Vector Convex Optimization Problems: Lexicographic Optimality and Solvability, Method of Solving

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Abstract

We have revealed conditions of existence and optimality of solutions of multicriterion problems of lexicographic optimization with an unbounded set of feasible solutions on the basis of applying properties of a recession cone of a convex feasible set, the cone which lexicographically puts in order a feasible set with respect to optimization criteria and local large tents, built at the frontier points of feasible set. Obtained conditions may be successfully used while developing algorithms for finding optimal solutions of mentioned problems of lexicographic optimization. A method for finding lexicographically optimal solutions of convex lexicographic problems has been constructed and substantiated on the basis of ideas of method of linearization and Kelley's cutting-plane method.

Keywords [1]

Vector criterion, lexicographic optimization, solvability, lexicographic optimality conditions, Pareto-optimal solutions, Slater's set, Kelly's cutting-plane method.

1. Introduction

Many problems of making multipurpose decisions in management, planning, design are formulated as multicriterion (vector) optimization problems. Among vector problems, lexicographic problems form a fairly wide and important class of optimization problems. Lexicographic ordering is used to establish rules of subordination and priority. Therefore, a significant number of problems, including the problems of optimization of complex systems, modeling of hierarchical structures, stochastic programming problems under risk conditions, problems of a dynamic nature, etc., can be represented in the form of lexicographic optimization problems [1-12]. The lexicographic approach to solving multicriterion problems consists in a strict ranking of criteria in terms of relative importance allowing optimizing a more important criterion at the expense of any losses for all other, less important criteria. Most often, such multicriterion problems arise when additional criteria are successively introduced into ordinary scalar optimization problems, which may not have a unique solution.

Possible methods for solving such problems include the use of a scalarization scheme or a convolution of a vector criterion for a one-stage solution [1-2]. In [2] it is proposed to use the simplex-method to find the lexicographic optimum of linear multicriterion optimization problems, in [7] − for linear maxmin problem. In [10], the problem of lexicographic optimization with convex criterial functions and linear constraints is reduced to a sequence of linear lexicographic problems by approximating the criteria functions. In single-criterion optimization, a number of extremum search algorithms are based on the use of the apparatus of duality theory. This issue is also of interest for multicriterion optimization problems. The article [11] investigates convex quadratic problems of lexicographic optimization on a set given by a system of linear inequalities, and questions of constructing problems that are dual to them. Dual problems to the original one are constructed using the Lagrange map, where the Lagrange multipliers are vector variables, the set of values of each of

II International Scientific Symposium «Intelligent Solutions» IntSol-2021, September 28–30, 2021, Kyiv-Uzhhorod, Ukraine
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them is a set of vectors in the space, the dimension of which is equal to the number of particular criteria with the lexicographic order introduced on it. An algorithm allowing reducing the solving of the initial lexicographic optimization problem by approximating a feasible set to solving a sequence of lexicographic problems of the linear programming is presented in this paper.

The present paper continues researches, presented in works [13-17]. The aim of the research presented in this article is to establish conditions for the lexicographic solvability of vector optimization problems with an unbounded feasible set and conditions for lexicographic optimality of solutions based on the use of the properties of a recessive cone of a convex feasible set [18], a cone lexicographically ordering the feasible set with respect to optimization criteria [2] and local tents [19], built at the boundary points of the admissible set, also to develop and substantiate a method for finding lexicographically optimal solutions to lexicographic problems of convex optimization based on the ideas of linearization methods and Kelly's cutting-planes [20].

2. Formulation of the problem

In the criterion space R^{ℓ}, we introduce a binary relation of the lexicographic order between vectors $z=(z_1,z_2,\dots,z_{\ell})$ and $z'=(z'_1,z'_2,\dots,z'_{\ell})$ such that:

$z \geq^{L} z' \Leftrightarrow (z=z') \vee \left(\exists j \in N_{\ell} : \forall i \in N_{j-1} \left(z_j > z'_j,\, z_i = z'_i\right)\right)$, where $N_0 = \varnothing$

Consider a lexicographic optimization problem of the following type:

$Z_L(F,X): \quad \max^{L}\{F(x) | x \in X\}$,

where $F(x) = (f_1(x),\dots,f_{\ell}(x))$, $\ell \geq 2$, $f_k(x) = \langle c_k, x\rangle$, $c_k \in R^n$, $k \in N_{\ell} = \{1,2,\dots,\ell\}$, $X = \left\{x \in R^n \middle| g^i(x) \leq 0, x \geq 0, i \in N_m\right\}$, $X \neq \varnothing$, $g^i(x), i \in N_m$, − convex functions.

In the problem of lexicographic optimization, particular criteria are ordered by importance. This gives rise to the concept of the lexicographic optimum.

Definition 1. A vector x is lexicographically preferable to a vector x' if one of the following ℓ conditions is met:

1) $f_1(x) > f_1(x')$;

2) $f_1(x) = f_1(x')$, $f_2(x) > f_2(x')$;

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ℓ) $f_j(x) = f_j(x')$, $j = 1,\dots,\ell-1$, $f_{\ell}(x) > f_{\ell}(x')$.

Definition 2. A vector x is equivalent to a vector x' if for each criterion the vectors have the same estimates, while $x \neq x'$.

By solving the problem $Z_L(F,X)$ we mean the search for elements of the set $L(F,X)$ of lexicographic optimal solutions, which we define in this way:

$L(F,X) = \{x \in X \mid \upsilon(x,F,X) = \varnothing\}$, where

$\upsilon(x,F,X) = \{x' \in X \mid \exists j \in N_{\ell} : f_j(x') > f_j(x) \wedge j = \min\{i \in N_{\ell} : f_i(x') \neq f_i(x)\}\}$. It follows directly from the definition of lexicographically optimal solutions that the set $L(F,X)$ can also be pecified using recurrence relations. Thus,

$$L_i(F,X) = \operatorname{Arg\,max}\{ f_i(x) : x \in L_{i-1}(F,X), i \in N_{\ell}, \tag{1}$$

where $\operatorname{Arg\,max}\{\cdot\}$ − is a set of all optimal solutions to the corresponding maximization problem, $L_0(F,X) = X$, $L_{\ell}(F,X) = L(F,X)$.

Reasonable inclusions of the sequence of sets follows from relations (1)

$$X \supseteq L_1(F,X) \supseteq L_2(F,X) \supseteq \dots \supseteq L_{\ell}(F,X) = L(F,X),$$

that is, each next particular criterion narrows the set of solutions obtained taking into account all the previous particular criteria.

It is known [1, 2], a set $L(F,X)$ can be defined as the result of solving a sequence ℓ of scalar convex programming problems $Z_{L_i}(F,X), i \in N_\ell,$. So, the problem $Z_L(F,X)$ can be viewed as a sequential optimization problem.

Let us note the important properties of problems $Z_{L_i}(F,X), i \in N_\ell,$ [18]: any local minimum (maximum) is a global minimum (maximum).

The definition of the lexicographically optimal solution to the problem implies the validity of such properties [8].

1. If for a feasible solution $x^0 \in X$ and $\forall x \in X \setminus \{x^0\}$ the inequality $f_1(x) < f_1(x^0)$ is carried out, then $x^0 \in L(F,X)$.

2. If for a feasible solution $x \in X$ $\exists x' \in X \setminus \{x\}$ is such that $f_1(x') > f_1(x),$ then $x \notin L(F,X)$.

According to [2], we introduce a definition.

Definition 3. A vector $z \in R^\ell$ is called lexicographically positive if its first nonzero component in ascending order of the component indices is positive.

We will denote the lexicographic positivity of the vector $z \in R^\ell$ as: $z >^L 0$, here $(>^L)-$ the sign of the relation is lexicographically larger.

A vector $z \in R^\ell$ is lexicographically larger than a vector $y \in R^\ell$ $z >^L y$ if the vector $(z-y)$ is lexicographically positive. With this ordering, any two vectors of the same dimension are comparable with each other.

So, for any vectors $a, b \in R^\ell$ $a >^L b$, if and only if $1 \le i \le \ell$ so that $a_i > b_i$ and if $i > 1$, then, the $a_k = b_k$, $k = 1,2,...,i-1$. Vector a is lexicographically not less than the vector b, $a \ge^L b$, if $a >^L b$ or, $a = b$, $(\ge^L)$ - the sign of the relation is lexicographically not less.

Definition 4. A solution $x^* \in X$ to a problem $Z_L(F,X)$ will be called lexicographically optimal if it is not worse than any other admissible solution $y \in X$ in understanding the relation $\ge^L$, that is, if $F(x^*) - F(y) \ge^L 0$.

So, for an arbitrary $x \in X$, the assertion is true

$$x \in L(F,X) \Leftrightarrow \{y \in X \mid F(y) >^L F(x)\} = \varnothing .$$

In terms of a lexicographic optimization problem, an arbitrarily small increase in a more important criterion is achieved at the expense of any losses according to other less important criteria.

3. Existence of lexicographically optimal solutions

The solvability of the problem of finding lexicographically optimal solutions on a feasible set X and the structure of the set of optimal solutions depend on the properties of the order of the preference relation, the structure of the feasible domain X, the nature of its elements, properties of the vector function $F(x)$, etc. According to [2], the finiteness of the set X is a sufficient condition for the existence of optimal solutions to the lexicographic problem of optimization. Also, the set $L(F,X)$ is not empty if the set of vector estimates $Y = \{F(x) \mid x \in X\}$ is bounded and closed. However, in the case of an infinite feasible region X, the set of lexicographically optimal solutions may be empty.

It is relevant to study the issues of solvability of lexicographic vector optimization problems in which the set of feasible solutions is not bounded and convex. The unboundedness of a convex set X means that $0^+ X \setminus \{0\} \ne \varnothing$, where $0^+ X = \{y \in R^n \mid \forall x \in X : x + ty \in X, t \ge 0\}$ is the recessive cone of the set X.

We will analyze the problem $Z_L(F,X)$ taking into account the properties of the recessive cone 0^+X [18] and the cone $K^L = \{x \in R^n \mid Cx >^L 0\}$ lexicographically ordering the feasible set with respect to the optimization criteria, which we will also call the cone of perspective [13] lexicographic directions of the problem $Z_L(F,X)$, since the transition from any point $x_1 \in R^n$ to the point $x_2 = x_1 + y$, where y belongs to the cone K^L, leads to the inequality $Cx_2 >^L Cx_1$, that is, to the lexicographic increase in the values of the vector criterion of the problem.

The cone K^L determining the lexicographic order in space R^ℓ is a convex cone of directions of lexicographically positive vectors and can be represented as a union of disjoint sets:

$$K^L = K_1 \bigcup K_2 \bigcup \ldots \bigcup K_\ell,$$

where $K_1 = \{x \in R^n \mid c_1 x > 0\}$,

$$K_2 = \{x \in R^n \mid c_1 x = 0, c_2 x > 0\},$$

$$\ldots,$$

$$K_\ell = \{x \in R^n \mid c_1 x = 0,\, c_2 x = 0, \ldots, c_{\ell-1} x = 0,\, c_\ell x > 0\}.$$

For an arbitrary, the statement [2] is true:

$$x \in L(F,X) \Leftrightarrow \left(x + K^L\right) \bigcap X = \varnothing. \qquad (2)$$

Continuing the study of the existence of various types of optimal solutions for vector optimization problems [14-17], started in work [2] for lexicographic problems, we will consider the necessary and sufficient conditions for the existence of lexicographically optimal solutions to the problem $Z_L(F,X)$. In the case of a convex closed unbounded feasible set X of the problem $Z_L(F,X)$, the theorem is valid.

Theorem 1. A necessary condition for the existence of lexicographically optimal solutions to the problem $Z_L(F,X)$ is the empty intersection of the cone K^L of promising lexicographic directions and the recessive cone 0^+X, that is,

$$K^L \bigcap 0^+ X = \varnothing. \qquad (3)$$

Proof. Let us suppose by contradiction that the set $L(F,X) \neq \varnothing$, but condition (3) is not satisfied, that is, the intersection of the cones K^L and 0^+X is not empty:. Then the following relations are true:

$$\left(x + K^L\right) \bigcap X \supseteq \left(x + K^L\right) \bigcap \left(x + 0^+ X\right) = x + \left(K^L \bigcap 0^+ X\right) \neq \varnothing.$$

Taking into account formula (2), we can conclude that the set $L(F,X) = \varnothing$. But this contradicts the condition of the theorem and thereby proves its validity.

The converse statement of the theorem is generally not true. In the monograph [2, p. 113] an example is given in which condition (3) is satisfied for an admissible set X, but the set of its extreme points is unbounded, and as a result, the set $L(F,X) = \varnothing$. The direction of the lexicographically positive vector will be called the lexicographically positive direction. The theorem is true [2, p. 113].

Theorem 2. Let V be a non-empty set of extreme points of a convex closed set X. If set V is a bounded, then the set X has a lexicographic maximum if and only if it is bounded in all lexicographically positive directions.

In our notation, under the conditions of Theorem 2, the set $L(F,X)$ is not empty if and only if condition (3) is satisfied. In the case of a convex, unbounded and polyhedral set, the corollary to Theorem 2 [2, p. 114] is true.

Corollary. A closed convex polyhedral set X has a lexicographic maximum if and only if it is bounded in all lexicographically positive directions.

Theorem 1 and the corollary to Theorem 2 imply the following theorem.

Theorem 3. Let the feasible set X of the problem $Z_L(F,X)$ be a closed convex polyhedral set. A necessary and sufficient condition for the existence of lexicographically optimal solutions to this problem is the fulfillment of equality (3).

It should be noted that the multifaceted condition of a convex closed unbounded set X is essential for the statement of the fact that condition (3) is a necessary and sufficient condition for the existence of lexicographically optimal solutions to the problem $Z_L(F,X)$.

4. Optimality conditions for solutions

Optimality conditions are an essential component of the mathematical theory of optimization, including vector optimization. Establishing the necessary and sufficient conditions for the optimality of solutions to vector problems is an urgent problem, since the knowledge of such conditions provides the basis for developing methods for testing the optimality of one or another chosen solution, as well as for constructing and developing effective optimization methods in order to find various sets of optimal solutions.

As is well known [3,13-17], if the criteria of a vector problem are equally important, then the solution of a vector problem is usually understood as finding a subset of one of such sets: $P(F,X)$ all Pareto-optimal (effective) solutions, $S\ell(F,X)$ Slater-optimal solutions. The following statements $\forall\, x \in X$ are true:

$$x \in P(F,X) \Leftrightarrow \{y \in X \mid F(y) \geq F(x), F(y) \neq F(x)\} = \varnothing,$$
$$x \in S\ell(F,X) \Leftrightarrow \{y \in X \mid F(y) > F(x)\} = \varnothing,$$

It's obvious that $L(F,X) \subseteq P(F,X) \subseteq S\ell(F,X)$.

According to Theorem 1 [3, p. 163], due to the linearity of the criterion functions of the problem $Z_L(F,X)$ and regardless of the structure of the feasible set X, Pareto-optimal and Slater-optimal solutions can constitute the entire feasible set, or be located only on its boundary. Therefore, taking into account the inclusions $L(F,X) \subseteq P(F,X) \subseteq S\ell(F,X)$ in establishing necessary and sufficient conditions for the lexicographic optimality of solutions to the problem, we will consider only the boundary points of the set X. We will denote a subset $\mathrm{Fr}B$ of boundary points of some set. Let us introduce the following sets for consideration:

$$N(y) = \{i \in N_m \mid g_i(y) = 0\},\ X(y) = \{x \in R^n \mid g_i(x) \leq 0,\ i \in N(y)\}.$$

Moreover, if, $g_i(x),\ i \in N(y),$ – are continuously differentiable functions in space R^n, we can define the set $Q(y) = \{x \in R^n \mid \langle \nabla g_i(y),\ x - y\rangle \leq 0,\ i \in N(y)\}$, where $\nabla g_i(y)$ – function gradient $g_i(x)$ at the point $y, i \in N(y)$. It's obvious that $\forall\, y \in \mathrm{Fr}\, X\colon N(y) \neq \varnothing$, $y + 0^+ X \subseteq X \subseteq X(y) \subseteq Q(y)$.

Theorem 4. Let $y \in \mathrm{Fr}\, X$. If $g_i(x), i \in N(y)$, – are continuously differentiable functions, then the relation

$$K^L \cap (Q(y) - y) = \varnothing \tag{4}$$

is a sufficient condition for the inclusion $y \in L(C,X)$. Moreover, if $\{\nabla g_i(y) \mid i \in N(y)\}$ –is a system of linearly independent vectors, then the relation

$$K_1 \cap (Q(y) - y) = \varnothing \tag{5}$$

is a necessary condition for the inclusion $y \in L(C,X)$.

Proof. The **sufficiency** of condition (4) of the theorem becomes obvious, taking into account the inclusion $X \subseteq Q(y)$, as well as formula (2).

Necessity. The requirement of linear independence of vectors $\{\nabla g_i(y) \mid i \in N(y)\}$ leads to the fulfillment of the relations: $\operatorname{int} Q(y) \neq \varnothing$, $\operatorname{int} Q(y) = \operatorname{ri} Q(y)$, where $\operatorname{ri} B$ is the relative interior of some set B. Let $y \in L(C,X)$, that is, according to formula (2)

$$\left(y + K^L\right) \bigcap X = \varnothing . \tag{6}$$

Let us suppose (by contradiction) that relation (5) is not fulfilled, that is $K_1 \bigcap \left(Q(y) - y\right) \neq \varnothing$, whence, by Corollary 6.3.2 with [18] $K_1 \bigcap \operatorname{int}\left(Q(y) - y\right) \neq \varnothing$. Taking into account also that under the conditions of this theorem the sum of the linear hulls of the cones K_1 and $\left(Q(y) - y\right)$ coincides with R^n, and according to Theorem 3.4 [21, p. 31], we conclude that the cones $K_1 \bigcup \{0\}$ and $\operatorname{int}\left(Q(y) - y\right)$ are inseparable, which are local tents [19, 21] at the point y of the sets $\left(y + K_1\right) \bigcup \{y\}$ and X, respectively. Moreover, each of these local tents is not a linear subspace in R^n, since the point $\{0\} \in R^n$ does not belong to their interiors, as well as taking into account Theorems 1.1 and 6.1 from [18]. Then, according to Theorem 1.3 from [21, p. 204] $\left((y + K_1) \bigcup \{y\}\right) \bigcap X \setminus \{y\} \neq \varnothing$, which contradicts condition (6) and thus it proves the necessity of satisfying relation (5) for any lexicographically optimal boundary point $y \in X$ under the conditions of the theorem. The proof of the theorem is finished.

5. Cutting plane method for solving lexicographic vector convex optimization problems

The search for solutions to the problem $Z_L\left(F, X\right)$ can be reduced to solving a sequence of lexicographic linear programming problems

$Z_L\left(F, X_p\right): \max^L \left\{F\left(x\right) \middle| x \in X_p\right\}$ on a polyhedral set

$X_p = \left\{x \in R^n \middle| \left\langle \nabla g^i\left(x^j\right), x - x^j \right\rangle + g^i\left(x^j\right) \leq 0,\ x \geq 0, i \in N_m, j = 0,1,\ldots,p\right\}$,

$x^j \in R_+^n, R_+^n = \left\{x \in R^n \middle| x_i \geq 0, i \in N_n\right\}$, containing the feasible domain X of the original problem.

Statement 1. The including $X \subset X_p$ is just.

Proof. The including follows directly from the construction of a polyhedral set X_p. Using the properties of a convex continuously differentiable function $h\left(x\right)$ for any $x, y \in R^n$, the inequality

$$\left\langle \nabla h\left(y\right), x - y \right\rangle + h\left(y\right) \leq h\left(x\right) . \tag{7}$$

According to (7), for some number $p > 0$ and any $x^j \in R_+^n$, $j = 1,\ldots,p$, we can write down

$$\left\langle \nabla g^i\left(x^j\right), x - x^j \right\rangle + g^i\left(x^j\right) \leq g^i\left(x\right), i \in N_m,\ j = 0,1,\ldots,p . \tag{8}$$

Since the inequalities, $g^i\left(x\right) \leq 0, i \in N_m$, are satisfied for arbitrary $x \in X$, then relation (8) implies the fulfillment of the inequalities

$$\left\langle \nabla g^i\left(x^j\right), x - x^j \right\rangle + g^i\left(x^j\right) \leq 0, i \in N_m,\ j = 0,1,\ldots,p, \tag{9}$$

that is $x \in X_p$, as required to be proved.

Theorem 4. [2, p. 190]. If a vector function F reaches a lexicographic maximum on a set X_p, then among the points of this maximum there is an extreme point of the set X_p.

From Theorem 4 it follows that a simplex algorithm can be used as an algorithm for directed enumeration of the extreme points of the set X_p to solve the problem $Z_L\left(F, X_p\right)$.

Finding lexicographically optimal solutions to the problem $Z_L\left(F, X_p\right)$ will be carried out by direct (lexicographic) search [2], which is reduced to solving maximization problems $Z\left(f_s, X_p\right): \max\left\{f_s(x) \middle| x \in X_p\right\}, s \in N_\ell$, in each of which the corresponding function of the lexicographically ordered vector criterion is maximized. The main idea of the proposed method is as follows. If the optimal solution to the problem $Z\left(f_s, X_p\right)$ is inadmissible in the problem $Z_L(F, X)$, then it is excluded from further consideration by adding a new linear constraint to the constraints of the problem $Z\left(f_s, X_p\right)$. Thus, this restriction cuts off the invalid solution, as well as part of the invalid problem domain, $Z_L(F, X)$ from all subsequent considerations. All the added constraints are correct cutting planes, that is, those that do not cut off any part of the feasible region of the convex problem $Z_L(F, X)$. If the optimal solution to a problem $Z\left(f_s, X_p\right)$ belongs to a set X, and it is the only optimal solution on this set, then the found solution is lexicographically optimal for the problem $Z_L(F, X)$.

6. Algorithm for solving the problem $Z_L(F, X)$

Initial step. Let $s=1, k=0.$ First we select an arbitrary point $x^k \in \operatorname{Fr} G$.

Ten we build a polyhedron $X_k = \left\{x \in R^n \middle| \left\langle \nabla g^i\left(x^k\right), x - x^k\right\rangle + g^i\left(x^k\right) \le 0, x \ge 0, i \in N_m\right\}$.

1. We will solve the problem

$$\max\left\{f_s(x) \middle| x \in X_k\right\}. \tag{10}$$

by the dual simplex algorithm [2]. Let $x^{k+1} = \arg\max\left\{f_s(x) \middle| x \in X_k\right\}$. If $x^{k+1} \in X$, and x^{k+1} – is the only optimal solution on the feasible set X, then $x^{k+1} = \arg\max^L\left\{F(x) \middle| x \in X\right\}$, insofar as $X \subseteq X_k$. The problem $Z_L(F, X)$ is solved.

2. If $x^{k+1} \in X$ and x^{k+1} – is not the only optimal solution on the feasible set X, we believe $\bar{f}_s = f_s\left(x^{k+1}\right), s = s+1,$

$X_{k+1} = \left\{x \in X_k \mid f_i(x) = \bar{f}_i, i = 1, 2, \ldots, s-1\right\}$ and pass over to step 1.

If $x^{k+1} \notin X$ we go to step 3.

3. We define the set $I_{k+1} = \left\{i \middle| g^i\left(x^{k+1}\right) > 0\right\}$ of constraint indices of problem $Z_L(F, X)$, which are violated at the point x^{k+1}. We will build a polyhedron X_{k+1}, adding to the constraints describing the set X_k, the inequality

$\left\langle \nabla g^i\left(x^{k+1}\right), x - x^{k+1}\right\rangle + g^i\left(x^{k+1}\right) \le 0$,

$i \in N_{k+1} = \left\{ j \in I_{k+1} \middle| g^j\left(x^{k+1}\right) = \max_{i \in I_{k+1}} g^i\left(x^{k+1}\right) \right\}$. We get a new multifaceted set $X_{k+1} = \left\{ x \in X_k \mid \left\langle \nabla g^i\left(x^{k+1}\right), x - x^{k+1}\right\rangle + g^i\left(x^{k+1}\right) \le 0, i \in N_{k+1} \right\}$.v

We go to step 1, believing $k = k+1$.

To solve auxiliary problems of linear optimization of the form (10) it is advisable to apply the dual simplex method [2], which allows using the solution obtained at the previous step as a basis for the updated feasible domain.

The convergence of the algorithm is established by the following theorem.

Theorem 5. If the functions $g^i(x)$, $i \in N_m$, are convex, continuously differentiable and the problem $Z_L(F,X)$ has a finite optimal solution, then the sequence of points generated by this algorithm converges to the lexicographically optimal solution to the problem $Z_L(F,X)$.

Proof. If the problem $Z_L(F,X)$ has a finite lexicographically optimal solution, then, starting from some number p_0, the sequence of points is contained in a bounded set $\left\{x^p\right\}$. Let $\left\{x^k\right\}$ be a subsequence of a sequence $\left\{x^p\right\}$ that converges to a point x^*. We will consider a subsequence $\left\{x^t\right\}$ of points for which the cutting hyperplane is generated with respect to the *i*-constraint of the form (9). If at each iteration a hyperplane is added with respect to the strongest (most violated) constraint, then, starting from some number $k \ge k_0$, the constraint is fulfilled $g^i\left(x^k\right) \le 0$, that is, x^k it belongs to the set of feasible solutions or the subsequence $\left\{x^t\right\}$ is infinite. In the case when the subsequence $\left\{x^t\right\}$ is infinite, the inequality holds for each $t' > t$, whence $\left\langle \nabla g^i\left(x^t\right), x^{t'} - x^t\right\rangle + g^i\left(x^t\right) \le 0$, following Cauchy-Bunyakovsky inequality, we obtain $g^i\left(x^t\right) \le \left\|\nabla g^i\left(x^t\right)\right\| \left\| x^{t'} - x^t\right\|$. Considering that $\left\| x^{t'} - x^t\right\| \to 0$, $\left\|\nabla g^i\left(x^t\right)\right\| \to \left\|\nabla g^i\left(x^*\right)\right\|$, from the last inequality it follows $g^i\left(x^t\right) \to g^i\left(x^*\right) \le 0$, that x^* is a feasible solution to the problem $Z_L(F,X)$. On the other hand, if $\bar{x}$ is the optimal solution to the problem $Z_L(F,X)$, then at each iteration of the algorithm the inequality $F\left(x^t\right) \ge^L F(\bar{x})$ is valid, whence we obtain $F\left(x^*\right) \ge^L F(\bar{x})$ at the passage to the limit. Hence x^* is the lexicographically optimal solution to the problem $Z_L(F,X)$. The theorem is proved.

The construction of the sequence $\left\{x^k\right\}$ in the proposed method is carried out in such a way that each of the points x^k is an no feasible point for the original problem. Therefore, the calculation process cannot be stopped even at rather large values s, this is possible only when we get an admissible point. Convergence to a lexicographically optimal solution is guaranteed by the algorithm in the case when the admissible set is convex.

7. Conclusions

The issues of existence and lexicographic optimality of solutions of vector convex optimization problems with linear functions of criteria and an unbounded feasible set have been investigated.

Based on the analysis of these problems, taking into account the properties of the cones of perspective lexicographic directions, of recessive directions and local tents at the boundary points of the feasible set, necessary and sufficient conditions for the existence and lexicographic optimality of solutions of the problems under study have been established. The obtained conditions can be successfully used in the development of algorithms for finding optimal solutions to these lexicographic optimization problems. Based on the ideas of linearization methods and Kelly's cutting planes, a method for finding lexicographically optimal solutions of convex lexicographic problems has been constructed and substantiated.

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