A deductive decision procedure for a restricted FTL

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The aim of this report is to present a new kind deductive procedure Sat for a restricted first-order linear temporal logic (FTL, in short). The proposed procedure Sat is a degenerate case of an ω-decidable-like procedure Sat_ω (see [2]) for restricted FTL. Different from ω-decidable procedure Sat_ω, the procedure Sat is decidable.

For simplicity, we assume that all the predicate symbols are flexible (i.e., change their value in time), but all the variables are rigid (i.e., with time-independent meanings). Besides, all predicate symbols have the same arity (for example all predicate symbols are 2-place only). We consider only skolemized formulas. We can consider occurrences of the "next" operator $\bigcirc$ only occuring in the formula $\bigcirc^k E$ (where E is an *elementary* formula i.e., an expression of the shape $P(t_1, \ldots, t_n)$, where P is a predicate symbol, t_i is a variable or a constant). For the sake of simplicity, we "eliminate" the "next" operator and the formula $\bigcirc^k E$ is abbreviated as E^k (i.e., as an elementary formula with the index k, which is called an *atomic* formula). Let us define the objects of consideration for Sat.

Definition 1 (kernel formulas, TD-sequents, induction-free TD-sequents). *The formulas of the form $\Box\forall\bar{x}(E(\bar{x}) \supset R(\bar{x}) \wedge P^l(\bar{b}))$ is a kernel formula, if $l > 0$, $E(\bar{x})$ is an elementary formula (called the premise of the kernel formula) $R(\bar{x})$ is an elementary formula (called isolated conclusion); $P^l(\bar{b})$ is an atomic formula (called constant conclusion, C-conclusion, in short); $\bar{x} = x_1, \ldots, x_n$; $\bar{b} = b_1, \ldots, b_n$; b_i is a free variable or a constant, $1 \leqslant i \leqslant n$, $n \geqslant 1$.*

A sequent S is TD-sequent, if $S = \Sigma, \Pi^1, \Box\Omega \to \Box^0 A$, where $\Sigma = 0$ or consists of elementary formulas; $\Pi^1 = \varnothing$ or consists of atomic formulas of the shape E^l $(l > 0)$; Σ, Π^1 is called parametrical formulas; $\Box\Omega$ consists of kernel formulas; $\Box^0 \in \{\varnothing, \Box\}$; $A = \exists\bar{y} \bigvee_{i=1}^{m} E_i(\bar{y})$, E_i is an atomic formula. If $\Box^0 = \varnothing$ then S is induction-free TD-sequent. Each TD-sequent must satisfy the following conditions:

(1) Non-repeating condition :

if $\Box\forall\bar{x}(E_i(\bar{x}) \supset R_i(\bar{x}) \wedge P_i^l(\bar{b})) \in \Box\Omega$ and $\Box\forall\bar{y}(E_j(\bar{y}) \supset R_j(\bar{y}) \wedge P_j^k(\bar{c})) \in \Box\Omega$ then $\forall i, j$ $E_i \neq E_j$ and $P_i \neq P_j$, if $i \neq j$.

(2) Saturation condition:

(a) for each elementary formula $Q(b)$ from Σ there must be the unique kernel formula $\Box\forall x(Q(\bar{x}) \supset R(\bar{x}) \wedge P^l(\bar{b}))$ from $\Box\Omega$; (b) for each atomic formula $P^k(\bar{b})$ from Σ, Π^1 there must be unique kernel formula $\Box\forall\bar{x}(M(\bar{x}) \supset R_1(\bar{x}) \wedge P^l(\bar{b}))$ from $\Box\Omega$ and $k < l$; (c) if $P^k(\bar{b})$, $P^l(\bar{c}) \in \Sigma, \Pi^1$, then $k \neq l$ and if $P^k(\bar{b}) \in \Sigma, \Pi^1$, then $P^k(\bar{c}) \notin \Sigma, \Pi^1$; (d) if $\Sigma, \Pi^1 = P(\bar{b})$, then $\Box\Omega = \Box\forall\bar{x}(P(\bar{x}) \supset R(\bar{x}) \wedge P^l(\bar{b}))$, $l \geq 1$; (e) let $\Gamma(\Delta)$ be the set of all predicate symbols from Σ, Π, Ω (from A, correspondingly), then $\Delta \subseteq \Gamma$.

(3) Periodic condition:

$\Box\Omega = \Box\forall\bar{x}_1(E(\bar{x}_1) \supset R_1(\bar{x}_1) \wedge E_1^{l_1}(\bar{b}_1)), \Box\forall\bar{x}_2(E_1(\bar{x}_2) \supset R_2(\bar{x}_1) \wedge E_2^{l_2}(\bar{b}_2)), \ldots, \Box\forall\bar{x}_n(E_{n-1}(\bar{x}_n) \supset R_n(\bar{x}_n) \wedge E_n(\bar{b}_n))$ and $E_n = E$.

So, TD-sequents do not satisfy, in general, the monodic condition from [1].

To define the separation rules $(ISIF)$ and (GIS) (see below) let us define the following operation $(+)$.

Definition 2 (operation $+$). *Let $S = \Sigma, \Pi^1, \Box\Omega \to \Box^0 A$ be a TD-sequent, and $E(\bar{b})$ be any elementary formula from Σ. Then $(E(\bar{b}))^+ := P^{n-1}(\bar{b}_i)$, where $P^n(\bar{b}_i)$ is a C-conclusion of a kernel formula $\Box\forall\bar{x}(E(\bar{x}) \supset R(\bar{x}) \wedge P^n(\bar{b}_i))$ from $\Box\Omega$.*

Let us define the infinitary calculus $G_{L\omega}$ with the help of which the proposed decision procedure is founded. Derivations in the calculus $G^*_{L\omega}$ are constructed in the bottom-up manner in the form of an infinite tree.

Definition 3 (calculi $G^*_{L\omega}$, G^*). *The calculus $G^*_{L\omega}$ is defined by the following postulates.*

The axiom $(\exists)$: $\Gamma, E_i(b_1, \ldots, b_m) \to \exists y_1 \ldots y_m \bigvee_{j=1}^{n} E_j(y_1, \ldots, y_m)$ $(m \leqslant n,\ m \geqslant 0,\ 1 \leqslant i \leqslant n)$.

The rules consist of the ω-type rule:

$$\frac{\Gamma \to A; \Gamma \to A^1; \ldots; \Gamma \to A^k; \ldots}{\Gamma \to \Box A} (\to \Box_\omega)$$

and the following (loop-free) integrated separation induction-free rule:

$$\frac{(\Sigma)^+, \Pi, \Box\Omega \to B^{k-1}}{\Sigma, \Pi^1, \Box\Omega \to B^k} \textit{(ISIF)}, \quad k > 0,$$

where $\Sigma = \varnothing$ or consists of elementary formulas; $\Pi^1 = \varnothing$ or consists of atomic formulas of the shape E^l $(l > 0)$; $\Box\Omega$ consists of kernel formulas; $B = \exists y_1, \ldots, y_n \bigvee_{i=1}^{m} E_i(\bar{y}_i)$ $(m \leqslant n)$, where operation $(+)$ is the same as in Definition 2.

The calculus G^* is obtained from $G^*_{L\omega}$ by dropping the ω-type rule $(\to \Box_\omega)$.

Theorem 1. *(a) The calculus $G_{L\omega}$ is sound and complete for TD-sequents. (b) The calculus G^* is a decision procedure.*

Let us define the generalized integrated separation rule (GIS) which is the main tool of the proposed deductive procedure Sat and which is applied to any non-induction-free TD-sequent.

Definition 4 (generalized integrated separation rule: (GIS), successful application of (GIS)). *Let $S = \Sigma, \Pi^1, \Box\Omega \to \Box B$ be a TD-sequent. Let $(\Sigma)^+$ mean the same as in definition of (ISIF), then the generalized integrated separation rule (GIS) is as follows:*

$$\frac{\Sigma, \Pi^1, \Box\Omega \to B;\ (\Sigma)^+, \Pi, \Box\Omega \to \Box B}{\Sigma, \Pi^1, \Box\Omega \to \Box B}\ (GIS).$$

If the left premise of (GIS), i.e., the sequent $S_1 = \Sigma, \Pi^1, \Box\Omega \to B$ is such that $G^* \vdash S_1$ we say that bottom-up application of (GIS) is successful.

Now we are going to define the basic part of Sat – the k-th resolvent (in symbols: $Re^k(S)$).

Definition 5 (similarity index, k-th resolvent: $Re^k(S)$, parametrical part of $Re^k(S)$). *Let $S = \Sigma, \Pi^1, \Box\Omega \to \Box B$ be a TD-sequent and $p_1, \dots, p_n$ indices of kernel C-conclusion formulas of S, then $p(S) = \sum_{i=1}^{n} p_i$ is similarity index of S. Let S be a TD-sequent, then the k-th resolvent of a TD-sequent S (in symbols: $Re^k(S)$) is defined in the following way: $Re^0(S) = S$. Let $Re^k(S) = S_k = \Sigma, \Pi^1, \Box\Omega \to \Box B$ then $Re^{k+1}(S)$ is defined in the following way.*

1. Let us bottom-up apply the rule (GIS) to S_k and S_{k1}, S_{k2} be the left and right premises of the application of (GIS).

2. If $G^* \nvdash S_{k1}$, then $Re^{k+1}(S) = \bot$ (false) and the calculation of $Re^{k+1}(S)$ is stopped.

3. Let $G^* \vdash S_{k1}$ (i.e., the bottom-up application of (GIS) is successful), then $Re^{k+1}(S) = S_{k2} = (\Sigma)^+, \Pi, \Box\Omega \to \Box B$; $(\Sigma)^+, \Pi$ is parametrical part of $Re^{k+1}(S)$.

4. If $Re^{k+1}(S) = S_{k2}$ and $k + 1 = p(S)$, then the calculation of $Re^{k+1}(S)$ is finished.

Analogously as in [2] we get the following

Lemma 1. *(a) Let S be a TD-sequent and all bottom-up applications of (GIS) in constructing $Re^k(S)$ are successful and $p = p(S)$ be similarity index of S. Then $Re^p(S) = S$.*

(b) The problem of calculation of $Re^p(S)$ is decidable.

Definition 6 (deductive procedure Sat, TD-sequent derivable by the help of Sat). *The deductive procedure Sat consists of decision procedure $Re^k(S)$. TD-sequent S is derivable by the help of Sat (in symbols: $Sat \vdash S$) if $Re^p(S) = S$, $p = p(S)$, $p(S)$ is the similarity index of S.*

Now let us introduce the "invariant calculus" IN.

Definition 7 (invariant calculus IN). *The calculus IN is obtained from the calculus $G^*_{L\omega}$ in the following way: (1) adding the logical rules $(\to \wedge)$, $(\wedge \to)$, $(\vee \to)$, $(\to \vee)$; (2) adding the axiom $\Gamma, \Box A \to \Box A^1$; (3) replacing the rule $(\to \Box_\omega)$ by the following rule*

$$\frac{\Gamma \to R;\ R \to R^1;\ R \to A}{\Sigma, \Pi^1, \Box\Omega \to \Box A}\ (\to \Box),$$

where $\Gamma = \Sigma, \Pi^1, \Box\Omega$ and invariant formula has the following shape $R = \bigvee_{i=1}^{n} \Gamma_i^\wedge \wedge \Box\Omega$, where $\Box\Omega$ is the kernel of the given TD-sequent $S = \Sigma, \Pi^1, \Box\Omega \to \Box A$; Γ_k is the parametrical part of k-th $Re^k(S)$, $n = p(S)$, i.e., the similarity index of S; $\Gamma_i^\wedge$ is the conjunction formulas from Γ_i.

Theorem 2. *Let S be TD-sequent then $G_{L\omega} \vdash S \iff Sat \vdash S \iff IN \vdash S$.*

From Theorems 1, 2 and Lemma 2(b) we get

Theorem 3. *The calculi Sat and IN are sound and complete and decidable for TD-sequents.*

Example 1. Let $S = E(b_2), \Box\Omega \to \Box\exists y(E(y) \vee P^1(y) \vee P(y))$, where $\Box\Omega = \Box\forall x(E(x) \supset R_1(x) \wedge P^2(b_1))$, $\Box\forall y(P(y) \supset R_2(y) \wedge E^1(b_2))$. Then the similarity index $p(S) = 2 + 1 = 3$. It is easy to verify that each calculation of $Re^k(S)$ is successful. Therefore the calculation $Re^k(S)$ stops when $k = 3$ and $Re^3(S) = S$. Hence $Sat \vdash S$. It is easy to verify that $Re^1(S) = P^1(b_1), \Box\Omega \to \Box A$; $Re^2(S) = P(b_1), \Box\Omega \to \Box A$; $Re^3(S) = E(b_2), \Box\Omega \to \Box A$. Therefore $R = (P^1(b_1) \vee P(b_1) \vee E(b_2)) \wedge \Box\Omega$. It is easy to verify that $IN \vdash S$.

References

1. Hodkinson I., Wolter F., Zakharyaschev M.: Decidable fragments of first-order temporal logics. (To appear in: *Annals of Pure and Applied Logic*).
2. Pliuškevičius R.: On an ω-decidable deductive procedure for non-Horn sequents of a restricted FTL. (To appear in: *Proceedings of First International Conference on Computational Logic*).