventeen basics. The Universal Boolean Algebras do not exist for most combinations from the operations of which only those basics can be constructed that are included in the combination. But Universal Boolean algebras are existing with signatures from the operations of which we can build the same set of basics.

Each algebra $U_i = \langle A, \Omega \rangle \in M_2$ is matched with a seventeen-dimensional Boolean vector $H_i = \{\alpha_1^i, \alpha_2^i, \ldots, \alpha_{17}^i\}$, where $\alpha_j^i = 1$, if the operations Ω can form j-basic and $\alpha_j^i = 0$ otherwise. The vector H_i is called the characteristic basis vector of algebra U_i. Denote by $B(U_i)$ the set of all basics of algebra $U_i = \langle A, \Omega_i \rangle$ from the operations that are included to Ω.

Definition 18. Algebras U_1 and $U_2 \in M_2$ are called basically equivalent $U_1 \overset{\sigma}{=} U_2$ if $B(U_1) = B(U_2)$. $U_1 \overset{\sigma}{=} U_2$ if and only if $H_1 = H_2$, where $\sigma -$ the equivalence relation.

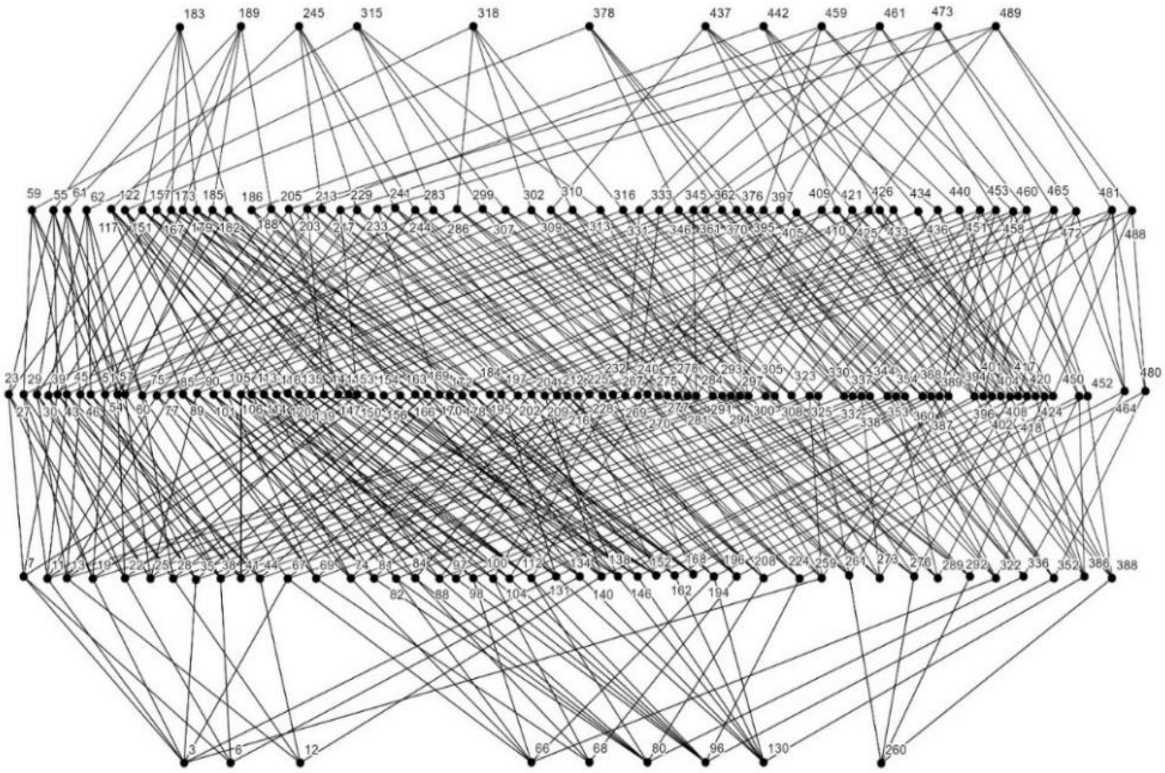


Figure 17: Ω-graph of threshold functionally algebras M_2^i

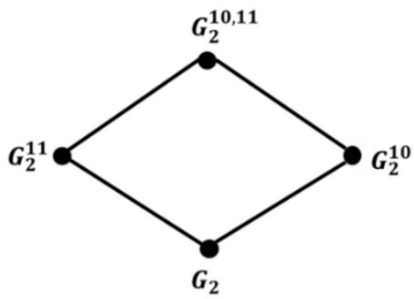


Figure 18: Ω-graphs of class algebras $M_2, M_2^{1-10}, M_2^{1-10,11}, G_2^{1-11}$

Let us construct a factor-grating M / σ by basis equivalence using characteristic basis vectors.

Algebras that come in the zero value element of the factor-grating have a characteristic basis vector $(0,0,\ldots,0)$. The maximum element of the factor-grating is an algebra U^* such that $M^* = (1,1,\ldots,1)$ the signature of the element includes all seventeen operations.

If algebras $U_1 = \langle A, \Omega_1 \rangle$ and $U_2 = \langle A, \Omega_2 \rangle$ have characteristic basis vectors $H_1 = \{\alpha_1^1, \alpha_2^1, \ldots, \alpha_{17}^1\}$ and $H_2 = \{\alpha_1^2, \alpha_2^2, \ldots, \alpha_{17}^2\}$, $U_1 \le U_2$ if and only if $H_1 \le H_2$, that is $\alpha_i^1 \le \alpha_i^2$, $\forall i = 1,2,\ldots,17$. Consider adjacent classes M_2^1 by basic equivalences σ.

The eighty-eight functionally incomplete algebras of class M_2^1 have a characteristic basis vector $(0,0,\ldots,0)$ [1]. Two adjacent classes exist that consist of ten algebras:

$K_{10}^1 = \{130, 138, 146, 162, 178, 386, 394, 402, 418, 434\}$, $K_{10}^2 = \{260, 261, 276, 292, 308, 388, 389, 406, 420, 436\}$
The classes K_{10}^1 and K_{10}^2 have isomorphic signature graphs.

Table 5
Isomorphism of graphs $G_i, i = \overline{1,9}$

$m=1$	260	273	289	261	276	292	388	277	293	305	308	389	401
$m=2$	130	152	168	138	146	162	386	154	170	178	184	394	402
$m=1$	404	417	420	309	405	421	433	436	437				
$m=2$	408	418	424	186	410	426	434	440	442				
$m=4$	96	41	97	104	168	224	289	352	105	169	225	232	297
$m=5$	80	25	88	81	152	208	273	336	89	153	209	216	281
$m=4$	417	424	480	233	361	425	481	488	489				
$m=5$	401	408	464	217	345	409	465	472	473				
$m=6$	12	25	41	152	168	13	28	44	140	29	45	57	60
$m=9$	3	25	41	273	289	11	19	35	259	27	43	57	54
$m=6$	156	169	172	184	408	424	61	157	173	185	188	440	189
$m=9$	281	291	297	305	401	424	59	283	299	307	313	443	315
$m=7$	6	13	38	140	268	388	60	204	284	325	404	205	316
$m=8$	6	11	35	138	131	386	54	202	150	323	402	203	182
$m=7$	12	14	44	196	276	30	77	269	294	332	420	286	333
$m=8$	3	7	38	332	146	29	75	267	163	330	418	151	331
$m=7$	68	22	69	261	292	46	141	270	300	389	452	302	397
$m=8$	66	19	67	279	169	39	139	163	166	387	450	167	395
$m=7$	260	28	76	262	324	54	197	278	308	396	62	310	436
$m=8$	30	22	74	134	194	51	195	166	178	394	55	179	434

The three classes consist of eight algebras: $K_8^1 = \{80,81,88,208,209,336,344,464\}$, $K_8^2 = \{96,97,104,224,225,352,360,480\}$, $K_8^3 = \{112,113,120,240,241,368,376,490\}$ have isomorphic signature graphs. The four classes consist of seven algebras: $K_7^1 = \{3,11,19,35,51,259,267\}$, $K_7^2 = \{12,13,28,44,60,140,141\}$, $K_7^3 = \{131,139,147,163,179,387,395\}$, $K_7^4 = \{268,269,284,300,316,396,397\}$. These classes are represented in Figure 21. The twenty-two classes $K_4^{t_1}, t_1 = 1,2,...,22$ consist of four algebras, the thirty classes $K_4^{t_2}, t_2 = 1,2,...,30$ includes two algebras and one hundred seventy-six classes $K_1^{t_3}, t_3 = 1,2,...,176$ – one algebra. Signature gratings of these classes are represented in Figure 22.

We will construct a basis grating of the factor class M_2/σ. The vertexes of the grating are encoded by the binary codes of the basics or the signature code of canonical algebras that come into the corresponding class. The edges are encoded by the number of basics or codes of operations that connect canonical algebras. Each element of the factor class has one canonical algebra, and the other algebras are free. The factor-grating can be constructed using a set of canonical algebras.

The number of basic algebras in each circle is given in Table 6.

Table 6
Number algebras of circle factor-grating

Circle	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number algebras	88	9	40	40	51	39	35	16	13	9	2	5	2	-	-	1

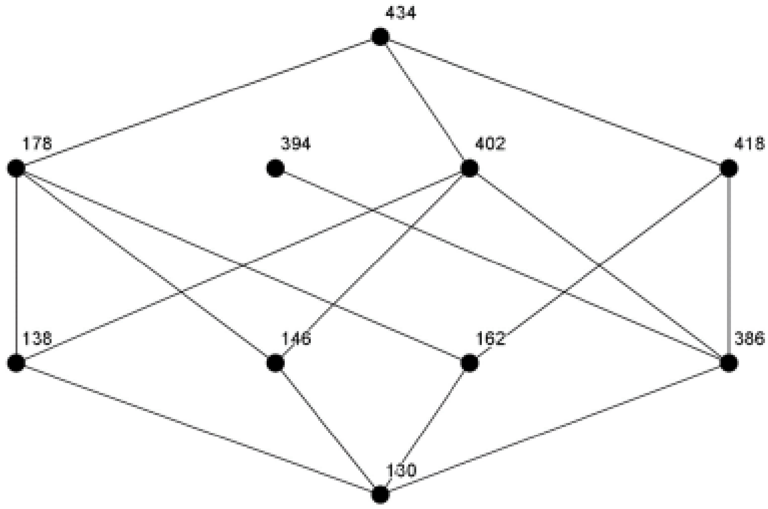


Figure 19: Signature Graph of class K_{10}^{1}

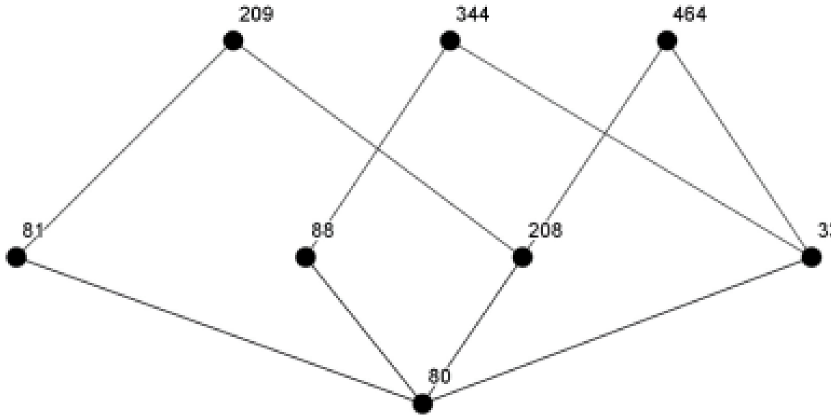


Figure 20: Signature Graph of class K_{8}^{1}

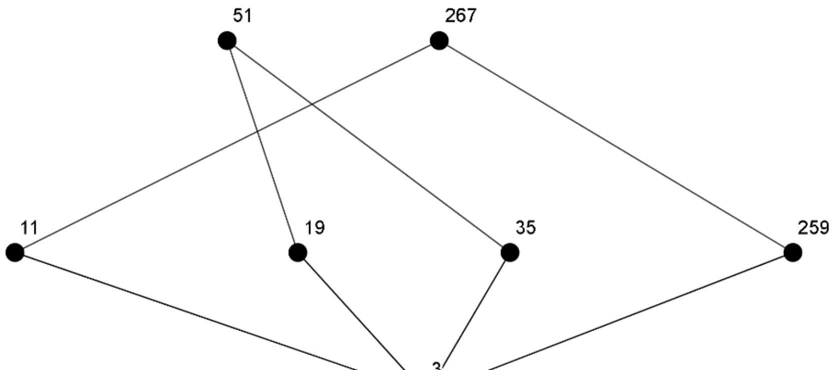


Figure 21: Signature Graph of class K_{7}^{1}

Since one-to-one correspondence exists between canonical algebras and basis vectors, the basis grating is isomorphic to the signature grating of canonical algebras represented in Figure 10.

From Table 2 it follows that power of the class M_2^1/σ is equal to two hundred sixty-five algebras. The basic gratings M_2^2/σ, M_2^3/σ, M_2^4/σ are isomorphic to grating M_2^1/σ. If these gratings combine into one basic grating M_2/σ, then the corresponding algebras U_k^i of grating M_2^i/σ, $i=1,2,3,4$ form sub gratings are represented in Figure 23.

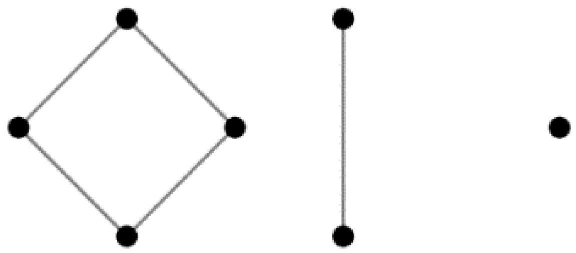


Figure 22: Signature Graph of classes $K_4^{t_1}, K_2^{t_2}, K_1^{t_3}$

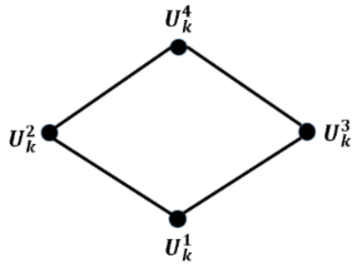


Figure 23: The Basic Grating M_2/σ

Theorem 3. The power of a class M_2/σ is equal to 1060 algebras.

Really $|M_2/\sigma| = |M_2^1/\sigma| + |M_2^2/\sigma| + |M_2^3/\sigma| + |M_2^4/\sigma| = 4 \cdot 265 = 1060$. From seventeen bases it is possible to form 2^{17} various combinations and only 1060 such combinations it is possible to find an algebra that has only those bases which are specified in the chosen combination.

9. Conclusion

The theory of Boolean functions is the foundation of modern discrete mathematics, mathematical logic, and computer science. This theory is used in combinatorics, theory of graphs, information and cryptology, theory of coding, and the theories such as machine learning, data mining, artificial intelligence, and neural networks. The complexity of algorithms for processing Boolean functions for a large number of variables requires the development of new methods in the analysis and representation of Boolean algebras. This paper has been using a description of Boolean algebras and their representation as graphs and lattices. That allows to significantly reduce the number of searches in the search algorithms for bases with specified characteristics. The construction of reliable technical systems involves the design of parallel circuits that use Boolean functions specified in different bases. The proposed work optimizes the search for algebras with given sets of basic.

10. References

[1] I. A Mych, V. V. Nykolenko, O. V. Vartsaba, Investigation of signature cube of universal boolean algebra. Scientific Bulletin of Uzhhorod University. Ser. Of Mathematics and Informatics 2(37) (2020) 157-167. doi: https://doi.org/10.24144/2616-7700.2017.1(30).79-86.2/

[2] I. A Mych, V. V. Nykolenko, O. V. Vartsaba, Structure of signature cube of boolean algebra. Scientific Bulletin of Uzhhorod University. Ser. Of Mathematics and Informatics, 1(38) (2020) 149-156. doi: https://doi.org/10.24144/2616-7700.2017.1(30).79-86.2/

[3] G. Birkhoff, On the structure of abstract algebras. Proceedings of the Cambridge philosophical society, volume 31 (1935) 433-454.

[4] A. Y. Maltcev, Algebraicheskie sistemy, Nauka, Moskva, 1970.

[5] A. H. Kurosh, Lekcii po obshchej algebra, Nauka, Moskva, 1973, pp. 3-393.

[6] R. Sykorskyi, Bulevy algebry, Myr, Moskva, 1969.

[7] V. M. Glushkov, G. E. Cejtlin, E. L. Yushchenko, Algebra. Yazyki. Programmirovanie, Nauk. Dumka, Kiev, 1974, pp. 65-112.

[8] YU. I. Zhuravlev, Korrektnye algebry nad mnozhestvami nekorrektnyh (evristicheskih) algoritmov, Kibernetika, 1977, №4, pp. 14-21, №6, pp. 21-27.

[9] S. S. Marchenkov, Osnovy teorii bulevyh funkcij, Fizmatlit, Moskva, 2014.

[10] R. Dzh. Uilson, Vvedenie v teoriyu grafov, Spb: OOO «Dialektika», 2019.