Classifying Fuzzy Subsumption in Fuzzy-$\mathcal{EL}+$

G. Stoilos[1], G. Stamou[1], and J.Z. Pan[2]

[1] Department of Electrical and Computer Engineering, National Technical University of Athens, Zographou 15780, Greece

[2] Department of Computing Science, The University of Aberdeen

Abstract. Fuzzy Description Logics (f-DLs) have been proposed as extensions of classical Description Logics able to handle imprecise and vague knowledge. Although several extensions to expressive DLs have been proposed today and many reasoning algorithms exist there is still no scalable and efficient reasoning system reported, mainly due to inherited computational complexity from crisp DLs and the lack for optimisation techniques for fuzzy DL reasoning algorithms. Following the paradigm of classical DLs, fuzzy extensions to tractable DLs have been proposed. In the current paper we present a fuzzy extension to the tractable DL $\mathcal{EL}+$, creating f-$\mathcal{EL}+$. Besides the syntax and the extended semantics we also provide a reasoning algorithm for f-$\mathcal{EL}+$. Interestingly, our algorithm supports classification over fuzzy subsumption for which scalable reasoning is not known.

1 Introduction

Fuzzy Description Logics (f-DLs) [12] have been proposed as formalisms capable of capturing and reasoning about imprecise and vague knowledge. Such extensions are particularly useful in knowledge based applications that face a considerable amount of imprecise and vague knowledge. Similarly to DLs, recent research in f-DLs is mainly focused on providing reasoning algorithms for very expressive fuzzy DLs, like the results in [11, 10]. Interestingly, there currently exist three f-DL reasoners, the tableaux based FiRE[3] [11], which supports f$_{KD}$-$\mathcal{SHIN}$ (a fuzzy extension of $\mathcal{SHIN}$ [5] with the fuzzy operators of Zadeh logic; see section 2), the mixed integer programming reasoner *fuzzyDL*[4] [13], which supports f$_{KD}$-$\mathcal{SHI}f(\mathcal{D})$ and f$_{L}$-$\mathcal{SHI}f(\mathcal{D})$ (fuzzy $\mathcal{SHI}f(\mathcal{D})$ with the fuzzy operators of Lukasiewicz logic) and the reduction to crisp DLs based one DELOREAN [3], which supports f$_{KD}$-$\mathcal{SROIQ}$ (a fuzzy extension of $\mathcal{SROIQ}$ [6]). Unfortunately, there is no report for scalable reasoning with any of the aforementioned systems. Moreover, it is obvious that regarding practical behavior they will perform even worse than crisp DL systems due to the addition of fuzziness. Thus, quite recently the focus in f-DLs was also shifted on fuzzy extensions to tractable DLs. First, Straccia presented a fuzzy version of the DL-Lite language (f-DL-Lite)

[3] `http://www.image.ece.ntua.gr/~nsimou`

[4] `http://gaia.isti.cnr.it/~straccia`

[15], while Pan et. al. [9] presented the very first efficient and scalable system for f-DL-Lite which is able to answer expressive fuzzy conjunctive queries over millions of data. Finally, Vojtás [16] presented a fuzzy extension of the DL $\mathcal{EL}$. His extension is quite different than the usual ones of fuzzy-DLs [12, 11] in that he interprets conjunction as a fuzzy aggregation rather than fuzzy intersection. Moreover, the approach is focused on query answering, and no algorithm for classification was presented.

In the current paper we will present a fuzzy extension to the tractable DL $\mathcal{EL}+$ [1]. Besides the syntax and semantics of fuzzy-$\mathcal{EL}+$ (f-$\mathcal{EL}+$) we will present a detailed reasoning algorithm for classifying f-$\mathcal{EL}+$ ontologies. An interesting feature is that we will extend concept axioms with degrees of truth, thus allowing for fuzzy subsumption [14]. It is well known that $\mathcal{EL}+$ is very good in performing classification tasks, thus providing a scalable algorithm for classifying fuzzy ontologies that use such axioms, as far as we know, hasn't been studied in the literature.

Besides the efficiency of the $\mathcal{EL}+$ DL, and thus our evident interest in it, our work on extending concept subsumption with fuzzy subsumption in a tractable DL, was heavily motivated by the area of ontology alignment [7] and view-based searching in Semantic portals [4]. For example in [4] the authors use the concept of fuzzy subsumption in order to create fuzzy mapping between concepts of an annotation ontology and and hierarchy of search views. For example the authors use the fuzzy mappings,

$$\begin{aligned} \mathsf{Diseases} &\sqsubseteq_{0.1} \mathsf{Food\&Diseases} \\ \mathsf{Nutrition} \sqcap \mathsf{Exercise} &\sqsubseteq_{0.9} \mathsf{Food\&Diseases} \end{aligned}$$

to fuzzy map the concept Diseases and the intersection of the concepts Nutrition and Exercise of an annotation ontology to the search view Food&Diseases. Additionally in the hierarchy of search views Food&Diseases is a sub-view of Nutrition (fuzzy subsumption to degree 1). By reasoning over such fuzzy mappings would be able to derive interesting inferences between concepts Nutrition and Exercise and super-views of Food&Diseases, like Nutrition. As was pointed in [4] the approach uses only lightweight ontologies to perform the mappings, compared to the expressivity of OWL DL, thus the $\mathcal{EL}+$ tractable DL and its mechanisms for classification is also clearly motivated.

In the current paper we contribute the the fuzzy DL literature by presenting the syntax, semantics and reasoning algorithm of the fuzzy DL language f-$\mathcal{EL}+$. The algorithm of f-$\mathcal{EL}+$ is the first ever to be reported offering for efficient classification of fuzzy knowledge bases, which allow for fuzzy subsumption.

2 Fuzzy Sets

Fuzzy set theory and fuzzy logic are widely used for capturing imprecise knowledge [8]. While in classical set theory an element either belongs to a set or not, in fuzzy set theory elements belong only to a certain degree. More formally, let X be a set of elements. A fuzzy subset A of X, is defined by a *membership function*

$\mu_A(x)$, or simply $A(x)$, of the form $\mu_A : X \rightarrow [0,1]$ [8]. This function assigns any $x \in X$ to a value between 0 and 1 that represents the degree in which this element belongs to X. In this new framework the classical set theoretic and logical operations are performed by special mathematical functions. More precisely *fuzzy complement* is a unary operation of the form $c : [0,1] \rightarrow [0,1]$, *fuzzy intersection* and *union* are performed by two binary functions of the form $t : [0,1] \times [0,1] \rightarrow [0,1]$ and $u : [0,1] \times [0,1] \rightarrow [0,1]$, called *t-norm* and *t-conorm* operations [8], respectively, and *fuzzy implication* also by a binary function, $\mathcal{J} : [0,1] \times [0,1] \rightarrow [0,1]$. In order to produce meaningful fuzzy complements, conjunctions, disjunctions and implications, these functions must satisfy certain mathematical properties. For example the operators must satisfy the following boundary properties, $c(0) = 1$, $c(1) = 0$, $t(1,a) = a$ and $u(0,a) = a$. Due to space limitations we cannot present all the properties that these functions should satisfy. The reader is referred to [8] for a comprehensive introduction. Nevertheless, it worths noting here that there exist two distinct classes of fuzzy implications, those of S-implications, given by the equation $\mathcal{J}(a,b) = u(c(a),b)$, and those of R-implications, given by $\mathcal{J}(a,b) = \sup\{x \in [0,1] \mid t(a,x) \leq b\}$. Examples of fuzzy operators are the Lukasiewicz negation, $c_L(a) = 1 - a$, t-norm, $t_L(a,b) = \max(0, a + b - 1)$, t-conorm $u_L(a,b) = \min(1, a + b)$, and implication, $\mathcal{J}_L(a,b) = \min(1, 1 - a + b)$, the Gödel norms $t_G(a,b) = \min(a,b)$, $u_G(a,b) = \max(a,b)$, and implication $\mathcal{J}_G(a,b) = b$ if $a > b$, $\mathcal{J}_G(a,b) = 1$ otherwise, and the Kleene-Dienes implication (KD-implication), $\mathcal{J}_{KD}(a,b) = \max(1 - a, b)$.

Similarly, we can define the notion of n-ary fuzzy relations, by membership functions of the form $R : \Delta^{\mathcal{I}} \times \ldots \times \Delta^{\mathcal{I}} \rightarrow [0,1]$. Given two fuzzy relations $R_1 : X \times Y \rightarrow [0,1]$ and $R_2 : Y \times Z \rightarrow [0,1]$ we define the $\sup -t$ composition as, $[R_1 \circ^t R_2](a,c) = \sup_{b \in Y}\{t(R(a,b), R(b,c))\}$. The operation of $\sup -t$ composition satisfies the following properties:

$$(R_1 \circ^t R_2) \circ^t R_3 = R_1 \circ^t (R_2 \circ^t R_3),$$
$$(R_1 \circ^t R_2)^- = (R_2^- \circ^t R_1^-)$$

Due to the associativity property we can extend the operation of $\sup -t$ composition to any number of fuzzy relations. In that case we will simply write $[R_1 \circ^t R_2 \circ^t \ldots \circ^t R_n](a,b)$.

3 The Fuzzy-$\mathcal{EL}+$ Language

In this section we introduce a fuzzy extension to the $\mathcal{EL}+$ DL. Our semantics as well as the reasoning algorithm will be tailored for the operators of the Gödel logic we call our language f$_G$-$\mathcal{EL}+$.

As usual *concept descriptions* are inductively defined by a set of *concept names* (CN) and a set of *role names* (RN) taken together with a set of *constructors* which help us construct such descriptions [2]. More precisely f$_G$-$\mathcal{EL}+$-concepts are defined by the following abstract syntax:

$$C, D ::= \top \mid \mathsf{A} \mid C \sqcap D \mid \exists r.C$$

where $\mathsf{A} \in \mathsf{CN}$.

As in the classical case, a f$_G$-$\mathcal{EL}+$ *ontology* consists of a finite set of *concept* and *role axioms*. But differently, and following Straccia [14], we allow for *fuzzy general concept inclusions* (f-GCIs) of the form $\langle C \sqsubseteq D, n\rangle$, where $n \in (0, 1]$ and C, D are f$_G$-$\mathcal{EL}+$-concepts. Intuitively, these axioms say that the degree of subsethood of C to D is at-least equal to n. On the other hand we choose not to fuzzify role axioms of $\mathcal{EL}+$. Thus, *role inclusions axioms* (RIAs) of f$_G$-$\mathcal{EL}+$ are defined by an axiom of the form $r_1 \circ \ldots \circ r_k \sqsubseteq s$. Note that we choose not to allow for fuzzy subsumptions for role inclusion axioms.

The semantics of fuzzy DLs are provided by a *fuzzy interpretation* [12]. A fuzzy interpretation consists of a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where the domain $\Delta^\mathcal{I}$ is a non-empty set of objects and $\cdot^\mathcal{I}$ is a fuzzy interpretation function, which maps,

- an individual a to an element of $a^\mathcal{I} \in \Delta^\mathcal{I}$,
- a concept name A to a membership function $\mathsf{A}^\mathcal{I} : \Delta^\mathcal{I} \to [0, 1]$, and
- a role name r to a membership function $r^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0, 1]$.

Using the fuzzy set theoretic operations, fuzzy interpretations can be extended to interpret f$_G$-$\mathcal{EL}+$-concepts. The complete set of semantics are depicted in Table 1, where $\mathcal{J}_G$ is the Gödel fuzzy implication. Our choice of operators is justified by the fact that Gödel implication is an R-implication and thus it holds that $\langle C \sqsubseteq D, 1\rangle$ iff $C^\mathcal{I}(a) \leq D^\mathcal{I}(a), \forall a \in \Delta^\mathcal{I}$. Intuitively, if C is fully subsumed by D, then the membership function of D is greater or equal than that of C in all cases. Additionally, this also has the effect that a concept C is always fully subsumed by itself.

Constructor	DL Syntax	Semantics
top	$\top$	$\top^\mathcal{I}(a) = 1$
conjunction	$C \sqcap D$	$(C \sqcap D)^\mathcal{I}(a) = \min(C^\mathcal{I}(a), D^\mathcal{I}(a))$
existential restriction	$\exists r.C$	$(\exists r.C)^\mathcal{I}(a) = \sup_{b \in \Delta^\mathcal{I}} \{\min(r^\mathcal{I}(a, b), C^\mathcal{I}(b))\}$
Fuzzy GCIs	$\langle C \sqsubseteq D, n\rangle$	$\inf_{a \in \Delta^\mathcal{I}} \mathcal{J}_G(C^\mathcal{I}(a), D^\mathcal{I}(a)) \geq n$
RIAs	$r_1 \circ \ldots \circ r_k \sqsubseteq s$	$[r_1^\mathcal{I} \circ^t \ldots \circ^t r_k^\mathcal{I}](a, b) \leq s^\mathcal{I}(a, b)$

Table 1. Syntax and Semantics of f$_G$-$\mathcal{EL}+$

The basic inference problem of f$_G$-$\mathcal{EL}+$ is *fuzzy concept subsumption*: A concept C is fuzzy subsumed by a concept D to a degree $n \in (0, 1]$ w.r.t. an f$_G$-$\mathcal{EL}+$ ontology $\mathcal{O}$, written $\langle C \sqsubseteq_\mathcal{O} D, n\rangle$ if $\inf_{a \in \Delta^\mathcal{I}} .\mathcal{J}_G(C^\mathcal{I}(a), D^\mathcal{I}(a)) \geq n$ for every model $\mathcal{I}$ of $\mathcal{O}$. Moreover we are also interested in the problem of *classifying* an f$_G$-$\mathcal{EL}$+ ontology which contains fuzzy-GCIs, i.e. compute all fuzzy subsumptions between concepts of the ontology.

4 Classifying with Fuzzy Subsumption

In the current section we will provide a detailed presentation of the algorithm for classifying fuzzy subsumption in f-$\mathcal{EL}$+ ontologies. As we will see in the following the algorithm for f$_G$-$\mathcal{EL}$+ is quite similar to the algorithm for classical $\mathcal{EL}$+ modulo the degrees of fuzzy-GCIs.

4.1 Normal Form for f-$\mathcal{EL}$+ Ontologies

Before applying the polynomial algorithm for classification a f-$\mathcal{EL}$+ ontology needs to be normalized [1].

Given an ontology $\mathcal{O}$, we write $\mathsf{CN}_\mathcal{O}^\top$ and $\mathsf{CN}_\mathcal{O}$ to denote the set of concept names with and without the top concept ($\top$), respectively. Then, an f$_G$-$\mathcal{EL}$+ ontology $\mathcal{O}$ is in *normal form* if

1. all fuzzy GCIs in $\mathcal{O}$ have one of the following forms, where $A_i \in \mathsf{CN}_\mathcal{O}^\top$ and $B \in \mathsf{CN}_\mathcal{O}$:

$$\langle A_1 \sqcap \ldots \sqcap A_k \sqsubseteq B, n\rangle$$
$$\langle A_1 \sqsubseteq \exists r.A_2, n\rangle$$
$$\langle \exists r.A_1 \sqsubseteq B, n\rangle, \text{ and}$$

2. all role inclusions are of the form $r \sqsubseteq s$ or $r_1 \circ r_2 \sqsubseteq s$.

As shown in [1] every $\mathcal{EL}$+ ontology $\mathcal{O}$ can be turned into a normalized one $\mathcal{O}'$ by exhaustively applying proper normalization rules, which introduce new concept and role names in the ontology. The complete set of normalization rules for f$_G$-$\mathcal{EL}$+ is described below.

Lemma 1. *A f$_G$-$\mathcal{EL}$+ ontology $\mathcal{O}$ is satisfiable iff the normalized one $\mathcal{O}$' is satisfiable.*

Theorem 1 ([1]). *Subsumption w.r.t. f-$\mathcal{EL}$+ ontologies can be reduced in linear time to subsumption w.r.t. normalized ontologies in f-$\mathcal{EL}$+.*

In the following, and without loss of generality, we assume that an input ontology $\mathcal{O}$ is in normal form.

NF1	$r_1 \circ \ldots \circ r_k \sqsubseteq s$	$\rightsquigarrow$	$r_1 \circ \ldots \circ r_{k-1} \sqsubseteq u$, $u \circ r_k \sqsubseteq s$
NF2	$\langle C_1 \sqcap \ldots \sqcap \hat{C} \sqcap \ldots \sqcap C_k \sqsubseteq C, n\rangle$	$\rightsquigarrow$	$\langle \hat{C} \sqsubseteq A, n\rangle$, $\langle C_1 \sqcap \ldots \sqcap A \sqcap \ldots \sqcap C_k \sqsubseteq C, n\rangle$
NF3	$\langle \exists r.\hat{C} \sqsubseteq D, n\rangle$	$\rightsquigarrow$	$\langle \hat{C} \sqsubseteq A, n\rangle$, $\langle \exists r.A \sqsubseteq D, n\rangle$
NF4	$\langle \hat{C} \sqsubseteq \hat{D}, n\rangle$	$\rightsquigarrow$	$\langle \hat{C} \sqsubseteq A, n\rangle$, $\langle A \sqsubseteq \hat{D}, n\rangle$
NF5	$\langle B \sqsubseteq \exists r.\hat{C}, n\rangle$	$\rightsquigarrow$	$\langle B \sqsubseteq \exists r.A, n\rangle$, $\langle A \sqsubseteq \hat{C}, n\rangle$
NF6	$\langle B \sqsubseteq C \sqcap D, n\rangle$	$\rightsquigarrow$	$\langle B \sqsubseteq C, n\rangle$, $\langle B \sqsubseteq D, n\rangle$

where $\hat{C}$, $\hat{D} \notin \mathsf{CN}_\mathcal{O}^\top$, C_i, C, D are arbitrary concept descriptions, $B \in \mathsf{CN}_\mathcal{O}^\top$, u denotes a *new* role name and A denotes a *new* concept name

Table 2. Normalization rules for f$_G$-$\mathcal{EL}+$

4.2 An Algorithm for Classification in f-$\mathcal{EL}+$

Let $\mathcal{O}$ be an f-$\mathcal{EL}+$ ontology in normal form. Our subsumption algorithm for normalized f$_G$-$\mathcal{EL}+$ ontologies can be restricted to subsumption checking between concept names. More precisely, $\langle C \sqsubseteq_\mathcal{O} D, n\rangle$ iff $\langle A \sqsubseteq_{\mathcal{O}'} B, n\rangle$, where $\mathcal{O}' = \mathcal{O} \cup \{\langle A \sqsubseteq C, n\rangle, \langle D \sqsubseteq B, n\rangle\}$ with A and B new concept names.

Let $\mathsf{RN}_\mathcal{O}$ be the set of all role names occurring in $\mathcal{O}$. The algorithm computes:

- A mapping S assigning to each concept name of $\mathsf{CN}_\mathcal{O}$ a subset $S(A)$ of $\mathsf{CN}_\mathcal{O}^\top \times [0,1]$, and
- A mapping R assigning to each role name r of $\mathsf{RN}_\mathcal{O}$ a ternary relation $R(r)$ which is a subset of $\mathsf{CN}_\mathcal{O}^\top \times \mathsf{CN}_\mathcal{O}^\top \times [0,1]$.

As we can see, due to the presence of fuzzy subsumptions we have extended the mappings $S(A), R(r)$ to range over subsets of $\mathsf{CN}_\mathcal{O}^\top \times [0,1]$ and $\mathsf{CN}_\mathcal{O}^\top \times \mathsf{CN}_\mathcal{O}^\top \times [0,1]$, respectively. As with crisp $\mathcal{EL}+$ intuitively, these mappings make implicit fuzzy subsumption relationships explicit in the sense that

- $\langle B, n\rangle \in S(A)$ implies $\langle A \sqsubseteq B, n\rangle$, and
- $\langle A, B, n\rangle \in R(r)$ implies $\langle A \sqsubseteq \exists r.B, n\rangle$.

The mappings are initialized as follows:

- $S(A) = \{A, \top\}$, for each $A \in \mathsf{CN}_\mathcal{O}$
- $R(r) = \emptyset$, for each $r \in \mathsf{RN}_\mathcal{O}$

Then, the sets $S(A)$ and $R(r)$ are extended by applying the completion rules shown in Table 3 until no more rules are applied.

Theorem 2. *The algorithm runs in polynomial time and it is sound and complete, i.e. after it terminates on input $\mathcal{O}$, we have for all $A, B \in \mathsf{CN}_\mathcal{O}^\top$, $n \in (0,1]$ that $\langle A \sqsubseteq_\mathcal{O} B, n\rangle$ iff $\langle B, n'\rangle \in S(A)$, for some $n' \in (0,1]$, with $n' \geq n$.*

R1	If $\langle A_1, n_1 \rangle \in S(X), \ldots, \langle A_l, n_l \rangle \in S(X)$, $\langle A_1 \sqcap \ldots \sqcap A_l \sqsubseteq B, k \rangle \in \mathcal{O}$ and $\langle B, m \rangle \notin S(X)$, where $m = \min(n_1, \ldots, n_l, k)$ then $S(X) := S(X) \cup \{\langle B, m \rangle\}$, where $m = \min(n_1, \ldots, n_l, k)$
R2	If $\langle A, n \rangle \in S(X), \langle A \sqsubseteq \exists r.B, k \rangle \in \mathcal{O}$, and $\langle X, B, m \rangle \notin R(r)$, where $m = \min(n, k)$ then $R(r) := R(r) \cup \{\langle X, B, m \rangle\}$, where $m = \min(n, k)$
R3	If $\langle X, Y, n_1 \rangle \in R(r), \langle A, n_2 \rangle \in S(Y), \langle \exists r.A \sqsubseteq B, n_3 \rangle \in \mathcal{O}$, and $\langle B, m \rangle \notin S(X)$, where $m = \min(n_1, n_2, n_3)$ then $S(X) := S(X) \cup \{\langle B, m \rangle\}$, where $m = \min(n_1, n_2, n_3)$
R4	If $\langle X, Y, n \rangle \in R(r), r \sqsubseteq s \in \mathcal{O}$, and $\langle X, Y, n \rangle \notin R(s)$ then $R(s) := R(s) \cup \{\langle X, Y, n \rangle\}$
R5	If $\langle X, Y, n_1 \rangle \in R(r), \langle Y, Z, n_2 \rangle \in R(s), r \circ s \sqsubseteq f \in \mathcal{O}$, and $\langle X, Z, m \rangle \notin R(f)$, where $m = \min(n_1, n_2)$ then $R(f) := R(f) \cup \{\langle X, Z, k \rangle\}$ where $m = \min(n_1, n_2)$

Table 3. Completion rules for f_G-$\mathcal{EL}$+

4.3 The Optimised Algorithm

As it was pointed in [1] although $\mathcal{EL}$+ is a tractable DL, in practice the above algorithm might fail to provide truly tractable, scalable and efficient reasoning. This is due to the fact that the application of rules is performed using a naive brute-force search. This effect is remedied by proposing a refined algorithm which is shown to provide truly scalable practical reasoning. The algorithm is realized by introducing a set of queues, one for each concept name, which intuitively guide the application of the expansion rules. In the following we sketch the necessary modifications to the $\mathcal{EL}$+ refined algorithm in order to also provide optimisations for the f_G-$\mathcal{EL}$+ algorithm.

Our entries of the queues are of the form

$$B_1, \ldots, B_m \rightarrow \langle B', n' \rangle \qquad \text{and} \qquad \langle \exists r.B, n \rangle$$

with $B_1, \ldots, B_m, B$ and B' concept names, r role name, $m \geq 0$ and $n, n' \in (0, 1]$. For $m = 0$ we simply write $\langle B', n' \rangle$. Intuitively,

- an entry $B_1, \ldots, B_m \rightarrow \langle B', n' \rangle \in \mathsf{queue}(A)$ means that $\langle B', k \rangle$, with $k = \min(n', n_1, \ldots, n_m)$ has to be added in $S(A)$ if $S(A)$ already contains information for $B_1, \ldots B_m$, i.e. entries $\langle B_1, n_1 \rangle, \ldots, \langle B_m, n_m \rangle$, and
- $\langle \exists r.B, n \rangle \in \mathsf{queue}(A)$ means that $\langle A, B, n \rangle$ has to be added to $R(r)$.

Similarly to the optimised algorithm of $\mathcal{EL}$+ we use the mapping $\hat{\mathcal{O}}$ from concepts to sets of queue entries as follows: For each concept name $A \in \mathsf{CN}_{\mathcal{O}}^{\top}$, $\hat{\mathcal{O}}(A)$ is the minimal set of queue entries such that:

– if $\langle A_1 \sqcap \ldots \sqcap A_m \sqsubseteq B, n\rangle \in \mathcal{O}$ and $A = A_i$, then

$$A_1, \ldots A_{i-1}, A_{i+1}, \ldots, A_m \rightarrow \langle B, n\rangle \in \hat{\mathcal{O}}(A) \text{ and}$$

– if $\langle A \sqsubseteq \exists r.B, n\rangle \in \mathcal{O}$, then $\langle \exists r.B, n\rangle \in \hat{\mathcal{O}}(A)$.

Similarly, for each concept $\exists r.A$, $\hat{\mathcal{O}}(\exists r.A)$ is the minimal set of queue entries such that, if $\exists r.A \sqsubseteq B \in \mathcal{O}$, then $\langle B, n\rangle \in \hat{\mathcal{O}}(\exists r.A)$.

Using the above changes the refined algorithm of $\mathcal{EL}+$ can be changed accordingly in order to also take into account fuzziness in subsumption axioms and provide an algorithm for processing the queue entries.

Theorem 3. *The refined algorithm runs in polynomial time and it is sound and complete, i.e. after it terminates on input $\mathcal{O}$, we have for all $A, B \in \mathsf{CN}_{\mathcal{O}}^{\top}$, $n \in (0, 1]$ that $\langle A \sqsubseteq_{\mathcal{O}} B, n\rangle$ iff $\langle B, n'\rangle \in S(A)$, for some degree $n' \in (0, 1]$, with $n' \geq n$.*

5 Conclusions

In the current paper we have presented a fuzzy extension of the well-known tractable DL $\mathcal{EL}+$, creating the f_G-$\mathcal{EL}+$. Besides the syntax and semantics we have also shown how the classification algorithm of $\mathcal{EL}+$ can be extended in order to provide reasoning for this fuzzy $\mathcal{EL}+$ DL. Our approach is interesting in several directions. On the one hand it presents a fuzzy extension of a tractable DL, which is expected to perform very well in practice compared to expressive fuzzy DL reasoning systems for which no report or evidence on scalable reasoning exists yet. On the other hand our algorithm is also able to handle and perform scalable and efficient classification over fuzzy subsumption using the semantics of the Gödel R-implication. To achieve this goal we have also investigated the refined (optimised) algorithm of $\mathcal{EL}+$ in the fuzzy case in order to provide an optimised algorithm for classifying with fuzzy subsumption.

Regarding future work we plan to implement the proposed optimised algorithm and apply an extensive evaluation in order to assess its practical performance. Additionally, we also plan to apply the algorithm in a real case scenario and more precisely in the scenario of view-based searching in Semantic portals [4] and assess the added value of it.

Acknowledgement.

The work of Giorgos Stoilos and Giorgos Stamou was partially supported by the European Commission under project X-Media (FP6-26978). The work of Giorgos Stoilos was partially supported by the Greek Secretariat of Research and Technology under project PENED 03ED475 2003.

References

1. F. Baader, C. Lutz, and B. Suntisrivaraporn. Is tractable reasoning in extensions of the description logic $\mathcal{EL}$ useful in practice? *Journal of Logic, Language and Information, Special Issue on Method for Modality (M4M), To Appear.*
2. F. Baader, D. McGuinness, D. Nardi, and P.F. Patel-Schneider. *The Description Logic Handbook: Theory, implementation and applications.* Cambridge University Press, 2002.
3. Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. Optimising the crisp representation of the fuzzy dl $\mathcal{SROIQ}$. In *Proc. of the 3nd International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 07), Busan, Korea*, 2007.
4. Markus Holi and Eero Hyvonen. Fuzzy view-based semantic search. In *Asian Semantic Web Conference*, 2006.
5. I. Horrocks and U. Sattler. A tableaux decision procedure for $\mathcal{SHOIQ}$. In *Proc. 19th Int. Joint Conf. on Artificial Intelligence (IJCAI 05)*, 2005.
6. Ian Horrocks, Oliver Kutz, and Ulrike Sattler. The even more irresistible $\mathcal{SROIQ}$. In *KR*, pages 57–67, 2006.
7. Pavel Shvaiko Jérôme Euzenat. *Ontology Matching*. Springer, 2007.
8. G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice-Hall, 1995.
9. J.Z. Pan, G. Stamou, G. Stoilos, and E. Thomas. Scalable querying services over fuzzy ontologies. In *Proceedings of the International World Wide Web Conference (WWW 2008), Beijing*, 2008.
10. G. Stoilos, G. Stamou, V. Tzouvaras, J.Z. Pan, and I. Horrocks. The fuzzy description logic f-$\mathcal{SHIN}$. In *Proc. of the International Workshop on Uncertainty Reasoning for the Semantic Web*, pages 67–76, 2005.
11. Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, Jeff Z. Pan, and Ian Horrocks. Reasoning with very expressive fuzzy description logics. *Journal of Artificial Intelligence Research*, 30(5):273–320, 2007.
12. U. Straccia. Reasoning within fuzzy description logics. *Journal of Artificial Intelligence Research*, 14:137–166, 2001.
13. U. Straccia. Description logics with fuzzy concrete domains. In *21st Conf. on Uncertainty in Artificial Intelligence (UAI-05)*, Edinburgh, 2005.
14. U. Straccia. Towards a fuzzy description logic for the semantic web. In *Proceedings of the 2nd European Semantic Web Conference*, 2005.
15. Umberto Straccia. Answering vague queries in fuzzy DL-Lite. In *Proceedings of the 11th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, (IPMU-06)*, pages 2238–2245, 2006.
16. Peter Vojtas. Proc. of the 2nd int. workshop on uncertainty reasoning for the semantic web, athens, georgia. In *$\mathcal{EL}$ Description Logics with aggregation of user preference concepts*, 2006.