Mathematical War Theories in Business: Lanchester's Equations Law, to the Commercial World

Evangelia-Eleni Priska [1], Dimitrios Kallivokas [1] and Miltiadis Chalikias [1]

[1] *University of West Attica, 250 Thivon & P. Ralli str, Egaleo, Greece*

Abstract

This paper deals with the connection of the most well-known mathematical theory of war, Lanchester's equations Law, to the commercial world. In order to evaluate a range of military conflicts throughout WWII, mathematical models in military operations research and combat modeling were frequently deployed. One of these, Lanchester's models describe the outcome of warfare between two similar forces without backups, for time [0, t).

Keywords

Mathematical war theories, Lanchester's equations law, Decision making, Commercial world, Industry

1. Introduction - Literature Review

One of the most important issues on the military field is the prediction of a battle outcome with ultimate goal the prediction of the winner. However, the scientific background was not always sufficient to bring such a result. Many great military philosophers, historians and generals tried to formulate principles, with which a Military Force could be led to victory. All those principles were theoretical and they had the form of rules. For this reason, Mathematical Models and Theories started to be used.

The first attempts, for not so theoretical approaches, started at the beginning of 20th century. At start, Differential Systems were used, that included mathematical relations in order to describe the loses and the size of two opponents military forces.

In 1902, American Jehu Valentine Chase, made a first attempt. This attempt took 70 years to be recognized. Chase studied the result of the battle between the gunboats of his time and he concluded that an important factor to predict the outcome of a combat was the number of the opponent's forces. So, he managed to create a system of 1st Class Ordinary Differential Equation. [1]

In 1905, American B. A. Fiske had published a very simple version of equations, that later were known as Lanchester's equation.[2]

Later, in 1914, English engineer Frederick William Lanchester worked with the same purpose and succeeded to establish mathematical relations between loses and the size of two opponent camps. At first, his main purpose was to understand the tactics that were applied by the English Navy and especially the Admiral Horace Nelson. Then he tried to predict the outcome of World War I, that had already become, and the air fights in general. His model is considered, until these days, very effective in many kinds of battle fields. In this way, he set the bases for the Mathematical War of Theories with the Quadratic Law, and that's why those differential systems, which are known as "Lanchester Mathematical Combat Models" [1]. They are considered as the most complete and extensive. Lanchester's equations have similarities with Richardson law as they both use differential equation systems [3].

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EMAIL: e.priska@gmail.com (A. 1); dimkalliv@yahoo.gr (A. 2); mchalikias@hotmail.com (A. 3)
ORCID: 0000-0003-1482-0926 (A. 3)

A year later, at 1915, unaware of Lanchester's efforts, Russian M. Osipov published an article to the tsarist military newspaper, where he presented his equations on historical events. His equations verified the equations of Lanchester. Soviets believed that Osipov discovered differential equations before Lanchester [4].

Later, after the completion of Interwar period and the Word War II, intensive efforts were made so these models can be verified and extended to more historical battles. For this reason, those models were revised with the introduction of both deterministic and not deterministic generalizations of differential equations. So, these relations enriched. The result was the connection among different military variables, such as the combat duration, surprise attack, nationality, leadership, level of military training, human loses etc.

Among these efforts were those of Lewis Fry Richardson in 1948. He had formulated the Mathematical Theory of Competitive Equipment. Through this, Lanchester's Theory of Mathematical Models came again to surface. The difference between those two theories is that Lanchester's theory describes combat between forces with the same equipment, in quantity and type, while Richardson tried to generalize this theory in a variety of forces. We could say that the two theories are completing each other [1].

Three years later, in 1951, P. M. Morse and G. E. Kimball tried to approach, in a deterministic way, the combats.

J. H. Engel, in 1954, with assistance of the Differential Equation's Model, managed to calculate the human loses of American Forces during the battle in Zima. Engel combined the Quality Theory of differential equations with the Lanchester' System of differential equations. In this way, he managed to predict the outcome of the above battle. With this project, he attracted many scientists to study even more Lanchester's Theory [1].

In 1961, R. L. Helmbold published some contributions on Deterministic Battle Models. Those contributions came up after studying and analyzing 92 combats. Two years later, he extended his studies to even more. From those studies, he concluded that the ratio of enemy's size and victory are connected [5],[6].

Eight years after Engel, in 1962, D. Willard studied 1500 combats that took place between 1618 and 1905. From those combats, only 1000 could contribute to study 's analysis. From his analysis, he came up to a generalization of the homogeneous differential equations that contained both Lanchester's linear model and quadratic model as well. After his analysis, D. Willard concluded that quadratic model should not be used on large scale [7]. Eighteen years later, in 1972, R. W. Samz came up to the same conclusions using less information than Engel [8].

In 1977, fifteen years after Willard's work, J. Fain studied and accomplished statistical analysis in 60 combats during World War II. She tried to apply methodology of Willard's study on small duration combats. Fain did not use the number of weapons that were used by each military force but only parts into which it was divided. After that, she concluded that the shorter the battle duration is, the better the data description could be done by the specific model [9],[10].

In 1985, for first time T. N. Dupuy referred to a Deterministic combat model. Through this model, he introduced a new methodology. With this study, he stated that the size of opposing force could be measured in different ways, depending on the division that could be achieved at the total weapons units of each force [11].

A decade after Fain's study, around 1988, Hartley studied different battles and concluded that only the Lanchester deterministic model is capable to efficiently describe the results of the historic recorded battles. Also, Hartley suggested a theory about Casualties. In 2001, D. S. Hartley III published a study relevant to the prediction of the surprise attacks and the battle's duration. [12].

2. Methodology of Lanchester's Models and Applications in Business

In the initial Lanchester's combat models, at time $t = 0$, two forces of equal martial aptitude, $R(t)$ and $G(t)$, were considered to commence a battle. $R(t)$ neutralizes g soldiers, while $G(t)$ neutralizes r soldiers [13]. The numbers g and r stand for R and G force efficiency coefficients, respectively [14]. In the initial state the below equations are applied:

$$\begin{cases} \dfrac{dG}{dt} = -gR \\ \dfrac{dR}{dt} = -rG \end{cases} \tag{1}$$

In other versions of the above model, it is possible to add additional variables, which for example, will describe how the number of soldiers, of the forces involved during the battle, increase or decrease [15][16].

Chalikias & Skordoulis [16] applied Lanchester's combat models to purchase type of cola in Greece, where they considered the accessible products to be the variables at time t for the two firms. Businesses A and B compete against each other and are the only ones (duopoly). It is assumed that the technology is the same for the two businesses and cannot be changed.

They used financial data from the period 2003 to 2007, to evaluate the coefficients of the model and compared them with real values.

Before them, Chintagunta και Vilcassim [17], utilized Lanchester's model in order to find out the results of promoting in duopoly. The same year, Erickson [18] used statistics to analyze the promoting techniques of two firms. In 1997, Fruchter & Kalish [19] utilized Lanchester's model to depict the elements of a duopoly.

In 2016, [20] extended the model to a 3x3 differential equation system for local mobile communications firms. The solution was:

$$\begin{cases} x(t) = -c_1 \dfrac{c}{a} e^{-t} - c_2 \dfrac{b}{a} e^{-t} + c_3 \dfrac{c}{a} e^{2t} \\ y(t) = c_2 e^{-t} + c_3 e^{2t} \dfrac{c}{b} \\ z(t) = c_1 e^{-t} + c_3 e^{2t} \end{cases} \tag{2}$$

In order to examine the best fitting results on real data, they applied suitable statistical analyses.

Later, the same research team expanded more the model to a 4x4 differential equation system for Greek banks, [21],[22] as bank sector is a really competitive sector and that bank competition is closely connected to the financial stability. They chose the four banks due to the fact that these banks gather more than 97% of the market share.

The research team used the data set of banks' profit between 2009 and 2015.

For the solution of the above system, it was used the Eigenvalue – Eigenvector method. The homogeneous linear system was converted in the matrix form:

$$\frac{dX}{dt} = A \cdot X \tag{3}$$

The previous research team along with Triantafyllou and Kallivokas [23] examined the case of healthcare companies in Athens Stock Exchange. They examined a nxn differential equation system.

The general solution of this system is:

$$\begin{gathered} x_1(t) = -\left(\sum_{j=1}^{n-1} c_j \frac{a_{n-j+1}}{a_1}\right) e^{-t} + c_n \frac{a_n}{a_1} e^{(n-1)t} \\ x_j(t) = c_{n-j+1} e^{-t} + c_n \frac{a_n}{a_j} e^{(n-1)t}, \text{ j = 2, …, n-1} \\ x_n(t) = c_1 e^{-t} + c_n e^{(n-1)t} \end{gathered} \tag{4}$$

For the set of the examined stock data, it has been estimated the function of time, in order for the above solution to fit in real data. There were used different functions of time for every stock due to different monotony of every stock.

In order to examine the best fitting results, they applied suitable statistical analyses (Wilcoxon test due to the fact that normality was not satisfied).

It must be mentioned that except Lanchester's and Richardson models many business applications have to do with probabilistic war theories [25][26].

3. Further Ideas

Moreover, generalized Lanchester's model showed up to portray the advancement of the fight conducted between military powers that use the same weapons with no help or working misfortune. The rate of misfortune of each side is relative to the number of remaining units of weapon of the rival side.

If there are two rival armed military forces [14] then we can assume that force 1 and force 2 have M_1 and M_2, respectively, different sorts of weapons $O\Sigma_1^{(1)}, O\Sigma_1^{(2)}, O\Sigma_1^{(3)}, \ldots, O\Sigma_1^{(M_1)}$ and $O\Sigma_2^{(1)}, O\Sigma_2^{(2)}, O\Sigma_2^{(3)}, \ldots, O\Sigma_2^{(M_2)}$.

This conducts to a system that can be described from the expression: $\dot{X} = \Theta X$ [14].

4. Research Results and Discussion

From the above applications of Lanchester's model, it was determined that such theory can be applied to modern businesses, due to the fact that firms compete with one another such as a warfare on a battlefield. In this case, we count their market share instead of the battles win [27].

As a result, models of mathematical war theories can be employed in cases involving modern enterprises, after necessary adjustments and the adoption of relevant theoretical conditions.

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