A Note on Counting Basic Choice Functions with Formal Concept Analysis

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Abstract
The paper aims at not only counting how many basic choice functions exist on a finite set of alternatives (all, non-empty, single-element valued) but shows how to do this with the help of Formal Concept Analysis. Moreover, we introduce the contextual representation of a choice function by considering the formal context of its map from $2^{A}$ to $2^{A}$. We also characterise these contexts as nominal scales of a certain size and build a lattice of all choice functions with their help. Last but not least, we study the asymptotic behaviour of those obtained and new counting formulas that do not have a closed form.

Keywords
Choice function, Concept lattice, Combinatorics, Asymptotic analysis

1. Introduction

Choice Theory is formalised with the help of Order Theory [1, 2] and has applications not only in Social Sciences but also in Artificial Intelligence, e.g. to model and learn preferences of agents [3, 4]. In particular, it deals with set-valued functions defined on a set of alternatives, i.e. variants that an individual or (rational) agent can choose based on her preferences or utility function [5, 6, 1].

In this paper, inspired by earlier works on choice functions and Lattice Theory [5, 6, 2] (including Formal Concept Analysis (FCA) as its applied branch [7, 8]), we characterise concept lattices induced by point-wise representations of choice functions considered as formal contexts and count basic choice functions (all, non-empty, single-element valued) for a fixed number of alternatives.

The previous work of Monjardet and Raderanirina [2] studies the space of all choice functions fulfilling certain axioms (called heredity, concordance, and outcast), which forms lattices if one of the axioms is fulfilled. The works of Revenko and Kuznetsov [9, 8] consider various axioms on set functions as (formal) attributes and perform attribute exploration [10] (an interactive semi-automatic procedure of hypotheses generation in terms of attribute implications and checking them by an expert) with functions on sets up to four elements. Not only choice functions were...
considered in [9, 8] since the choice functions are intensive, but extensity property was also included.

Other related works on FCA and Choice Theory include learning individual and collective preferences [11], enchaining consensus voting procedures [12], for example, in consensus clustering [13], studying games on concept lattices [14, 15], and attribute ranking in formal concepts with Shapley values [16].

The paper is organised as follows. In Section 2 we give basic definitions from FCA and for considered families of choice functions. Section 3 contains our main results split in three subsections on the proposed conceptual representation of choice functions, three counting formulas, and their asymptotic behaviour, respectively. The last section concludes the paper.

2. Basic Notions

2.1. Formal Concept Analysis

We recall several definitions from Formal Concept Analysis [7], an applied branch of modern Lattice Theory. We reproduce basic definitions from our tutorial [17], for more details see also textbook [18].

A formal context \( \mathbb{K} = (G, M, I) \) consists of two sets \( G \) and \( M \) and a relation \( I \) between \( G \) and \( M \). The elements of \( G \) are called the objects and the elements of \( M \) are called the attributes of the context. The notation \( g I m \) or \((g, m) \in I \) means that the object \( g \) has attribute \( m \).

A special type of context defined on any set \( S \) is used in the next section: the nominal scale \( \mathbb{N}_S := (S, S, =) \).

For \( A \subseteq G \) and \( B \subseteq M \), let

\[
A' := \{ m \in M \mid (g, m) \in I \text{ for all } g \in A \}
\]

\[
B' := \{ g \in G \mid (g, m) \in I \text{ for all } m \in B \}.
\]

These operators are called derivation operators or concept-forming operators for \( \mathbb{K} = (G, M, I) \).

Proposition 1. Let \( (G, M, I) \) be a formal context, for subsets \( A, A_1, A_2 \subseteq G \) and \( B \subseteq M \) we have

1. \( A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1' \) (antimonotony of’’),
2. \( A_1 \subseteq A_2 \Rightarrow A_2'' \subseteq A_1'' \) (monotony of’’’’),
3. \( A \subseteq A'' \) (extensity of’’’’),
4. \( A' = A''' \) (hence, \( A'' = A'''' \), i.e. idempotency of’’’’),
5. \( (A_1 \cup A_2)' = A_1' \cap A_2' \),

Similar properties hold for subsets of attributes.

Note that traditionally \( \{g\}' \) and \( \{m\}' \) are written as \( g' \) and \( m' \) for brevity.

For \( \mathbb{K} = (G, M, I) \), the operators \((\cdot)' : 2^G \rightarrow 2^G, (\cdot)'' : 2^M \rightarrow 2^M\) are closure operators, i.e. idempotent, extensive, and monotone.

A formal concept of a formal context \( \mathbb{K} = (G, M, I) \) is a pair \((A, B)\) with \( A \subseteq G, B \subseteq M, A' = B \) and \( B' = A \). The sets \( A \) and \( B \) are called the extent and the intent of the formal concept \((A, B)\),
respectively. The subconcept-superconcept relation is given by \((A_1, B_1) \leq (A_2, B_2)\) iff \(A_1 \subseteq A_2\) \((B_2 \subseteq B_1)\).

The set of all formal concepts of a context \(K\) together with the order relation \(\leq\) forms a complete lattice called the concept lattice of \(K\) and denoted by \(\mathcal{B}(K)\).

### 2.2. Choice Functions

A choice function on a set \(A\) is defined as map \(C : 2^A \rightarrow 2^A\) such that \(C(A) \subseteq A\) (intensity property).

In what follows, we adopt terminology from [1]. Let \(\mathcal{A}\) be the set of all non-empty subsets of \(A\), while \(\mathcal{C}\) be the set of all choice functions on \(A\). The subset \(\mathcal{C}^+\) of \(\mathcal{C}\) contains only non-empty choice functions, i.e. \(C(X) \neq \emptyset\) for all \(X \in \mathcal{A}\).

The set of all single-valued functions \(\hat{\mathcal{C}}\) contains \(\hat{C}\) such that \(|\hat{C}(X)| = 1\) for all \(X \in \mathcal{A}\).

### 3. Main Results

#### 3.1. Conceptual Representation

Let us form the context representing a choice function as follows \(K_C := (G, M, I)\) with \(G := 2^A\), \(M := 2^A, I \subseteq 2^A \times 2^A\), where for \(g \in G, m \in M\), \(g I m\) iff \(C(g) = m\). It is clear that the domain of \(C\), \(\text{dom}(C)\), is \(G\), while \(\text{range}(C) \subseteq M\).

Contexts representing non-empty and single-valued functions are denoted \(K_C^+ := (\mathcal{A}, \mathcal{A}, \mathcal{I}^+_C)\) and \(K_{\hat{C}} := (\mathcal{A}, \mathcal{A}, \mathcal{I}_{\hat{C}})\), respectively, where \(g I^+_C m \iff C^+(g) = m\) and for the last context \(g I_{\hat{C}} m \iff \hat{C}(g) = m\) and \(|m| = 1\).

**Proposition 2.** Let \(C \in \mathcal{C}, C^+ \in \mathcal{C}^+, \hat{C} \in \hat{\mathcal{C}}\) and \(|A| = n\), then the concept lattices of \(K_C = (2^A, 2^A, I_C), K_C^+ = (\mathcal{A}, \mathcal{A}, I^+_C)\) and \(K_{\hat{C}} = (\mathcal{A}, \mathcal{A}, I_{\hat{C}})\) are isomorphic to the lattices of nominal scales \(N_n = ([k], [k], =)\) where \(k = |\text{range}(F)|\) for \(F \in \{C, C^+, \hat{C}\}\) and 1) \(1 \leq k \leq 2^n\), 2) \(n \leq k \leq 2^n - 1\), and 3) \(k = n\), respectively.

**Proof.** 1) \(\text{range}(C)\) may vary from \(\{\emptyset\}\) set to \(2^A\), which means that the number of \(m \in 2^A\) such that \(m' \neq \emptyset\) varies from 1 to \(2^n\). Equality 2) follows from the condition \(\forall g \in \mathcal{A} : |g| = 1 \Rightarrow g' = \{g\}\) (by intensity of \(C(g)\)). Equality 3) follows from the previous condition and condition \(\forall g \in \mathcal{A} \exists a \in A : g' = \{a\} \wedge a \in g\).

The interpretation of concepts in such lattices is straightforward. Let \(m \in M = 2^A\), then \((m', m)\) contains the image \(m\) as the intent and its preimage \(m'\) (or the fibre \(C^{-1}(\{m\})\)), the set all of sets that mapped to \(\{m\}\) as the extent. Note that \(\{m\} = \{m\}\) and there are no other concepts than \((m', m)\), \((G, G')\) and \((M', M)\).

The following example is inspired by our previous work on how university entrants are choosing their departments [19].

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\(^1\)We use \([n]\) for \(\{1, 2, \ldots, n\}\)
Example 1. Let us consider a set $A$ with three alternatives $a_1$ (Computer Science faculty), $a_2$ (Mathematical faculty), and $a_3$ (Faculty of Economics). It is known that if an individual $S$ has preferences represented by a binary relation $P$, then they can be rationalised by a choice function under certain conditions [1]. Since the choice is not necessarily effective (a single-alternative outcome), our individual may choose two alternatives $C(A) = \{a_1, a_2\}$ out of three.

<table>
<thead>
<tr>
<th>$K_C$</th>
<th>$\emptyset$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_1, a_2$</th>
<th>$a_1, a_3$</th>
<th>$a_2, a_3$</th>
<th>$a_1, a_2, a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
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<td>$a_1$</td>
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<td>$a_2$</td>
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<td>$a_1, a_2$</td>
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<td>$\times$</td>
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<td>$a_1, a_3$</td>
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<tr>
<td>$a_2, a_3$</td>
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<td>$a_1, a_2, a_3$</td>
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<td>$\times$</td>
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</tbody>
</table>

Figure 1: An example context for $K_C$ and its concept lattice diagram.

Definitely, she loves Mathematics and Computer Science so the choice between those two is not final, $C(\{a_1, a_2\}) = \{a_1, a_2\}$. When only a single faculty out of the last two is available, she chooses it. However, when only the faculty of Economics is offered, she refuses and probably takes a year gap (it might be a very pity that there is no choice among the favourite faculties). However, when educational tracks for mathematics and economics are compared, she might decide to apply both. So, the choices might seem to be not fully rational (in terms of common sense), but they are in line with the definition of $C(\cdot)$.

The line diagram of the corresponding concept lattice $\mathfrak{B}(K_C)$ on the left in Fig. 1 is drawn in Concept Explorer. We use the so-called reduced labelling when nodes (representing concepts) are labelled with object names when objects are first time added to the extent of a concept (when we go from the bottom concept to the topmost one) and attribute names when attributes are first time added to the intent of a concept (when we go in top-to-bottom direction). Note that we use shorthand $CS, M,$ and $E$ in the attribute labels (the latter denote choices on all alternative subsets), and $a_1, a_2,$ and $a_3$ in the object labels (the latter denote the subsets of all the alternatives).

Note that our attributes are sets of alternatives and $\{a_3\}, \{a_1, a_3\},$ and $\{a_1, a_2, a_3\}$ can be eliminated from the $K_C$ without affecting the lattice structure. The obtained concept lattice is isomorphic to the so-called diamond lattice $M_5$.

The lattice of a choice function can be defined via point-wise intersection and union. Let us order objects of $G = 2^A$ first by their cardinality and lexicographically for sets of equal cardinality such that $g_0 = \emptyset, \ldots, g_{2^n-1} = A$. Now, every choice function $C$ is represented by its point-wise vector of images $im(C) = \left(\bigcup g_0', \bigcup g_1', \ldots, \bigcup g_{2^n-1}'\right)^C$. Note that $g_0' = \{\emptyset\}$.

For the example in Fig. 1, we have $im(C) = \left(\emptyset, \{a_1\}, \{a_2\}, \emptyset, \{a_1, a_2\}, \{a_1\}, \{a_2, a_3\}, \{a_1, a_2\}\right)$.

\footnote{we use $\bigcup$ as a set unfolding operation since $g_0' = \{m\}$ and $C(g_0) = \bigcup g_0'$}
For two functions $C_1$ and $C_2$, the supremum and infimum of their point-wise vectors of images

$$im(C_1) = (\bigcup g^I_{01} \cap \bigcup g^I_{02} \cap \ldots \cap \bigcup g^I_{0n}, \ldots, \bigcup g^I_{12} \cap \bigcup g^I_{13} \cap \ldots \cap \bigcup g^I_{1n}, \ldots, \bigcup g^I_{m2} \cap \bigcup g^I_{m3} \cap \ldots \cap \bigcup g^I_{mn})$$

(primes are taken in the respective contexts) are defined as follows:

$$im(C_1) \vee im(C_2) = (\bigcup \bigcup g^I_{i1} \cup \bigcup g^I_{i2})_{i=0}^{n−1},$$

$$im(C_1) \wedge im(C_2) = (\bigcup \bigcup g^I_{i1} \cap \bigcup g^I_{i2})_{i=0}^{n−1}.$$

Their existence is guaranteed by set intersection and union on images of choice functions.

Let $A = [n]$, then triple $𝔏(𝒞) = (𝐼 𝑚(𝒞), \vee, \wedge)$ forms a lattice with $0 = (∅)$ and $1 = ([n])$, while $𝔖 = (𝐼 𝑚(𝒞^+), \wedge)$ is an upper-semilattice and $𝐴 = (𝐼 𝑚(𝒪), ≤)$ forms an antichain with respect to the point-wise set inclusion of components $im(\hat{C}_1) \leq im(\hat{C}_2) ⇔ \bigcup g^I_{i1} \subseteq \bigcup g^I_{i2}$ for $i ∈ [2^n − 1])$.

### 3.2. Counting Cardinalities

Let us prove the following proposition on the cardinality of $𝒰_n$, $𝒰_n^+$, $𝒰_n'$ where $|A| = n$.

**Proposition 3.**

$$|𝒰_n| = 2^{2^n−1}$$

$$|𝒰_n^+| = \prod_{k=1}^{n} (2^k − 1)(k)$$

$$|𝒰_n'| = \prod_{k=1}^{n} k(k)$$

Note that (1) and (2) have been proven in [20] according to [1] (where they are given without proof). We give our proof of (1) and (2) with the help of FCA.

**Proof.** 1) Let us consider $1 = (\emptyset, \{1\}, \ldots, [n])$ it corresponds to $K_0, 2^A, 2^A, l_\emptyset$, where $C_0(X) = X$ for $X ⊆ A$ and $l_\emptyset : = :$. For each other choice function, $im(C)$ is below 1 in the lattice $𝒰(𝒪)$, which means that $\bigcup g^I_{i1} \subseteq \bigcup g^I_{i2}$, where $'i$ is taken in the $K_C$. Thus each row of $K_C$ has $|\bigcup g^I_{i2}|$ variants and the choice of each row is independent (we are ready for the product rule).

$$\prod_{k=0}^{2^n−1} 2^{|\bigcup g^I_{k}|} = \prod_{X⊆A} 2^{|X|} = \prod_{k=0}^{n} 2^k(k)$$

The last step is due to the presence of each set of size $k \binom{n}{k}$ times. The sum $\sum_{k=0}^{n} k(k)$ equals $n2^{n−1}$.

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$^{3}Im(𝒪) = \{im(C) \mid C ∈ 𝒪\}$
Table 1
Counting sequences for $|\mathcal{C}_n^+$|, $|\mathcal{C}_n|$ and $|\widehat{\mathcal{C}}_n|$ up to $n = 5$

<table>
<thead>
<tr>
<th>Formula</th>
<th>OEIS sequence</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\mathcal{C}_n</td>
<td>$</td>
<td><a href="https://oeis.org/A061301">https://oeis.org/A061301</a></td>
<td>2</td>
<td>16</td>
<td>4096</td>
</tr>
<tr>
<td>$</td>
<td>\mathcal{C}_n^+</td>
<td>$</td>
<td>–</td>
<td>1</td>
<td>3</td>
<td>189</td>
</tr>
<tr>
<td>$</td>
<td>\widehat{\mathcal{C}}_n</td>
<td>$</td>
<td><a href="https://oeis.org/A229333">https://oeis.org/A229333</a></td>
<td>1</td>
<td>2</td>
<td>24</td>
</tr>
</tbody>
</table>

2) Now, we are not allowed to consider $\bigcup g_{\mathcal{I} \mathcal{I}d} \subseteq \emptyset$, which implies subtraction of 1 (i.e. $2^k - 1$) when counting variants for the choice of a row in the context $K_{\mathcal{C}^+} = (\mathcal{A}, \mathcal{A}, \mathcal{I} \mathcal{C}^+ \subseteq)$. Here $I_{\mathcal{I}d} \subseteq \mathcal{A} \times \mathcal{A}$ so $g_0$ (also $m_0$) is excluded and the product starts with $k = 1$.

3) Here, compared to the previous case, since we can choose only single-element sets among all $m \in \mathcal{A}$, $2^k - 1$ is simply replaced by $k$.

Note that Monjardet and Raderanirina [2] claim that the lattice of all choice functions on a set of alternatives $\mathcal{A}$ is Boolean (i.e. atomistic and distributive) with $n2^{n-1}$ atoms, which directly implies the proof of (1). Some authors also rediscovered this value without addressing prior works by Monjardet and Raderanirina [21].

We also note that the beginning values by equations 1 and 3 are listed in OEIS: see integer sequences https://oeis.org/A061301 and https://oeis.org/A229333, respectively.

Before we go to the asymptotic analysis, let us also list some beginning values of these sequences by equations 1–3 in Table 1.

3.3. Asymptotic Analysis

The values represented by equations 2 and 3 have no closed-form formulas but are smaller than the size of the whole space of choice functions. Our goal here is to figure out their asymptotic behaviour to better understand how the sizes of the posets, $|\mathcal{C}_n^+|$, $|\mathcal{C}_n|$, and $|\widehat{\mathcal{C}}_n|$, interrelated.

**Proposition 4.**

$$\log_2 |\mathcal{C}_n^+| = n2^{n-1} + O(2^n n^{-1/2})$$

**Proof.** Let us apply $\log_2$ to the product (2).

$$\log_2 |\mathcal{C}_n^+| = \sum_{k=1}^{n} \binom{n}{k} \log_2 2^k + \sum_{k=1}^{n} \binom{n}{k} \log_2 (1 - 1/2^k)$$

The first sum equals (1), while the second is more laborious since it has no closed form. Since $\log_2 x \leq x - 1$ for all $x > 0$, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} \log_2 (1 - 1/2^k) \leq \sum_{k=1}^{n} \binom{n}{k} (-1/2^k) = -\left(\frac{3}{2}\right)^n + 1$$
Since $1/2 < (1 - 1/2^k) < 1$ for $k \geq 1$, we have $-2^n \leq \sum_{k=1}^{n} \binom{n}{k} \log_2(1 - 1/2^k)$. However, we can do better with the lower bound if we pull out the maximal binomial coefficient, i.e. the middle (or central) binomial coefficient.

$$\sum_{k=1}^{n} \binom{n}{k} \log_2(1 - 1/2^k) \geq \binom{n}{\lfloor n/2 \rfloor} \sum_{k=1}^{\lfloor n/2 \rfloor} \log_2(1 - 1/2^k)$$

$$\sum_{k=1}^{\infty} \log_2(1 - 1/2^k) = \log_2 \prod_{k=1}^{\infty} (1 - 1/2^k) = \log_2 \phi(1/2) = -1.79192$$

where

$$\phi(q) = (q)_\infty \equiv (q; q)_\infty = \prod_{k=1}^{\infty} (1 - q^k)$$

is the Euler function [22], and $(q)_\infty$ and $(q; q)_\infty$ are $q$-Pochhammer symbols [23].

The variable term $\binom{n}{\lfloor n/2 \rfloor}$ is $O(2^{n \log_2 n})$ since for even $n$, we have $\binom{n}{\lfloor n/2 \rfloor} = \sqrt{2/\pi \cdot 2^n n^{-1/2}}$ [24] and the following inequalities are known $\frac{\sqrt{2}}{n} \leq \binom{n}{\lfloor n/2 \rfloor} < \sqrt{2/\pi \cdot 2^n n^{-1/2}}$ [25].

**Proposition 5.**

$$\lim_{n \to \infty} \frac{|\mathcal{C}_n^+|}{|\mathcal{C}_n|} = \prod_{k=1}^{\infty} (1 - 1/2^k)^{(\frac{n}{2})} \text{ diverges to zero.}$$

**Proof.** From the proof of the previous proposition it follows that

$$\phi(1/2) \sqrt{2/\pi \cdot 2^{n^{-1/2}}} \leq \prod_{k=1}^{n} (1 - 1/2^k)^{(\frac{n}{2})} \leq 2^{-(\frac{1}{2})^{n-1}}$$

where $\phi(1/2) \approx 0.2888$.

When $n$ tends to $\infty$, both sides tend to $0$, and since no terms of the partial product are zeros, the whole product is said to diverge to zero [26, 27].

**Proposition 6.**

$$\log_2|\mathcal{C}_n| = \Theta(2^n \log_2 n)$$

**Proof.** To prove the statement we need to show that there are constants $c_1, c_2 > 0$, such that $c_1 2^n \log_2 n \leq \log_2|\mathcal{C}_n| \leq c_2 2^n \log_2 n$ for all $n > n_0$.

We can pull out the largest value that $\log_2 n$ takes

$$\log_2|\mathcal{C}_n| = \sum_{k=1}^{n} \binom{n}{k} \log_2 k \leq \log_2 n \sum_{k=1}^{n} \binom{n}{k} = (2^n - 1) \log_2 n.$$

For the lower bound we can split the sum into two parts as follows:

$$\sum_{k=1}^{n} \binom{n}{k} \log_2 k \geq \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k} \log_2 k + \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n}{k} \log_2 \frac{n}{2} \geq \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n}{k} \log_2 \frac{n}{2} \geq$$
\[ \geq \log_2 \frac{n}{2} \sum_{k=1}^{n} \binom{n}{k}/2 = \frac{1}{2}(\log_2 n - 1)(2^n - 1). \]

The last result can be sharpened to \( \log_2 |\widehat{\mathcal{C}}_n| = 2^n \log_2 n(1 + O(1/\log_2 n)) \). By changing \( k \) to \( n - k \) we get \( \sum_{k=1}^{n} \binom{n}{k} \log_2 k = \sum_{k=0}^{n-1} \binom{n}{k} \log_2(n - k) \) and pull out \( \log_2 n \), which gives us the term \((2^n - 1)\log_2 n\) and the remaining term is
\[
\sum_{k=0}^{n-1} \binom{n}{k} \log_2 (1 - k/n) \leq -\frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} = -\sum_{k=0}^{n-1} \binom{n-1}{k-1} = 1 - 2^{n-1}.
\]

4. Conclusion

Monjardet and Raderanirina [2] inform that not all spaces of choice functions with given properties have been explored in the sense that concrete counting formulae exist while a few beginning values are known.

For example, the lattice of choice functions satisfying hereditary axiom has size \(|\mathcal{L}(\mathcal{C}_Hn)| = D_{n-1}\)^4, where \(D_n\) is the \(n\)-th Dedekind number [2]. And thus we get the new value \(|\mathcal{L}(\mathcal{C}_H10)|\) with recently obtained \(D_9\) [28]^4 (with FCA):

\[28638657766829841112846915166759849881236610].\]

We hope to continue this work on combinatorial properties of choice functions with FCA tools for their representation and counting and perform asymptotic analysis (if necessary).

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References


https://oeis.org/A000372


