Lattice Theoretical Analysis of Dung-style AFs – Information and Reachability Order

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Abstract

The distinction between explicit and implicit information is essential in knowledge representation. In case of argumentation frameworks (AFs) the latter comes to light if dynamics are considered. The study of dynamic involvements is one of the most active fields in argumentation theory in recent years. In this paper, we further contribute to this topic via introducing two new orderings between AFs motivated by their carried implicit information. The orderings, called information and reachability order, are analyzed under lattice theoretical aspects.

Keywords

Abstract Argumentation, Dynamics, Orderings, Lattice Theory

1. Introduction

The groundbreaking contribution of Dungs seminal paper [1] was the idea that the evaluation of arguments can be done on an abstract level solely based on their interactions. This means, the concrete logical structure of arguments or the reason why an argument attacks another does not matter and can thus be disregarded. Consequently, arguments and attacks can be represented as nodes and edges in a directed graph which are known as argumentation frameworks (AFs). So-called extensions, i.e. jointly acceptable subsets of the arguments are then determined by argumentation semantics [2].

The set of extensions of an AF F can be seen as the *explicit information* of F. In contrast, the *implicit information* of F comes to light if F undergoes dynamic changes. Both concepts come along with an induced notion of equivalence, namely *ordinary* or *strong equivalence*, respectively. Especially the latter has been studied for many nonmonotonic formalisms such as logic programs [3], causal theories [4], as well as logics in general [5, 6].

The topic of comparing frameworks has received some attention in recent years [7, 8]. In this paper we focus on orderings between AFs induced by their implicit information. More precisely, we introduce and formally analyse two new orderings, so-called *information* and *reachability order*. Both concepts use the notion of a kernel which can be seen as a uniquely determined

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representative of AFs sharing the same implicit information [9]. We consider the classical Dung semantics and perform a lattice theoretical analysis of both orderings.

2. Background

2.1. Abstract Argumentation

We fix a non-finite background set $\mathscr{U}$, so-called universe. An argumentation framework (AF) [1] is a directed graph $F = (A,R)$ where $A \subseteq \mathscr{U}$ represents a set of arguments and $R \subseteq A \times A$ models *attacks* between them. In this paper we consider finite AFs only and we use $\mathscr{F}$ for the set of all these graphs. For a given AF $G = (B,S)$ we use $A(G), R(G)$ and $L(G)$ to refer to its arguments, attacks and self-defeating arguments, i.e. $A(G) = B$, $R(G) = S$ and $L(G) = \{a \in A(G) \mid (a,a) \in R(G)\}$. The union $F \sqcup G$ of two AFs $F = (A,R)$ and $G = (B,S)$ is given as $(A \cup B, R \cup S)$. Analogously, the intersection $F \sqcap G$ is defined as $(A \cap B, R \cap S)$. Moreover, subgraph relation $G \sqsubseteq F$ holds iff $A \subseteq B$ and $R \subseteq S$. For two arguments $a,b \in A$, if $(a,b) \in R$ we say that *a attacks b* as well as *a attacks* (the set) E given that $b \in E \subseteq A$. We say a set E *defends* an argument a or a set E' if any attacker of a respective E' is attacked by some argument of E. A set E is conflict-free in F (for short, $E \in cf(F)$) iff for no $a,b \in E$, $(a,b) \in R$.

A *semantics* is a function $\sigma : \mathscr{F} \to 2^{2^{\mathscr{U}}}$ with $F \mapsto \sigma(F) \subseteq 2^A$. This means, given an AF $F = (A,R)$ a semantics returns a set of subsets of A. These subsets are called σ-*extensions*. In this paper we consider so-called *stable*, *admissible*, *complete*, *grounded* and *preferred* semantics (abbr. *stb*, *ad*, *co*, *gr*, *pr*) already introduced by Phan Minh Dung in 1995 [1].

Definition 1. Let $F = (A,R)$ be an AF and $E \in cf(F)$.

1. $E \in stb(F)$ iff $E \in cf(A)$ and E attacks any $a \notin A \setminus E$.
2. $E \in ad(F)$ iff E defends all its elements,
3. $E \in co(F)$ iff $E \in ad(F)$ and for any a defended by E we have, $a \in E$,
4. $E \in gr(F)$ iff E is $\subseteq$-minimal in $co(F)$ and
5. $E \in pr(F)$ iff E is $\subseteq$-maximal in $co(F)$.

2.2. Lattice Theory

In this section we recall some basic notions from lattice theory [10].

A *partial order* $\leq$ on a set M is a binary relation which is reflexive, antisymmetric and transitive. The pair $(M, \leq)$ is called *partially ordered set*. Finite partial ordered sets can be represented via so-called *Hasse-diagrams*, i.e. a drawing of its transitive reduction.

Definition 2. Given a partial ordered set $(M, \leq)$ and a subset $X \subseteq M$. An element $m \in M$ is called

- *upper bound* of X iff $x \leq m$ for all $x \in X$, and
- *supremum/join* of X ($\sup(X)$) iff it is the $\leq$-least upper bound of X,

- *lower bound* of X iff $m \leq x$ for all $x \in X$, and
- *infimum/meet* of X ($\inf(X)$) iff it is the $\leq$-greatest lower bound of X.

Remember that joins and meets need not to exist in general. There might be no upper/lower bounds at all, or no greatest/lowest bounds among them.

Definition 3. A partial order $(M, \leq)$ is called a *lattice* iff joins and meets exist for any two-element sets $\{m_1, m_2\} \subseteq M$. If only one of those is guarenteed we call it *join-semilattice* or *meet-semilattice*, respectively.

3. Strong Equivalence and Induced Orders

3.1. Characterizing Strong Equivalence

In case of propositional logic we have that sharing the same models guarantees intersubstitutability in any logical context without loss of information. This property does not transfer to mainstream non-monotonic logics [5, 6]. It is not hard to find two AFs F and G possessing the same σ-extensions but differ semantically if augmented by a further AF H. We say that both frameworks are *strongly equivalent* if the latter is impossible. Consider the following formal definition.

Definition 4. Given a semantics σ. Two AFs F and G are *strongly equivalent w.r.t.* σ (for short, $F \equiv_s^\sigma G$) iff for each AF H we have, $\sigma(F \sqcup H) = \sigma(G \sqcup H)$.

Note that strong equivalence is indeed an equivalence relation, i.e. a binary relation being reflexive, symmetric and transitive. In view of the fact that strong equivalence is defined semantically it was a quite surprising result that it can be decided by looking at the syntax only [9]. Oikarinen and Woltran introduced so-called *kernels* of an AF F which are (informally speaking) graphs without redundant attacks w.r.t. any possible expansion. They showed that syntactical identity of suitably chosen kernels characterizes strong equivalence.

Definition 5. Let $\sigma \in \{stb, ad, co, gr\}$. For any AF $F = (A, R)$ we define the σ-*kernel* $F^{k(\sigma)} = \left(A, R^{k(\sigma)}\right)$ as:

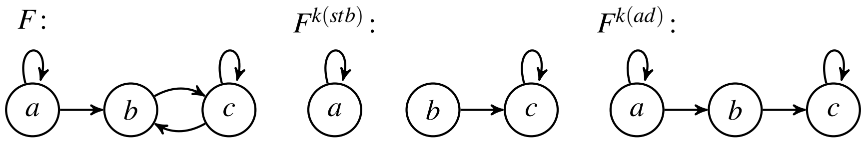
1. $R^{k(stb)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R\}$,
2. $R^{k(ad)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R, \{(b,a),(b,b)\} \cap R \neq \emptyset\}$,
3. $R^{k(co)} = R \setminus \{(a,b) \mid a \neq b, (a,a),(b,b) \in R\}$,
4. $R^{k(gr)} = R \setminus \{(a,b) \mid a \neq b, (b,b) \in R, \{(a,a),(b,a)\} \cap R \neq \emptyset\}$.

The following characterization results for finite AFs are taken from [9]. An exhaustive overview of characterization theorems for different expansion notions in abstract argumentation can be found in [11]. Note that the admissible kernel is even strong enough to characterize preferred semantics.

Theorem 1. *Given two AFs F and G. We have,*

1. $F \equiv_s^\sigma G$ *iff* $F^{k(\sigma)} = G^{k(\sigma)}$ *for any* $\sigma \in \{stb, co, gr\}$,
2. $F \equiv_s^\tau G$ *iff* $F^{k(ad)} = G^{k(ad)}$ *for any* $\tau \in \{ad, pr\}$.

Example 1. Consider F as depicted below. According to Definition 5 we obtain the stable kernel $F^{k(stb)}$ as well as the admissible kernel $F^{k(ad)}$ as graphically presented. Note that redundancy regarding admissible semantics implies redundancy w.r.t. stable semantics.



3.2. Properties of Kernels

We start with some basic properties which will be frequently used throughout the paper.

Proposition 2 (cf. [9, 12])**.** *Given* $k \in \{k(stb), k(ad), k(co), k(gr)\}$. *For any AF F we have:*

1. $A(F) = A\left(F^k\right)$, *(node-preserving)*
2. $L(F) = L\left(F^k\right)$, *(loop-preserving)*
3. $R(F^k) \subseteq R(F)$ *and* *(no further attacks)*
4. $\left(F^k\right)^k = F^k$. *(idempotency)*

We proceed with the following two useful properties stating that subgraphs of kernels as well as intersections of kernels are already free of redundancy. Due to the limited space we will from now on present proofs for stable semantics only.

Proposition 3. *For any two AFs F, G and kernel* $k \in \{k(stb), k(ad), k(co), k(gr)\}$ *we have: If* $G \sqsubseteq F^k$, *then* $G^k = G$.

Proof. First, $G^k \sqsubseteq G$ is given by Proposition 2. Suppose towards a contradiction that $G \not\sqsubseteq G^k$. Since arguments and loops are preserved we deduce an attack $(a,b) \in R(G) \setminus R(G^k)$. Consider $k = k(stb)$. Since $(a,b) \notin R(G^k)$ we infer $(a,a) \in R(G)$. The assumption $G \sqsubseteq F^k$ yields $(a,b), (a,a) \in R(F^k)$ which is impossible in case of stable kernel. The other kernels can be handled in a similar way. □

Proposition 4. *For AFs F, G and* $k \in \{k(stb), k(ad), k(co), k(gr)\}$, $F^k \sqcap G^k = \left(F^k \sqcap G^k\right)^k$.

Proof. Since $F^k \sqcap G^k \sqsubseteq F^k$ we apply Proposition 3 and obtain $\left(F^k \sqcap G^k\right)^k = F^k \sqcap G^k$. □

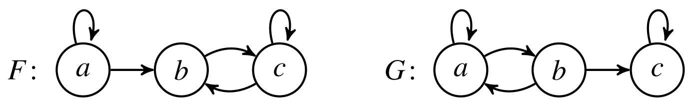
3.3. Information Order

In the following we want to compare AFs with regard to their carried implicit semantical information. This means, we want to consider semantically relevant syntactical material only. Thus, the standard ordering $\sqsubseteq$ is not appropriate as it compares syntactical information without checking its meaningfulness. A suitable candidate is instead to compare the associated kernels of two AFs since kernels do not contain semantical redundant information. Hence, we will define the *information order* directly on the level of equivalence classes $|F|_\sigma = \{G \mid F \equiv_s^\sigma G\}$ since all elements possess the same kernel.

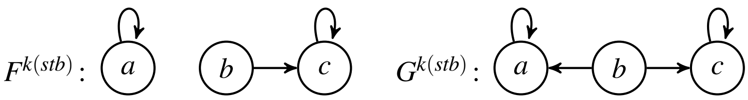
Definition 6. Given a semantics $\sigma \in \{stb, ad, co, gr\}$ and two AFs F and G. The *information order* $\leq_i^\sigma$ on the associated equivalence classes is defined as:

$$|F|_\sigma \leq_i^\sigma |G|_\sigma \quad \textit{iff} \quad F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}.$$

Example 2. Consider again AF F from Example 1 as well as the AF G as depicted below. We observe that neither $F \sqsubseteq G$, nor $G \sqsubseteq F$.



However, if considering the associated stable kernels we are able to compare their carried information. More precisely, we have $F^{k(stb)} \sqsubseteq G^{k(stb)}$ implying that their corresponding equivalence classes are ordered as $|F|_{stb} \leq_i^{stb} |G|_{stb}$.



Before continuing we have to show that $\leq_i^\sigma$ does not depend on the particular choice of representatives.

Proposition 5. *For any $\sigma \in \{stb, co, gr, ad\}$, $\leq_i^\sigma$ is well-defined.*

Proof. Given $\sigma \in \{stb, co, gr, ad\}$ and $|F|_\sigma \leq_i^\sigma |G|_\sigma$ for some AFs F, G. Assume now $F' \in |F|_\sigma$ as well as $G' \in |G|_\sigma$. We have to show $|F'|_\sigma \leq_i^\sigma |G'|_\sigma$.
By Definition 6 we have $F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$. Since $F' \in |F|_\sigma$ as well as $G' \in |G|_\sigma$ we obtain $F' \equiv_s^\sigma F$ and $G' \equiv_s^\sigma G$. According to Theorem 1 we derive $F'^{k(\sigma)} = F^{k(\sigma)}$ and $G'^{k(\sigma)} = G^{k(\sigma)}$. Hence $F'^{k(\sigma)} \sqsubseteq G'^{k(\sigma)}$ is implied showing $|F'|_\sigma \leq_i^\sigma |G'|_\sigma$ as required.

□

The next proposition states that the induced information order is indeed a partial order as claimed.

Proposition 6. *For any $\sigma \in \{stb, ad, co, gr\}$, $\leq_i^\sigma$ is a partial order.*

Proof.

1. reflexive: For any AF F we have, $F^{k(\sigma)} = F^{k(\sigma)}$. Hence, $F^{k(\sigma)} \sqsubseteq F^{k(\sigma)}$ is implied justifying $|F|_\sigma \leq_i^\sigma |F|_\sigma$.

2. antisymmetric: Consider two AFs F and G, s.t. $|F|_\sigma \leq_i^\sigma |G|_\sigma$ as well as $|G|_\sigma \leq_i^\sigma |F|_\sigma$. For antisymmetry we have to prove $|F|_\sigma = |G|_\sigma$.
   The given ordering entails $F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$ as well as $G^{k(\sigma)} \sqsubseteq F^{k(\sigma)}$. Thus, $F^{k(\sigma)} = G^{k(\sigma)}$. Consequently, $F \equiv_s^\sigma G$ and thus, $F \in |G|_\sigma$ proving $|F|_\sigma = |G|_\sigma$.

3. transitive: Consider three AFs F, G and H s.t. $|F|_\sigma \leq_i^\sigma |G|_\sigma$ and $|G|_\sigma \leq_i^\sigma |H|_\sigma$. By definition we obtain $F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$ and $G^{k(\sigma)} \sqsubseteq H^{k(\sigma)}$. Since $\sqsubseteq$ is itself a partial order we derive $F^{k(\sigma)} \sqsubseteq H^{k(\sigma)}$ proving $|F|_\sigma \leq_i^\sigma |H|_\sigma$.

□

For illustration purpose we present the Hasse diagram for the information order regarding stable semantics. We restrict ourselves to AFs containing the arguments a and b only. Note that there are $2^4 = 16$ syntactically different AFs. However, regarding strong equivalence we obtain 9 different equivalence classes only. Some equivalence classes consist of one element, e.g. $|F|_{stb}$ for $F = (\{a,b\},\{(a,b)\})$. In contrast, the AF $G = (\{a,b\},\{(a,a),(b,b)\})$ induces an equivalence class with 4 elements.

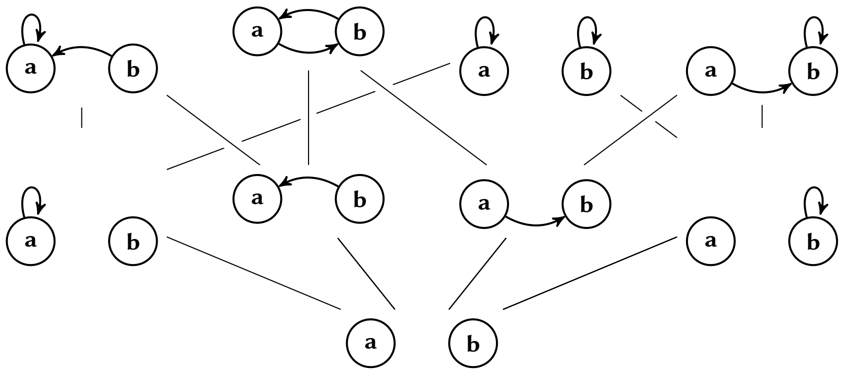


Figure 1: Hasse diagram for $\leq_i^{stb}$

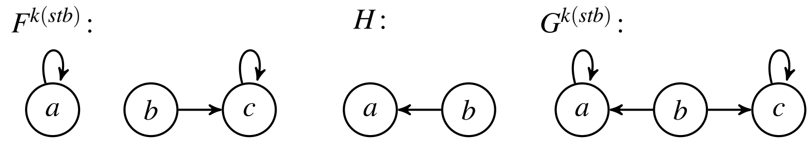
3.4. Reachability Order

Let us turn now to another reasonable ordering. Consider a debate where the current state is represented by the AF F. One interesting question is whether a certain scenario G (and hence a certain output $\sigma(G)$) is reachable, or even more telling, not reachable in future. This means, is it possible to reach G from F via adding further information encoded by H. Clearly, semantical redundant information of F and G does not matter. Consequently, we will define the so-called *reachability order* on the level of strong equivalence classes.

Definition 7. Given a semantics $\sigma \in \{stb, ad, co, gr\}$ and two AFs F and G. The *reachability order* $\leq_r^\sigma$ on the associated equivalence classes is defined as:

$$|F|_\sigma \leq_r^\sigma |G|_\sigma \quad \textit{iff} \quad \exists H : \left(F^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = G^{k(\sigma)}.$$

Example 3. Let us reconsider Example 2. Regarding the information order we found $|F|_{stb} \leq_i^{stb} |G|_{stb}$ since $F^{k(stb)} \sqsubseteq G^{k(stb)}$. The following AF H justifies the same relation w.r.t. reachability, i.e. $|F|_{stb} \leq_r^{stb} |G|_{stb}$. Note that $\left(F^{k(stb)} \sqcup H\right)^{k(stb)} = \left(G^{k(stb)}\right)^{k(stb)} = G^{k(stb)}$.



We will see that the above observation is no coincidence (cf. Proposition 9). However, before turning to more interesting results we have to show well-definedness as well as the defining properties of a partial order.

Proposition 7. *For any* $\sigma \in \{stb, co, gr, ad\}$, $\leq_r^\sigma$ *is well-defined.*

Proof. Given $\sigma \in \{stb, co, gr, ad\}$ and $|F|_\sigma \leq_r^\sigma |G|_\sigma$ for some AFs F, G. Assume now $F' \in |F|_\sigma$ as well as $G' \in |G|_\sigma$. We have to show $|F'|_\sigma \leq_r^\sigma |G'|_\sigma$.
By Definition 7 we have the existence of an AF H, s.t. $\left(F^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = G^{k(\sigma)}$. Since $F' \in |F|_\sigma$ as well as $G' \in |G|_\sigma$ we obtain $F' \equiv_s^\sigma F$ and $G' \equiv_s^\sigma G$. According to Theorem 1 we derive $F'^{k(\sigma)} = F^{k(\sigma)}$ and $G'^{k(\sigma)} = G^{k(\sigma)}$. Consequently, $\left(F'^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = \left(F^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = G^{k(\sigma)} = G'^{k(\sigma)}$. Thus, $|F'|_\sigma \leq_r^\sigma |G'|_\sigma$ as required. $\square$

Proposition 8. *For any* $\sigma \in \{stb, ad, co, gr\}$, $\leq_r^\sigma$ *is a partial order.*

Proof. Given $\sigma \in \{stb, ad, co, gr\}$.

1. reflexive: For any AF F, $F^{k(\sigma)} = F^{k(\sigma)}$. Consequently, the empty AF $H = (\emptyset, \emptyset)$ serves as a witness for $|F|_\sigma \leq_r^\sigma |F|_\sigma$ as $\left(F^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = \left(F^{k(\sigma)}\right)^{k(\sigma)} = F^{k(\sigma)}$.

2. antisymmetric: Consider two AFs F and G, s.t. $|F|_\sigma \leq_r^\sigma |G|_\sigma$ as well as $|G|_\sigma \leq_r^\sigma |F|_\sigma$. For antisymmetry we have to prove $|F|_\sigma = |G|_\sigma$, i.e. $F^{k(\sigma)} = G^{k(\sigma)}$.
The given ordering entails the existence of two AFs H_1 and H_2, s.t. $\left(F^{k(\sigma)} \sqcup H_1\right)^{k(\sigma)} = G^{k(\sigma)}$ (1) and $\left(G^{k(\sigma)} \sqcup H_2\right)^{k(\sigma)} = F^{k(\sigma)}$ (2). Both equations immediately entail $A(F) \subseteq A(G)$, $L(F) \subseteq L(G)$ as well as $A(G) \subseteq A(F)$, $L(G) \subseteq L(F)$. Thus, the initial frameworks and their associated kernels share the same arguments as well as loops.
Towards a contradiction we suppose $(a,b) \in R\left(F^{k(\sigma)}\right) \setminus R\left(G^{k(\sigma)}\right)$. First of all, we derive $a \notin L(F)$. However, according to (1) (a,b) must be deleted in $F^{k(\sigma)} \sqcup H_1$ if kernelized. This means, $a \in L(H_1)$ has to hold. However, this would imply $a \in L(G)$ contradicting $L(F) = L(G)$.
The case $(a,b) \in R\left(G^{k(\sigma)}\right) \setminus R\left(F^{k(\sigma)}\right)$ can be shown in a similar way. Just use equation (2) instead of (1).

3. transitive: Consider three AFs F, G and H s.t. $|F|_\sigma \preceq_r^\sigma |G|_\sigma$ and $|G|_\sigma \preceq_r^\sigma |H|_\sigma$. By definition we obtain the existence of two AFs I_1 and I_2, s.t. $\left(F^{k(\sigma)} \sqcup I_1\right)^{k(\sigma)} = G^{k(\sigma)}$ and $\left(G^{k(\sigma)} \sqcup I_2\right)^{k(\sigma)} = H^{k(\sigma)}$. We have to show $|F|_\sigma \preceq_r^\sigma |H|_\sigma$, i.e. the existence of a witness I_3, s.t. $\left(F^{k(\sigma)} \sqcup I_3\right)^{k(\sigma)} = H^{k(\sigma)}$.

   Consider $I_3 = I_1 \sqcup I_2$. Due to Proposition 2 we have $A\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right) = A\left(H^{k(\sigma)}\right)$ as well as $L\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right) = L\left(H^{k(\sigma)}\right)$. We will show now $R\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right) = R\left(H^{k(\sigma)}\right)$.

   Consider $\sigma = stb$.

   a) Let $(a,b) \in R\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right)$. We infer $(a,a) \notin R\left(F^{k(\sigma)}\right), R(I_1), R(I_2)$ and hence, $(a,a) \notin R\left(\left(F^{k(\sigma)} \sqcup I_1\right)^{k(\sigma)}\right) = R\left(G^{k(\sigma)}\right)$.

      i. Let $(a,b) \in R\left(F^{k(\sigma)} \sqcup I_1\right)$: Since $(a,a) \notin R\left(F^{k(\sigma)} \sqcup I_1\right)$ we deduce $(a,b) \in R\left(\left(F^{k(\sigma)} \sqcup I_1\right)^{k(\sigma)}\right) = R\left(G^{k(\sigma)}\right)$. Since further $(a,a) \notin R(I_2)$ we have $(a,b) \in R\left(\left(G^{k(\sigma)} \sqcup I_2\right)^{k(\sigma)}\right)$ justifying $(a,b) \in R\left(H^{k(\sigma)}\right)$.

      ii. Let $(a,b) \in R(I_2)$: Since $(a,a) \notin R\left(G^{k(\sigma)}\right), R(I_2)$ we deduce $(a,b) \in R\left(\left(G^{k(\sigma)} \sqcup I_2\right)^{k(\sigma)}\right)$ hence $(a,b) \in R\left(H^{k(\sigma)}\right)$.

   b) Consider now $(a,b) \in R\left(H^{k(\sigma)}\right) = R\left(\left(G^{k(\sigma)} \sqcup I_2\right)^{k(\sigma)}\right)$.

      This entails again that $(a,a) \notin R\left(G^{k(\sigma)}\right), (I_2)$. Since $G^{k(\sigma)} = \left(F^{k(\sigma)} \sqcup I_1\right)^{k(\sigma)}$ we derive $(a,a) \notin R\left(F^{k(\sigma)}\right), R(I_1)$ as well.

      i. Let $(a,b) \in R\left(G^{k(\sigma)}\right) = R\left(\left(F^{k(\sigma)} \sqcup I_1\right)^{k(\sigma)}\right)$: Due to Proposition 2 we have $(a,b) \in R\left(\left(F^{k(\sigma)} \sqcup I_1\right)\right)$. Since $(a,a) \notin R\left(F^{k(\sigma)}\right), R(I_1), R(I_2)$ we infer $(a,b) \in R\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right)$.

      ii. Let $(a,b) \in R(I_2)$: Since $(a,a) \notin R\left(F^{k(\sigma)}\right), R(I_1), R(I_2)$ we immediately infer $(a,b) \in R\left(\left(F^{k(\sigma)} \sqcup I_1 \sqcup I_2\right)^{k(\sigma)}\right)$.

□

Let us consider the Hasse diagram for reachability regarding stable semantics (Figure 2). Again, we restrict ourselves to AFs containing the arguments a and b only. We observe that the reachability order looks quite different compared to information order. In particular, we have much less incomparable classes. However, a closer look reveals that the information order is part of the reachability order. In the following we will formally prove this observation.

Proposition 9. *For any two AFs F and G and any semantics $\sigma \in \{stb, ad, co, gr\}$ we have: If $|F|_\sigma \preceq_i^\sigma |G|_\sigma$, then $|F|_\sigma \preceq_r^\sigma |G|_\sigma$.*

Proof. Given $|F|_\sigma \leq_i^\sigma |G|_\sigma$. Consequently, $F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$. In order to show $|F|_\sigma \leq_r^\sigma |G|_\sigma$ we have to present a witnessing AF H, s.t. $\left(F^{k(\sigma)} \sqcup H\right)^{k(\sigma)} = G^{k(\sigma)}$. Consider $H = G^{k(\sigma)}$. Since $F^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$ we deduce $\left(F^{k(\sigma)} \sqcup G^{k(\sigma)}\right)^{k(\sigma)} = \left(G^{k(\sigma)}\right)^{k(\sigma)} = G^{k(\sigma)}$ concluding the proof. □

4. Lattice-theoretical Analysis

We now consider the question whether there exist bounds, meets and joins w.r.t. the introduced orders.

4.1. Information Order

4.1.1. Lower Bounds and Meets

The following proposition shows that lower bounds as well as meets always exist. A lower bound contains information which is shared by both frameworks. The meet can thus be understood as the maximum of shared information.

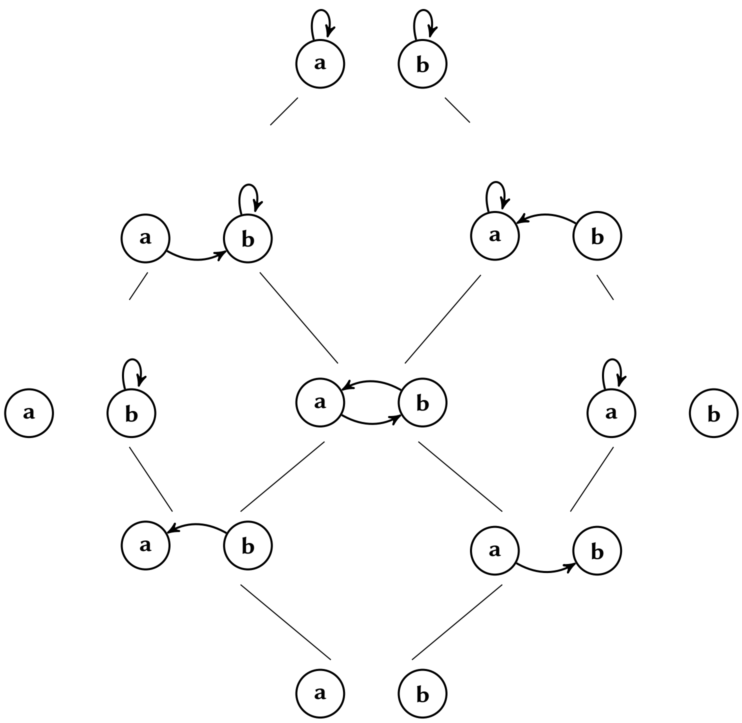


Figure 2: Hasse diagram for $\leq_r^{stb}$

Proposition 10. *For two AFs F, G and semantics* $\sigma \in \{stb, ad, co, gr\}$ *we have*

$$\inf_i^\sigma(|F|_\sigma, |G|_\sigma) = \left|F^{k(\sigma)} \sqcap G^{k(\sigma)}\right|_\sigma$$

Proof.

1. well-definedness: If $F' \in |F|_\sigma$ and $G' \in |G|_\sigma$ we obtain $F^{k(\sigma)} = F'^{k(\sigma)}$ and $G^{k(\sigma)} = G'^{k(\sigma)}$. Consequently, $\left|F^{k(\sigma)} \sqcap G^{k(\sigma)}\right|_\sigma = \left|F'^{k(\sigma)} \sqcap G'^{k(\sigma)}\right|_\sigma$.

2. lower bound: Consider two AFs F, G and $H = F^{k(\sigma)} \sqcap G^{k(\sigma)}$. Since $H \sqsubseteq F^{k(\sigma)}$ and $H \sqsubseteq G^{k(\sigma)}$ we deduce $H^{k(\sigma)} \sqsubseteq F^{k(\sigma)}$ as well as $H^{k(\sigma)} \sqsubseteq G^{k(\sigma)}$ (Prop. 3). Hence $|H|_\sigma \leq_i^\sigma |F|_\sigma$ and $|H|_\sigma \leq_i^\sigma |G|_\sigma$ is implied.

3. meet: Assume towards contradiction that H is not the greatest lower bound. Then there has to exist H', s.t. $|H|_\sigma \lneq_i^\sigma |H'|_\sigma \leq_i^\sigma |F|_\sigma, |G|_\sigma$. This means, $\left(F^{k(\sigma)} \sqcap G^{k(\sigma)}\right)^{k(\sigma)} \sqsubsetneq H'^{k(\sigma)} \sqsubseteq F^{k(\sigma)} \sqcap G^{k(\sigma)} \sqsubseteq F^{k(\sigma)}$. Applying Prop. 3 again we obtain $\left(F^{k(\sigma)} \sqcap G^{k(\sigma)}\right)^{k(\sigma)} = F^{k(\sigma)} \sqcap G^{k(\sigma)}$. Thus, $H'^{k(\sigma)}$ can not exist proving that H is indeed the greatest lower bound.

□

4.1.2. Upper Bounds and Joins

In contrast to lower bounds as well as meets we may show the conditional existence of upper bounds as well as joins only. More precisely, a join exists whenever there is at least one upper bound. Interestingly, upper bounds regarding the information order coincide with so-called *models* in *Dung-logics* firstly considered in [13]. It will be part of future work to study further relations between the introduced information order and already established Dung logics. For the moment, we just use some main results from [13] in order to prove the following non-trivial assertions. First, there are upper bounds for two AFs if the union of their corresponding kernels does not contain any redundant attacks. Secondly, if upper bounds exist, the mentioned union serves as a join.

Proposition 11. *For two AFs F, G and semantics* $\sigma \in \{stb, ad, co, gr\}$ *we have:*

$$\{|H|_\sigma \mid |F|_\sigma, |G|_\sigma \leq_i^\sigma |H|_\sigma\} \neq \emptyset \quad \textit{iff} \quad \left(F^{k(\sigma)} \sqcup G^{k(\sigma)}\right)^{k(\sigma)} = F^{k(\sigma)} \sqcup G^{k(\sigma)}$$

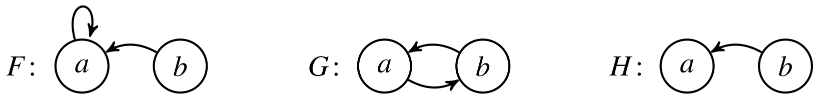
Proof. Combine Theorem 5 and Lemma 6 from [13]. □

Proposition 12. *For two AFs F, G and semantics* $\sigma \in \{stb, ad, co, gr\}$ *we have:*

$$\sup_i^\sigma(|F|_\sigma, |G|_\sigma) = \left|F^{k(\sigma)} \sqcup G^{k(\sigma)}\right|_\sigma \quad \textit{iff} \quad \{|H|_\sigma \mid |F|_\sigma, |G|_\sigma \leq_i^\sigma |H|_\sigma\} \neq \emptyset$$

Proof. It is easy to see that $\sup_i^\sigma$ is well-defined as it uses the kernelized versions only. The claim follows directly if combing Theorem 5 and Lemma 6 from [13]. □

Example 4. Consider the following three AFs which can also be found in Figure 1.



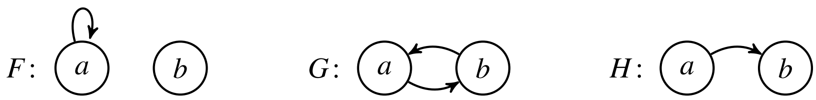
We observe that $F = F^{k(stb)}$, $G = G^{k(stb)}$ and $H = H^{k(stb)}$. First, the join of F and H exists as $\left(F^{k(stb)} \sqcup H^{k(stb)}\right)^{k(stb)} = \left(F^{k(stb)}\right)^{k(stb)} = F^{k(stb)} = F^{k(stb)} \sqcup H^{k(stb)}$. Calculating the join reveals $\sup_i^{stb}(|F|_{stb}, |H|_{stb}) = \left|F^{k(stb)} \sqcup H^{k(stb)}\right|_{stb} = \left|F^{k(stb)}\right|_{stb} = |F|_{stb}$. However, in case of F and G there are no upper bounds as $(a,b) \in R\left(F^{k(stb)} \sqcup G^{k(stb)}\right) \setminus R\left(\left(F^{k(stb)} \sqcup G^{k(stb)}\right)^{k(stb)}\right)$.

4.2. Reachability Order - Bounds, Joins and Meets

4.2.1. Lower Bounds and Meet

The following Example shows that the intersection of classical kernels does not serve for finding the meet.

Example 5. Consider the following three AFs also depicted in Figure 2.



We have $F = F^{k(stb)}$, $G = G^{k(stb)}$ and $H = H^{k(stb)}$. Moreover, $F^{k(stb)} \sqcap G^{k(stb)} = (\{a,b\},\emptyset) = H'$. Clearly $|H'|_{stb}$ is a lower bound for $|F|_{stb}$ and $|G|_{stb}$ (confer Figure 2). However, $|H'|_{stb}$ is not the meet as $|H'|_{stb} \not\geq_r^{stb} |H|_{stb}$ and $|H|_{stb} \leq_r^{stb} |F|_{stb}, |G|_{stb}$ holds.

For the considered semantics we have that the defined kernels represent $\sqsubseteq$-least elements within one equivalence class. However, any equivalence class even possesses a $\sqsubseteq$-greatest element (for more details we refer to the Handbook of Formal Argumentation [11]). In case of stable semantics we may define such an element as $F^{k^+(stb)} = (A(F), R(F) \cup \{(a,b) \mid a \neq b, (a,a) \in R(F)\})$. This means, instead of deleting all redundant attacks, we add every single one of them. Note that the positive version of a kernel is also uniquely determined. The corresponding kernel versions for the other considered semantics are defined in the same fashion. Due to the limited space we will omit them as well as all subsequent proofs.

Proposition 13. *For two AFs F, G and semantics $\sigma \in \{stb, ad, co, gr\}$ we have:*

$$\inf_r^{\sigma}(|F|_\sigma, |G|_\sigma) = \left|F^{k^+(\sigma)} \sqcap G^{k^+(\sigma)}\right|_\sigma$$

4.2.2. Upper Bounds and Join

Example 6. Considering again the AFs F and G depicted in Example 4. We observed that for those two AFs no join under information order exists, because $R\left(F^{k(stb)} \sqcup G^{k(stb)}\right) \neq R\left(\left(F^{k(stb)} \sqcup G^{k(stb)}\right)^{k(stb)}\right)$. However, inspecting Figure 2 reveals that there is candidate for a join w.r.t. reachability, namely $\left|\left(F^{k(stb)} \sqcup G^{k(stb)}\right)^{k(stb)}\right|_{stb}$.

The following proposition shows that our guess happens to be the join in general.

Proposition 14. *For two AFs F, G and semantics $\sigma \in \{stb, ad, co, gr\}$ we have:*

$$\sup_r^\sigma(|F|_\sigma, |G|_\sigma) = \left| \left(F^{k(\sigma)} \sqcup G^{k(\sigma)}\right)^{k(\sigma)} \right|_\sigma$$

5. Conclusion and Related Work

We introduced two orderings on equivalence classes of AFs, namely the information and reachability order. We showed that AFs under $\leq_i^\sigma$ form a semi-lattice as a join does not always exist. However, AFs under $\leq_r^\sigma$ form a full-lattice. The topic of comparing frameworks has received some attention in recent years. In the report by Skiba [7], the ordering of arguments of an AF is examined by means of ranking based semantics. The concept of single arguments is extended to sets of arguments. The paper by Sakama and Inoue [8] considers two orderings over AFs. Both orders compare whole extension sets and establish a relation between two AFs F and G if for any extension of F (G) there is a superset being an extension of G (F). This can be seen as possible orderings regarding explicit information. Finally, they lift their orderings to dynamic environments and provide a connection to strong equivalence of AFs. We mention that our newly introduced orderings are incomparable with those considered in [8]. We leave a detailed comparison for future work.

Regarding future work there is one quite interesting question, namely the relation between our information order and the Dung logics considered in [13]. From a computational point of view there is one further interesting question, namely how to decide efficiently whether two AFs are in reachability relation. By definition we have to check the existence of a certain expansion. The considered four kernels serve for different argumentation semantics. However, in future we plan to extend our analysis to further semantics not covered by these kernels like the recently introduced family of semantics based on weak admissibility [14].

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