A Note on Counting Basic Choice Functions with Formal Concept Analysis

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Abstract

The paper aims at not only counting how many basic choice functions exist on a finite set of alternatives (all, non-empty, single-element valued) but shows how to do this with the help of Formal Concept Analysis. Moreover, we introduce the contextual representation of a choice function by considering the formal context of its map from 2^A to 2^A. We also characterise these contexts as nominal scales of a certain size and build a lattice of all choice functions with their help. Last but not least, we study the asymptotic behaviour of those obtained and new counting formulas that do not have a closed form.

Keywords

Choice function, Concept lattice, Combinatorics, Asymptotic analysis

1. Introduction

Choice Theory is formalised with the help of Order Theory [1, 2] and has applications not only in Social Sciences but also in Artificial Intelligence, e.g. to model and learn preferences of agents [3, 4]. In particular, it deals with set-valued functions defined on a set of alternatives, i.e. variants that an individual or (rational) agent can choose based on her preferences or utility function [5, 6, 1].

In this paper, inspired by earlier works on choice functions and Lattice Theory [5, 6, 2] (including Formal Concept Analysis (FCA) as its applied branch [7, 8]), we characterise concept lattices induced by point-wise representations of choice functions considered as formal contexts and count basic choice functions (all, non-empty, single-element valued) for a fixed number of alternatives.

The previous work of Monjardet and Raderanirina [2] studies the space of all choice functions fulfilling certain axioms (called heredity, concordance, and outcast), which forms lattices if one of the axioms is fulfilled. The works of Revenko and Kuznetsov [9, 8] consider various axioms on set functions as (formal) attributes and perform attribute exploration [10] (an interactive semi-automatic procedure of hypotheses generation in terms of attribute implications and checking them by an expert) with functions on sets up to four elements. Not only choice functions were

Published in Sergei O. Kuznetsov, Amedeo Napoli, Sebastian Rudolph (Eds.): The 11th International Workshop "What can FCA do for Artificial Intelligence?", FCA4AI 2023, co-located with IJCAI 2023, August 20 2023, Macao, S.A.R. China, Proceedings, pp. 47–56.

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considered in [9, 8] since the choice functions are intensive, but extensity property was also included.

Other related works on FCA and Choice Theory include learning individual and collective preferences [11], enchaining consensus voting procedures [12], for example, in consensus clustering [13], studying games on concept lattices [14, 15], and attribute ranking in formal concepts with Shapley values [16].

The paper is organised as follows. In Section 2 we give basic definitions from FCA and for considered families of choice functions. Section 3 contains our main results split in three subsections on the proposed conceptual representation of choice functions, three counting formulas, and their asymptotic behaviour, respectively. The last section concludes the paper.

2. Basic Notions

2.1. Formal Concept Analysis

We recall several definitions from Formal Concept Analysis [7], an applied branch of modern Lattice Theory. We reproduce basic definitions from our tutorial [17], for more details see also textbook [18].

A *formal context* $\mathbb{K} = (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G are called the *objects* and the elements of M are called the *attributes* of the context. The notation gIm or $(g, m) \in I$ means that the object g has attribute m.

A special type of context defined on any set S is used in the next section: the *nominal scale* $\mathbb{N}_S := (S, S, =)$.

For $A \subseteq G$ and $B \subseteq M$, let

$$A' := \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}$$

$$B' := \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}.$$

These operators are called *derivation operators* or *concept-forming operators* for $\mathbb{K} = (G, M, I)$.

Proposition 1. *Let (G, M, I) be a formal context, for subsets $A, A_1, A_2 \subseteq G$ and $B \subseteq M$ we have*

1. $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ *(antimonotony of $'$),*
2. $A_1 \subseteq A_2 \Rightarrow A_1'' \subseteq A_2''$ *(monotony of $''$),*
3. $A \subseteq A''$ *(extensity of $''$),*
4. $A' = A'''$ *(hence, $A'''' = A''$, i.e. idempotency of $''$),*
5. $(A_1 \cup A_2)' = A_1' \cap A_2'$,

Similar properties hold for subsets of attributes.

Note that traditionally $\{g\}'$ and $\{m\}'$ are written as g' and m' for brevity.

For $\mathbb{K} = (G, M, I)$, the operators $(\cdot)'' : 2^G \to 2^G$, $(\cdot)'' : 2^M \to 2^M$ are closure operators, i.e. idempotent, extensive, and monotone.

A *formal concept* of a formal context $\mathbb{K} = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The sets A and B are called the extent and the intent of the formal concept (A, B), respectively. The *subconcept-superconcept relation* is given by $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$ $(B_2 \subseteq B_1)$.

The set of all formal concepts of a context $\mathbb{K}$ together with the order relation $\leq$ forms a complete lattice called the *concept lattice* of $\mathbb{K}$ and denoted by $\underline{\mathfrak{B}}(\mathbb{K})$.

2.2. Choice Functions

A choice function on a set A is defined as map $C : 2^A \to 2^A$ such that $C(A) \subseteq A$ (intensity property).

In what follows, we adopt terminology from [1]. Let $\mathscr{A}$ be the set of all non-empty subsets of A, while $\mathscr{C}$ be the set of all choice functions on A. The subset $\mathscr{C}^+$ of $\mathscr{C}$ contains only non-empty choice functions, i.e. $C(X) \neq \emptyset$ for all $X \in \mathscr{A}$.

The set of all single-valued functions $\widehat{\mathscr{C}}$ contains $\hat{C}$ such that $|\hat{C}(X)| = 1$ for all $X \in \mathscr{A}$.

3. Main Results

3.1. Conceptual Representation

Let us form the context representing a choice function as follows $\mathbb{K}_C := (G, M, I)$ with $G := 2^A$, $M := 2^A$, $I \subseteq 2^A \times 2^A$, where for $g \in G$, $m \in M$, gIm iff $C(g) = m$. It is clear that the domain of C, $dom(C)$, is G, while $range(C) \subseteq M$.

Contexts representing non-empty and single-valued functions are denoted $\mathbb{K}_{C^+} := (\mathscr{A}, \mathscr{A}, I_{C^+})$ and $\mathbb{K}_{\hat{C}} := (\mathscr{A}, \mathscr{A}, I_{\hat{C}})$, respectively, where $gI_{C^+}m \iff C^+(g) = m$ and for the last context $gI_{\hat{C}}m \iff \hat{C}(g) = m$ and $|m| = 1$.

Proposition 2. *Let $C \in \mathscr{C}$, $C^+ \in \mathscr{C}^+$, $\hat{C} \in \widehat{\mathscr{C}}$ and $|A| = n$, then the concept lattices of $\mathbb{K}_C = (2^A, 2^A, I_C)$, $\mathbb{K}_{C^+} = (\mathscr{A}, \mathscr{A}, I_{C^+})$ and $\mathbb{K}_{\hat{C}} = (\mathscr{A}, \mathscr{A}, I_{\hat{C}})$ are isomorphic to the lattices of nominal scales $N_n = ([k], [k], =)$[1] where $k = |range(F)|$ for $F \in \{C, C^+, \hat{C}\}$ and 1) $1 \leq k \leq 2^n$, 2) $n \leq k \leq 2^n - 1$, and 3) $k = n$, respectively.*

Proof. 1) $range(C)$ may vary from $\{\emptyset\}$ set to 2^A, which means that the number of $m \in 2^A$ such that $m' \neq \emptyset$ varies from 1 to 2^n. Equality 2) follows from the condition $\forall g \in \mathscr{A} : |g| = 1 \Rightarrow g' = \{g\}$ (by intensity of $C(g)$). Equality 3) follows from the previous condition and condition $\forall g \in \mathscr{A}\, \exists a \in A : g' = \{a\} \land a \in g$.

□

The interpretation of concepts in such lattices is straightforward. Let $m \in M = 2^A$, then (m', m) contains the image m as the intent and its preimage m' (or the fibre $C^{-1}(\{m\})$, the set all of sets that mapped to $\{m\}$) as the extent. Note that $\{m\}'' = \{m\}$ and there are no other concepts than (m', m), (G, G') and (M', M).

The following example is inspired by our previous work on how university entrants are choosing their departments [19].

[1] We use $[n]$ for $\{1, 2, \dots, n\}$

Example 1. *Let us consider a set A with three alternatives a_1 (Computer Science faculty), a_2 (Mathematical faculty), and a_3 (Faculty of Economics). It is known that if an individual S has preferences represented by a binary relation P, then they can be rationalised by a choice function under certain conditions [1]. Since the choice is not necessarily effective (a single-alternative outcome), our individual may choose two alternatives $C(A) = \{a_1, a_2\}$ out of three.*

$\mathbb{K}_C$	$\emptyset$	a_1	a_2	a_3	a_1, a_2	a_1, a_3	a_2, a_3	a_1, a_2, a_3
$\emptyset$	×							
a_1		×						
a_2			×					
a_3	×							
a_1, a_2					×			
a_1, a_3		×						
a_2, a_3							×	
a_1, a_2, a_3					×			

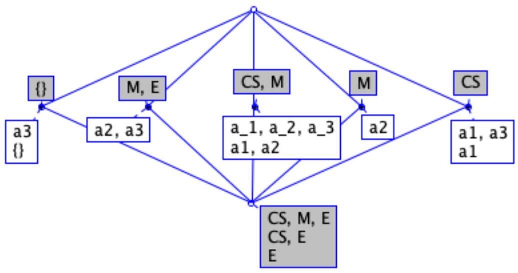

Figure 1: An example context for $\mathbb{K}^C$ and its concept lattice diagram.

Definitely, she loves Mathematics and Computer Science so the choice between those two is not final, $C(\{a_1, a_2\}) = \{a_1, a_2\}$. When only a single faculty out of the last two is available, she chooses it. However, when only the faculty of Economics is offered, she refuses and probably takes a year gap (it might be a very pity that there is no choice among the favourite faculties). However, when educational tracks for mathematics and economics are compared, she might decide to apply both. So, the choices might seem to be not fully rational (in terms of common sense), but they are in line with the definition of $C(\cdot)$.

The line diagram of the corresponding concept lattice $\mathfrak{B}(\mathbb{K}_C)$ on the left in Fig. 1 is drawn in Concept Explorer. We use the so-called reduced labelling when nodes (representing concepts) are labelled with object names when objects are first time added to the extent of a concept (when we go from the bottom concept to the topmost one) and attribute names when attributes are first time added to the intent of a concept (when we go in top-to-bottom direction). Note that we use shorthand CS, M, and E in the attribute labels (the latter denote choices on all alternative subsets), and a_1, a_2, and a_3 in the object labels (the latter denote the subsets of all the alternatives).

Note that our attributes are sets of alternatives and $\{a_3\}$, $\{a_1, a_3\}$, and $\{a_1, a_2, a_3\}$ can be eliminated from the $\mathbb{K}_C$ without affecting the lattice structure. The obtained concept lattice is isomorphic to the so-called diamond lattice M_5.

The lattice of a choice function can be defined via point-wise intersection and union. Let us order objects of $G = 2^A$ first by their cardinality and lexicographically for sets of equal cardinality such that $g_0 = \emptyset, \ldots, g_{2^n-1} = A$. Now, every choice function C is represented by its point-wise vector of images $im(C) = (\bigcup g_0', \bigcup g_1', \ldots, \bigcup g_{2^n-1}')$[2]. Note that $g_0' = \{\emptyset\}$.

For the example in Fig. 1, we have $im(C) = (\emptyset, \{a_1\}, \{a_2\}, \emptyset, \{a_1, a_2\}, \{a_1\}, \{a_2, a_3\}, \{a_1, a_2\})$.

[2]we use $\bigcup$ as a set unfolding operation since $g_i' = \{m_j\}$ and $C(g_i) \equiv \bigcup g_i'$

For two functions C_1 and C_2, the supremum and infimum of their point-wise vectors of images

$$im(C_1) = (\bigcup g_0^{I_1}, \bigcup g_1^{I_1}, \dots, \bigcup g_{2^n-1}^{I_1}) \text{ and } im(C_2) = (\bigcup g_0^{I_2}, \bigcup g_1^{I_2}, \dots, \bigcup g_{2^n-1}^{I_2})$$

(primes are taken in the respective contexts) are defined as follows:

$$im(C_1) \bigvee im(C_2) = (\bigcup g_i^{I_1} \cup \bigcup g_i^{I_2})_{i=0}^{2^n-1},$$

$$im(C_1) \bigwedge im(C_2) = (\bigcup g_i^{I_1} \cap \bigcup g_i^{I_2})_{i=0}^{2^n-1}.$$

Their existence is guaranteed by set intersection and union on images of choice functions.

Let $A = [n]$, then triple $\mathfrak{L}(C) = (Im(\mathscr{C}), \bigvee, \bigwedge)$[3] forms a lattice with $\mathbf{0} = (\emptyset)_{i=0}^{2^n-1}$ and $\mathbf{1} = (\emptyset, \{1\}, \dots, [n])$, while $\mathfrak{S} = (Im(\mathscr{C}^+), \bigwedge)$ is an upper-semilattice and $\mathfrak{A} = (Im(\widehat{\mathscr{C}}), \leq)$ forms an antichain with respect to the point-wise set inclusion of components $\leq$ ($\forall \widehat{C}_1, \widehat{C}_2 \in \widehat{\mathscr{C}} : im(\widehat{C}_1) \leq im(\widehat{C}_2) \iff \bigcup g_i^{I_1} \subseteq \bigcup g_i^{I_2}$ for $i \in [2^n - 1]$).

3.2. Counting Cardinalities

Let us prove the following proposition on the cardinality of $\mathscr{C}_n$, $\mathscr{C}_n^+$, $\widehat{\mathscr{C}_n}$ where $|A| = n$.

Proposition 3.

$$|\mathscr{C}_n| = 2^{n2^{n-1}} \tag{1}$$

$$|\mathscr{C}_n^+| = \prod_{k=1}^{n} (2^k - 1)^{\binom{n}{k}} \tag{2}$$

$$|\widehat{\mathscr{C}_n}| = \prod_{k=1}^{n} k^{\binom{n}{k}} \tag{3}$$

Note that (1) and (2) have been proven in [20] according to [1] (where they are given without proof). We give our proof of (1) and (2) with the help of FCA.

Proof. 1) Let us consider $\mathbf{1} = (\emptyset, \{1\}, \dots, [n])$ it corresponds to $\mathbb{K}_{C_{id}} = (2^A, 2^A, I_{id})$, where $C_{id}(X) = X$ for $X \subseteq A$ and $I_{id} :==$. For each other choice function, $im(C)$ is below $\mathbf{1}$ in the lattice $\mathfrak{L}(\mathscr{C})$, which means that $\bigcup g_i{}' \subseteq \bigcup g_i^{I_{id}}$, where $'$ is taken in the $\mathbb{K}_C$. Thus each row of $\mathbb{K}_C$ has $|2^{\bigcup g_i^{I_{id}}}|$ variants and the choice of each row is independent (we are ready for the product rule).

$$\prod_{k=0}^{2^n-1} 2^{|\bigcup g_i^{I_{id}}|} = \prod_{X \subseteq A} 2^{|X|} = \prod_{k=0}^{n} 2^{k\binom{n}{k}}$$

The last step is due to the presence of each set of size k $\binom{n}{k}$ times. The sum $\sum_{k=0}^{n} k\binom{n}{k}$ equals $n2^{n-1}$.

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[3] $Im(\mathscr{C}) = \{im(C) \mid C \in \mathscr{C}\}$

Table 1
Counting sequences for $|\mathscr{C}_n^+|$, $|\mathscr{C}_n|$, and $|\widehat{\mathscr{C}_n}|$ up to $n = 5$

Formula	OEIS sequence	1	2	3	4	5
$\lvert\mathscr{C}_n\rvert$	https://oeis.org/A061301	2	16	4096	4294967296	1208925819614629174706176
$\lvert\mathscr{C}_n^+\rvert$	–	1	3	189	26254935	3926548231524629155625
$\lvert\widehat{\mathscr{C}_n}\rvert$	https://oeis.org/A229333	1	2	24	20736	309586821120

2) Now, we are not allowed to consider $\bigcup g_i^{I_{id}} = \emptyset$, which implies subtraction of 1 (i.e. $2^k - 1$) when counting variants for the choice of a row in the context $\mathbb{K}_{C^+} = (\mathscr{A}, \mathscr{A}, I_{C^+} \subseteq)$. Here $I_{id} \subseteq \mathscr{A} \times \mathscr{A}$ so g_0 (also m_0) is excluded and the product starts with $k = 1$.

3) Here, compared to the previous case, since we can choose only single-element sets among all $m \in A$, $2^k - 1$ is simply replaced by k.

□

Note that Monjardet and Raderanirina [2] claim that the lattice of all choice functions on a set of alternatives A is Boolean (i.e. atomistic and distributive) with $n2^{n-1}$ atoms, which directly implies the proof of (1). Some authors also rediscovered this value without addressing prior works by Monjardet and Raderanirina [21].

We also note that the beginning values by equations 1 and 3 are listed in OEIS: see integer sequences https://oeis.org/A061301 and https://oeis.org/A229333, respectively.

Before we go to the asymptotic analysis, let us also list some beginning values of these sequences by equations 1–3 in Table 1.

3.3. Asymptotic Analysis

The values represented by equations 2 and 3 have no closed-form formulas but are smaller than the size of the whole space of choice functions. Our goal here is to figure out their asymptotic behaviour to better understand how the sizes of the posets, $|\mathscr{C}_n^+|$, $|\mathscr{C}_n|$, and $|\widehat{\mathscr{C}_n}|$, interrelated.

Proposition 4.

$$\log_2 |\mathscr{C}_n^+| = n2^{n-1} + O(2^n n^{-1/2})$$

Proof. Let us apply $\log_2$ to the product (2).

$$\log_2 |\mathscr{C}_n^+| = \sum_{k=1}^{n} \binom{n}{k} \log_2 2^k + \sum_{k=1}^{n} \binom{n}{k} \log_2(1 - 1/2^k)$$

The first sum equals (1), while the second is more laborious since it has no closed form. Since $\log_2 x \leq x - 1$ for all $x > 0$, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} \log_2(1 - 1/2^k) \leq \sum_{k=1}^{n} \binom{n}{k} (-1/2^k) = -\left(\frac{3}{2}\right)^n + 1$$

Since $1/2 \le (1-1/2^k) < 1$ for $k \ge 1$, we have $-2^n \le \sum\limits_{k=1}^{n} \binom{n}{k} \log_2(1-1/2^k)$. However, we can do better with the lower bound if pull out the maximal binomial coefficient, i.e. the middle (or central) binomial coefficient.

$$\sum_{k=1}^{n} \binom{n}{k} \log_2(1-1/2^k) \ge \binom{n}{\lfloor n/2 \rfloor} \sum_{k=1}^{n} \log_2(1-1/2^k) \ge \binom{n}{\lfloor n/2 \rfloor} \sum_{k=1}^{\infty} \log_2(1-1/2^k)$$

$$\sum_{k=1}^{\infty} \log_2(1-1/2^k) = \log_2 \prod_{k=1}^{\infty} (1-1/2^k) = \log_2 \phi(1/2) \approx -1.79192, \text{ where}$$

$\phi(q) \equiv (q)_\infty \equiv (q;q)_\infty = \prod\limits_{k=1}^{\infty} (1-q^k)$ is the Euler function [22], and $(q)_\infty$ and $(q;q)_\infty$ are q-Pochhammer symbols [23].

The variable term $\binom{n}{\lfloor n/2 \rfloor}$ is $O(2^n n^{-1/2})$ since for even n, we have $\binom{n}{n/2} = \sqrt{\frac{2}{\pi n}} 2^n \big(1 + O(1/n)\big)$ [24] and the following inequalities are known $\frac{2^n}{n} \le \binom{n}{\lfloor n/2 \rfloor} < \sqrt{2/\pi} \cdot 2^n n^{-1/2}$ [25]. $\square$

Proposition 5.

$$\lim_{n\to\infty} \frac{|\mathscr{C}_n^+|}{|\mathscr{C}_n|} = \prod_{k=1}^{\infty} (1 - \frac{1}{2^k})^{\binom{n}{k}} \textit{ diverges to zero.}$$

Proof. From the proof of the previous proposition it follows that

$$\phi(1/2)^{\sqrt{2/\pi} \cdot 2^n n^{-1/2}} \le \prod_{k=1}^{n} (1 - \frac{1}{2^k})^{\binom{n}{k}} \le 2^{-\left(\frac{3}{2}\right)^n + 1} \text{ where } \phi(1/2) \approx 0.2888.$$

When n tends to ∞, both sides tend to 0, and since no terms of the partial product are zeros, the whole product is said to diverge to zero [26, 27]. $\square$

Proposition 6.

$$\log_2 |\widehat{\mathscr{C}_n}| = \Theta(2^n \log_2 n) \,.$$

Proof. To prove the statement we need to show that there are constants $c_1, c_2 > 0$, such that $c_1 2^n \log_2 n \le \log_2 |\widehat{\mathscr{C}_n}| \le c_2 2^n \log_2 n$ for all $n > n_0$.

We can pull out the largest value that $\log_2 n$ takes

$$\log_2 |\widehat{\mathscr{C}_n}| = \sum_{k=1}^{n} \binom{n}{k} \log_2 k \le \log_2 n \sum_{k=1}^{n} \binom{n}{k} = (2^n - 1) \log_2 n.$$

For the lower bound we can split the sum into two parts as follows:

$$\sum_{k=1}^{n} \binom{n}{k} \log_2 k \ge \sum_{k=1}^{\lfloor n/2 \rfloor - 1} \binom{n}{k} \log_2 k + \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n}{k} \log_2 \frac{n}{2} \ge \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n}{k} \log_2 \frac{n}{2} \ge$$

$$\geq \log_2 \frac{n}{2} \sum_{k=1}^{n} \binom{n}{k} / 2 = \frac{1}{2} (\log_2 n - 1)(2^n - 1) \, .$$

$\square$

The last result can be sharpened to $\log_2 |\widehat{\mathscr{C}_n}| = 2^n \log_2 n\big(1 + O(1/log_2 n)\big)$. By changing k to $n-k$ we get $\sum\limits_{k=1}^{n} \binom{n}{k} \log_2 k = \sum\limits_{k=0}^{n-1} \binom{n}{k} \log_2 (n-k)$ and pull out $\log_2 n$, which gives us the term $(2^n - 1) \log_2 n$ and the remaining term is

$$\sum_{k=0}^{n-1} \binom{n}{k} \log_2 (1 - k/n) \leq -\frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} k = -\sum_{k=0}^{n-1} \binom{n-1}{k-1} = 1 - 2^{n-1} \, .$$

4. Conclusion

Monjardet and Raderanirina [2] inform that not all spaces of choice functions with given properties have been explored in the sense that concrete counting formulae exist while a few beginning values are known.

For example, the lattice of choice functions satisfying hereditary axiom has size $|\mathfrak{L}(\mathscr{C}_n^H)| = (D_{n-1})^n$, where D_n is the n-th Dedekind number [2]. And thus we get the new value $|\mathfrak{L}(\mathscr{C}_{10}^H)|$ with recently obtained D_9 [28][4] (with FCA):

$$286386577668298411128469151667598498812366^{10}.$$

We hope to continue this work on combinatorial properties of choice functions with FCA tools for their representation and counting and perform asymptotic analysis (if necessary).

Acknowledgments

This study was implemented in the Basic Research Program's framework at HSE University.

We would like to thank the anonymous reviewers for relevant suggestions and the OEIS maintainers. We also would like to thank Lev P. Shibasov and Fuad T. Aleskerov for the inspirational lectures on Analysis and Choice Theory, respectively. Last but not least, we are grateful to Dmitry V. Vinogradov for useful remarks on the notation.

References

[1] F. Aleskerov, D. Bouyssou, B. Monjardet, Utility maximization, choice and preference, volume 16, Springer Science & Business Media, 2007.

[2] B. Monjardet, V. Raderanirina, Lattices of choice functions and consensus problems, Social Choice and Welfare 23 (2004) 349–382. doi:10.1007/s00355-003-0251-9.

[4] https://oeis.org/A000372

[3] C. Boutilier, I. Caragiannis, S. Haber, T. Lu, A. D. Procaccia, O. Sheffet, Optimal Social Choice Functions, Artif. Intell. 227 (2015) 190–213. doi:10.1016/j.artint.2015.06.003.

[4] G. Pigozzi, A. Tsoukiàs, P. Viappiani, Preferences in artificial intelligence, Annals of Mathematics and Artificial Intelligence 77 (2016) 361–401. doi:10.1007/s10472-015-9475-5.

[5] H. Moulin, Choice functions over a finite set: A summary, Social Choice and Welfare 2 (1985) 147–160. doi:10.1007/BF00437315.

[6] M. Aizerman, F. Aleskerov, Voting operators in the space of choice functions, Mathematical Social Sciences 11 (1986) 201–242.

[7] B. Ganter, R. Wille, Formal Concept Analysis - Mathematical Foundations, Springer, 1999. doi:10.1007/978-3-642-59830-2.

[8] A. Revenko, S. O. Kuznetsov, Attribute exploration of properties of functions on sets, Fundam. Informaticae 115 (2012) 377–394. doi:10.3233/FI-2012-660.

[9] A. Revenko, S. O. Kuznetsov, Attribute exploration of properties of functions on ordered sets, in: M. Kryszkiewicz, S. A. Obiedkov (Eds.), Proceedings of the 7th International Conference on Concept Lattices and Their Applications, Sevilla, Spain, October 19-21, 2010, volume 672 of *CEUR Workshop Proceedings*, CEUR-WS.org, 2010, pp. 313–324. URL: https://ceur-ws.org/Vol-672/paper28.pdf.

[10] B. Ganter, Attribute exploration with background knowledge, Theoretical Computer Science 217 (1999) 215–233. doi:10.1016/S0304-3975(98)00271-0, oRDAL'96.

[11] S. Obiedkov, Parameterized ceteris paribus preferences over atomic conjunctions under conservative semantics, Theoretical Computer Science 658 (2017) 375–390. doi:10.1016/j.tcs.2016.01.035.

[12] F. Domenach, A. Tayari, Implications of Axiomatic Consensus Properties, in: B. Lausen, D. Van den Poel, A. Ultsch (Eds.), Algorithms from and for Nature and Life, Springer International Publishing, Cham, 2013, pp. 59–67.

[13] A. Bocharov, D. Gnatyshak, D. I. Ignatov, B. G. Mirkin, A. Shestakov, A lattice-based consensus clustering algorithm, in: M. Huchard, S. O. Kuznetsov (Eds.), Proc. of CLA 2016, volume 1624 of *CEUR Workshop Proceedings*, CEUR-WS.org, 2016, pp. 45–56. URL: https://ceur-ws.org/Vol-1624/paper4.pdf.

[14] U. Faigle, M. Grabisch, A. Jiménez-Losada, M. Ordóñez, Games on concept lattices: Shapley value and core, Discrete Applied Mathematics 198 (2016) 29–47. doi:https://doi.org/10.1016/j.dam.2015.08.004.

[15] K. Maafa, L. Nourine, M. S. Radjef, Algorithms for computing the Shapley value of cooperative games on lattices, Discret. Appl. Math. 249 (2018) 91–105. doi:10.1016/j.dam.2018.03.022.

[16] D. I. Ignatov, L. Kwuida, On Shapley value interpretability in concept-based learning with formal concept analysis, Ann. Math. Artif. Intell. 90 (2022) 1197–1222. doi:10.1007/s10472-022-09817-y.

[17] D. I. Ignatov, Introduction to formal concept analysis and its applications in information retrieval and related fields, in: P. Braslavski, N. Karpov, M. Worring, Y. Volkovich, D. I. Ignatov (Eds.), RuSSIR 2014, volume 505 of *Communications in Computer and Information Science*, Springer, 2014, pp. 42–141. doi:10.1007/978-3-319-25485-2_3.

[18] B. Ganter, S. A. Obiedkov, Conceptual Exploration, Springer, 2016. doi:10.1007/978-3-662-49291-8.

[19] N. Romashkin, D. I. Ignatov, E. Kolotova, How university entrants are choosing their department? mining of university admission process with FCA taxonomies, in: M. Pechenizkiy, T. Calders, C. Conati, S. Ventura, C. Romero, J. C. Stamper (Eds.), Proceedings of the 4th International Conference on Educational Data Mining, Eindhoven, The Netherlands, July 6-8, 2011, www.educationaldatamining.org, 2011, pp. 229–234. URL: http://educationaldatamining.org/EDM2011/wp-content/uploads/proc/edm2011_paper31_short_Romashkin.pdf.

[20] V. Raderanirina, Treillis et agrégation de familles de Moore et de fonctions de choix, These de doctorat Université Paris 1 (2001).

[21] F. Echenique, Counting combinatorial choice rules, Games and Economic Behavior 58 (2007) 231–245. URL: https://www.sciencedirect.com/science/article/pii/S0899825606000431. doi:https://doi.org/10.1016/j.geb.2006.03.009.

[22] E. W. Weisstein, Euler Function, From MathWorld–A Wolfram Web Resource, 2023. URL: https://mathworld.wolfram.com/EulerFunction.html.

[23] B. C. Berndt, q-Series and Theta-Functions, Springer New York, New York, NY, 1991, pp. 11–86. doi:10.1007/978-1-4612-0965-2_2.

[24] D. E. Knuth, I. Vardi, R. Richberg, 6581. The Asymptotic Expansion of the Middle Binomial Coefficient, The American Mathematical Monthly 97 (1990) 626–630. URL: http://www.jstor.org/stable/2324649.

[25] Z.-H. Sun, Inequalities for binomial coefficients, 2013. arXiv:1310.0353.

[26] B. P. Demidovich, Problems in Mathematical Analysis, American First Edition ed., Mir Publishers, 1989.

[27] H. Jeffreys, B. Jeffreys, Methods of Mathematical Physics, Cambridge Mathematical Library, 3 ed., Cambridge University Press, 1999. doi:10.1017/CBO9781139168489.

[28] C. Jäkel, A computation of the ninth Dedekind Number, 2023. arXiv:2304.00895.