The existential theories of term algebras with the Knuth-Bendix orderings are decidable

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We consider term algebras with simplification orderings (which are monotonic, and well-founded). Solving constraints (quantifier-free formulas) in term algebras with this kind of orderings has several important applications like pruning search space in automated deduction and proving termination and confluence of term rewriting systems. Two kinds of ordering are normally used in automated deduction: Knuth-Bendix ordering and various versions of recursive path orderings. There exists extensive literature on solving recursive path ordering constraints [1, 4], but no algorithms for solving Knuth-Bendix ordering constraints are known. We proved that the problem of solving Knuth-Bendix ordering constraints is decidable and NP–hard.

Let us briefly describe the proof, for the full version we refer to [3]. We consider term algebras in a finite signature Σ with at least one constant, denoted $\mathrm{TA}(\Sigma)$. Let us now define Knuth-Bendix orderings on $\mathrm{TA}(\Sigma)$ [2]. The definition of Knuth-Bendix ordering is parametrized by a *weight function* on Σ, i.e., a function $w : \Sigma \to \mathbb{N}$, and a linear ordering $\gg$ on Σ. We require from the weight function the following: if $w(f) = 0$ and f is unary, then f must be the greatest w.r.t. $\gg$ in Σ, and weights of constants are positive. We define the weight of a ground term as a sum of weights of functors occurring in the term. Given a weight function w and a linear ordering $\gg$ on Σ, the *Knuth-Bendix ordering* on $\mathrm{TA}(\Sigma)$ is the binary relation $>_{KB}$ defined as follows. For any ground terms $g(t_1, \ldots, t_n)$ and $h(s_1, \ldots, s_k)$ we have $g(t_1, \ldots, t_n) >_{KB} h(s_1, \ldots, s_k)$ if

1. $|g(t_1, \ldots, t_n)| > |h(s_1, \ldots, s_k)|$

or

2. $|g(t_1, \ldots, t_n)| = |h(s_1, \ldots, s_k)|$ and one of the following holds:

   (a) $g \gg h$ or

   (b) $g = h$ and for some $1 \leq i \leq n$ we have $t_1 = s_1, \ldots, t_{i-1} = s_{i-1}$ and $t_i >_{KB} s_i$.

To prove the theorem we first extend our term algebra with the natural numbers with addition and the weight function on terms. We show how to transform arbitrary quantifier-free formula into an equivalent disjunction of conjunctions such that all occurring terms in the formula are variables. Then we introduce formulas which express that there exists at least n terms of the weight x, where n is a fixed parameter and show how to write them as linear Diophantine equations on weights of terms. Using obtained formulas we transform initial formula into an equivalent constraint which consists of linear Diophantine equations on weights of terms. Then we show that satisfiability of that constraint is equivalent to satisfiability of the systems of linear Diophantine equations over the natural numbers.

References

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