On approximate strategyproof mechanisms for Hedonic Games and the Group Activity Selection Problem

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Abstract
Hedonic Games and the Group Activity Selection Problem are two well-established models to describe the coalition formation of strategic agents. While most of the literature has been focusing on the existence and computation of stable outcomes, under the demanding assumption of complete knowledge of agents’ preferences, very little has been made on the elicitation of true preferences.

With this paper, we provide an overview of strategyproof mechanisms for the aforementioned games with particular attention on mechanisms having a good approximation guarantee in terms of social welfare. Moreover, we outline future challenges in this area.

Keywords
Strategyproof Mechanisms, Hedonic Games, Group Activity Selection Problem, Coalition Formation, Additively Separable, Approximation Mechanisms, Mechanisms without Money

1. Introduction
In many real-world scenarios, individuals prefer to perform tasks or activities by gathering into groups rather than being on their own. Simply think about researchers or employees working on different projects, politicians forming parties, people attending social events, and so forth. The increasing interest in economic, political, and social contexts in which individuals attempt common goals by splitting into groups led to the definition of Coalition Formation Games.

Coalition Formation Games model multi-agent systems where selfish agents form coalitions and have preferences over the possible outcomes of the game. More specifically, an outcome of a Coalition Formation Game is a partition of the agents into disjoint coalitions, and agents express their satisfaction with an outcome through preference relations. When agents’ preferences only depend on the coalition they belong to, and not on how the other agents aggregate, we talk about Hedonic Games (HGs), introduced by Dreze and Greenberg in [2]. Hedonic Games have been widely studied [3] and, according to the properties of agents’ preferences or other possible constraints, numerous classes have been defined.
While on the one hand, Hedonic Games offer a model sophisticated enough to describe a large variety of settings, on the other hand, they are not able to capture the satisfaction that agents have for the task they are eventually supposed to perform together. To this aim, the Group Activity Selection Problem (GASP), a proper generalization of the Hedonic Games, was introduced in [4]. Here, a set of possible tasks (or activities) is available, and the agents in a coalition have to perform a common task, one for each coalition. Thus, agents’ preferences are based on both the coalition they belong to and the activity they are supposed to perform.

In the Hedonic Games and Group Activity Selection Problem literature, different approaches have been considered, that can be broadly attributed to one of the two following research scenarios: i) the understanding of which kind of outcomes the selfish behavior of the agents leads to or ii) the elicitation of the agents’ real preferences while maintaining good properties for the designed outcome of the game.

Regarding the first research direction, agents’ preferences are known in advance and one of the main goals is to understand the outcomes that may be reached by the agents, both from an existential and an algorithmic perspective. To this aim, several stability concepts have been defined and studied which are based either on individual [5, 6, 7, 8, 9] or group deviations [10, 11, 12, 7, 13, 14].

In addition, when agents’ preferences are expressed by utilities, it is possible to measure the quality of stable outcomes by comparing their social welfare to the optimum (the maximum achievable one) [15, 16, 17, 18].

Regarding the second research direction, agents’ preferences are private information and must be communicated to a designer who splits agents accordingly. Being agents selfish, they may misreport their preferences so as to maximize their satisfaction with the final outcome of the game. Then the designer, solely based on the reported preferences, must compute an outcome of the game in order to induce, without any monetary request, a truthful behavior of the agents while satisfying good properties, for instance, stability or a good approximation of the optimum. This is known as the Mechanism Design without Money framework. In particular, in this setting, the outcome of the game is computed by an algorithm commonly called mechanism; a mechanism is said to be a strategyproof if no agent has an incentive to misreport her private information.

We focus on strategyproof mechanisms guaranteeing a good approximation to the optimum in the context of HGs and the GASP. The paper is organized as follows: In Section 2, we give an overview of the state of the art. In Section 3, we point out the limits of the current literature and establish which are the key challenges in this area.

2. State of the Art

In this section, we give an overview of the literature on strategyproof mechanisms with good guarantees with respect to the optimum. We assume that agents’ preferences are expressed by means of utilities and that the utility of an agent for a given outcome can be univocally determined by the individual values given to the other agents and the activities. Such values are a private information of the agents and have to be communicated to the central authority so as to compute an outcome of the game. More formally, we assume that, given a set of agents $N$ and
a set of activities $A$, every agent $i$ has a weight $v_{ij}$ for any other agent $j \in N \setminus \{i\}$ and a preference $p_a$ for each activity $a \in A^1$. Hence, the utility an agent has for being a member of a specific group and participating in a certain activity can be computed according to the weights she has for her coalition members and the preference she has for the activity they are performing. We will describe specific classes that fit this definition in the subsequent paragraphs.

We evaluate the quality of an outcome by its utilitarian welfare, that is, the sum of agents’ utilities, and measure the performances of the provided strategyproof mechanisms by comparing the social welfare they achieve (deterministically or in expectation) to the optimal one. In principle, a strategyproof mechanism may have an unbounded approximation guarantee to the optimum; whenever a mechanism has a bounded approximation guarantee we simply call it a bounded mechanism.

In what follows, having a lower bound of value $\alpha$ means no strategyproof mechanism can achieve an approximation better than $\alpha$. In turn, we say we have an upper bound of value $\alpha$ if a strategyproof mechanism guaranteeing an $\alpha$-approximation exists. Moreover, whenever a lower bound is $> 1$ means that any mechanism returning the optimum is not strategyproof.

Additively Separable Hedonic Games. In additively separable HGs (ASHGs) an agent $i$ values a coalition $C$, such that $i \in C$, by simply summing up the weights she has for each member, that is, $u_i(C) = \sum_{j \in C \setminus \{i\}} v_{ij}$. In [19, 20], the problem of achieving good approximate mechanisms with respect to the maximum utilitarian welfare has been considered under several assumptions on the weights $v_{ij}$, see Table 1. First of all, the authors showed that there is a dichotomy between the cases where $v_{ij} \in [-1, 1]$ and $v_{ij} \in [0, 1]$: In the former, no bounded deterministic/randomized strategyproof mechanism exists, while in the latter, returning the optimum, that is achieved by gathering all agents in the same coalition, is strategyproof. Given the general impossibility result, the authors further restrict the attention to the scenario where agents equally like or dislike, or simply don’t care about any other agent, i.e., $v_{ij} \in \{-1, 0, 1\}$. In this case, non-trivial bounded randomized/deterministic mechanisms together with theoretical lower bounds have been provided. Nonetheless, there exists a linear gap between the obtained lower and upper bounds both for deterministic and randomized mechanisms, and closing these gaps remains so far an open problem.

Table 1

<table>
<thead>
<tr>
<th>$v_{ij}$</th>
<th>$[-1, 1]$</th>
<th>$[0, 1]$</th>
<th>$[-1, 0, 1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.B.</td>
<td>$\infty$</td>
<td>1</td>
<td>$O(n^2)$, $O(n)^*$</td>
</tr>
<tr>
<td>L.B.</td>
<td>$\Omega(n)$, $2 - \epsilon$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^1$HGs can be seen as a special case of the GASP where there are arbitrarily many activities, so it is possible to form any partition of agents, and the activity that agents are performing has no impact on their utility. We therefore do not refer to preferences towards activities in the case of HGs.
Fractional Hedonic Games. Similarly to ASHGs, in fractional HGs (FHGs), the utility an agent \( i \in C \subseteq N \) derives in being included in the coalition \( C \) is obtained by summing up the weights she has for the other participants and then normalizing by the coalition size; formally, \( u_i(C) = \frac{\sum_{j \in C \setminus \{i\}} v_{ij}}{|C|} \). The work [20] gives a complete picture of approximate strategyproof mechanisms for FHGs by making a case distinction between the possible values for the weights \( v_{ij} \) and we showcase them in Table 2. Similarly to the ASHGs case, the freedom of choosing both negative and positive weights in the interval [-1,1] does not allow bounded mechanisms, not even by means of randomization. In turn, given the more involved structure of utilities, only a linear approximation is possible for positive weights, and the provided deterministic mechanism has been proven to be optimal. The special cases where weights are in \{-1, 0, 1\} and \{0, 1\} have also been considered and a constant approximation, at least in expectation, is possible. In these two cases, whether the provided mechanisms are the best possible remains an open problem.

Table 2

<table>
<thead>
<tr>
<th>( v_{ij} ) ∈ ([-1, 1])</th>
<th>([0, 1])</th>
<th>([-1,0,1])</th>
<th>([0,1])</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.B.</td>
<td>( \infty^* )</td>
<td>( O(n), 8^* )</td>
<td>2</td>
</tr>
<tr>
<td>L.B.</td>
<td>( \frac{2}{2} )</td>
<td>( 2 - \epsilon, \frac{3}{2} - \epsilon^* )</td>
<td>( \frac{6}{5} )</td>
</tr>
</tbody>
</table>

Friends and Enemies Games. Considering the impossibility result for general ASHGs, a natural question was to understand whether there exist other special classes of ASHGs, as for the case \( v_{ij} \in \{-1, 0, 1\} \), where bounded mechanisms are possible. An interesting HG class where agents split the others into friends and enemies, the so-called Friends and Enemies Games, has been introduced by [21] with respect to two different types of preference profiles: Friends Appreciation (FA) and Enemies Aversion (EA). Under FA, agents prefer coalitions with a higher number of friends. When the number of friends is the same, they prefer a coalition with a smaller number of enemies. Conversely, under EA, agents always prefer coalitions with a smaller number of enemies and, in case of a tie, the ones with a bigger number of friends. FA and EA have been shown to belong to the ASHGs class for suitable choices of weights. Namely, for FA profiles it is sufficient to set \( v_{ij} \in \{-1, n\} \) while, for EA, \( v_{ij} \in \{-n, 1\} \).

For these two specific classes, the existence of strategyproof mechanisms outputting core or weak core stable solutions is well-established. However, these results are either inefficient or require exponential time to be executed. In [22], a complete picture of both positive and negative results on approximate strategyproof mechanisms is provided and we summarize their main results in Table 3. For FA preference profiles it is well-known that splitting the agents into maximal strongly connected components, in the directed graphs where agents are the nodes and directed edges represent friendship relationships, is strategyproof and core stable [23]; however, this mechanism is particularly inefficient in approximating the utilitarian welfare. In turn, [22]
shows that splitting agents into maximal weakly connected components is still strategyproof and guarantees an approximation linear in the number of agents; however, such mechanism does not necessarily provide core stable outcomes. Furthermore, a 4-approximation is possible when relying on randomization. Concerning EA profiles, repeatedly extracting and creating a coalition with a maximum clique, in the undirected graph of mutual friendship, is strategyproof and provides a weak core stable outcome \([23]\); moreover, it gives a constant approximation w.r.t. the optimal utilitarian welfare \([22]\). Unfortunately, such a procedure is not computationally efficient. In turn, via a matching algorithm, it is possible to achieve a strategyproof mechanism having linear approximation in the number of agents.

### Table 3

<table>
<thead>
<tr>
<th>FA profiles</th>
<th>Deterministic</th>
<th>Randomized</th>
<th>EA profiles</th>
<th>Poly-time</th>
<th>Exp-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.B.</td>
<td>(n)</td>
<td>4</td>
<td>U.B.</td>
<td>((1 + \sqrt{2})n)</td>
<td>1 + (\sqrt{2})</td>
</tr>
<tr>
<td>L.B.</td>
<td>2</td>
<td>&gt; 1</td>
<td>L.B.</td>
<td>(\Omega(n))</td>
<td>&gt; 1</td>
</tr>
</tbody>
</table>

**Additively Separable Group Activity Selection Problem.** In this paragraph, we present the results obtained in \([24]\) on approximate strategyproof mechanisms for the *additively separable GASP* (ASGASP). Here, the utility of an agent \(i \in C \subseteq N\) for participating in an activity \(a\) with the coalition \(C\) is given by the sum of the weights for the other coalition members and the preference she has for the activity \(a\), that is, \(u_i(C, a) = p_{ia} + \sum_{j \in C \setminus \{i\}} v_{ij}\). The provided results take into account the possible values of the preferences among the activities and of the individual weights between the agents. Considering the impossibility result for ASHG's when weights may be both positive and negative reals, we can conclude that no bounded strategyproof mechanism exists for the ASGASP as well. Therefore, the authors distinguished the analysis considering non-negative or binary values towards agents/activities. A complete picture of the results is given in Table 4.

In the case of non-negative preferences, it is possible to show that, even if \(v_{ij} \in \{0, 1\}\), no deterministic mechanism can achieve a bounded approximation ratio. In turn, when relying on randomized mechanisms, a simple \(k\)-approximate mechanism, where \(k\) is the number of activities, exists. Such a result is indeed possible by selecting uniformly at random to which activity all agents will be assigned; this makes the mechanisms independent of the agents’ preferences for the activity and therefore no misreport can be successful. In fact, the main reason for the impossibility of a bounded deterministic mechanism is the freedom of agents in selecting high values for the most preferred activity. In contrast, when agents have limited expressiveness on their preferences over activities, bounded deterministic mechanisms are possible. Specifically, if agents’ preferences are boolean, a \(k\)-approximate deterministic mechanism exists; however, it was only possible to show an \(\Omega(\sqrt{k})\) lower bound. An interesting fact is that the \(k\)-approximate mechanism is indeed tight when restricting the attention on anonymous mechanisms\(^2\). Finally,

\(^2\)Anonymous means that the computation of the outcome does not depend on the agents’ identities.
when the weights towards other agents are also binary, a 2-approximate deterministic, and therefore randomized, mechanism is attainable.

Table 4

<table>
<thead>
<tr>
<th></th>
<th>Lower and upper bounds of strategyproof mechanisms for the additively separable GASP. L.B. and U.B. stand for lower and upper bounds respectively. Randomized mechanisms are denoted by *, results for anonymous mechanisms are denoted by †, and ( k ) is the number of activities.</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_{ij} \in \mathbb{R}_{\geq 0} )</td>
<td>( \in \mathbb{R}_{\geq 0} )</td>
<td>{0,1}</td>
</tr>
<tr>
<td>U.B.</td>
<td>( \infty )</td>
<td>( k^* )</td>
</tr>
<tr>
<td>L.B.</td>
<td>( 2 - \frac{2}{k+1} )</td>
<td>( 4/3^* )</td>
</tr>
<tr>
<td>Non-negative preferences: ( p_{ia} \in \mathbb{R}_{\geq 0} )</td>
<td></td>
<td></td>
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</table>

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<tr>
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<tbody>
<tr>
<td>( v_{ij} \in \mathbb{R}_{\geq 0} )</td>
<td>( \in \mathbb{R}_{\geq 0} )</td>
<td>{0,1}</td>
</tr>
<tr>
<td>U.B.</td>
<td>( k )</td>
<td>( 2 )</td>
</tr>
<tr>
<td>L.B.</td>
<td>( \Omega(\sqrt{k}) )</td>
<td>( \Omega(k)^\dagger )</td>
</tr>
<tr>
<td>Boolean preferences: ( p_{ia} \in {0,1} )</td>
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</table>

3. Future Challenges

As we already mentioned, very little is known about mechanisms design for HGs and the GASP. Other than good approximation guarantees, strategyproof mechanisms or manipulability aspects, that is, non-strategyproofness of a certain allocation rule, are considered for example in [25, 26].

In this section, we draw a landscape of compelling future research directions in the sense of approximate strategyproof mechanisms. We anyway consider the goal of finding strategyproof mechanisms guaranteeing some stability notion clearly of interest.

Dealing with the impossibility results. We have seen there exists a number of settings where bounded strategyproof mechanisms are not possible. In these circumstances, strategyproofness turns out to be very demanding and it looks natural to question whether a more relaxed notion may lead to a bounded approximation. More specifically, one could allow the agents to misreport their private information up to some extent, e.g. the distance from their actual private information can be bounded by some constant or the number of misreported values is limited. On the other hand, the reason why bounded mechanisms are not possible in some contexts can be attributed to the fact that agents have too much freedom in their declarations. Since agents do not incur any penalty for arbitrarily raising their values, this allows them to manipulate not only optimal but also approximate solutions. Monetary transfers could disincentivize such kinds of manipulations and, therefore, considering mechanisms with payments is another way we may hope to circumvent impossibility results.

Closing gaps. Although some of the provided mechanisms are bounded, the question if their guarantee is the best attainable in conjunction with strategyproofness remains open. In the case of the ASGASP, with binary preferences and arbitrary weights, a matching lower bound was possible only by assuming anonymity while, in general, the gap remains open. We suspect that anonymity is indeed not needed to show a lower bound of \( k \) and that the provided mechanism is the best possible. Such a belief is driven by what happened in the study of strategyproof
mechanisms for machine scheduling. Here, some tasks must be split among machines and machines can declare their completion time for each task; the goal is to minimize the make-span. In [27] the authors provide an \( m \)-approximate mechanism, where \( m \) is the number of machines, together with a lower bound of 2. Subsequently, this gap has been closed for anonymous mechanisms: no anonymous mechanism can have an approximation better than \( m \) [28]. Only recently, the lower bound has been shown to be \( m \) (without the anonymity condition) both with or without money [29]. It would be interesting therefore to understand if a similar result is possible for the ASGASP; moreover, it is also worth considering anonymous mechanisms in the context of HGs as a first step for tightening bounds. We finally want to highlight that while, on the one hand, the mechanisms have been designed to work in polynomial time (if not specified otherwise), on the other hand, the obtained lower bounds do not depend on the time complexity. Hence, it might be possible to establish whether the provided upper bounds are the best possible for poly-time mechanisms.

**Considering new classes of games.** Other than the aforementioned, there exist other HGs classes that are suitable for the study of approximate strategyproof mechanisms. For example, social distance [30] and distance hedonic [31] games are two generalizations of (simple) FHGs where agents lay in a social network and derive their utility according to their induced distance from the other agents in their coalition. In such games, agents can only hide their connections to other agents rather than provide the values they retain by being in a certain coalition. This intuitively suggests that the agents do not have much room for manipulation which makes attractive the idea of studying approximate strategyproof mechanisms in this setting. In the paper [32], strategyproof and approximately envy-free mechanisms for ASHGsWith bounded coalition size are considered. Their main contribution is a mechanism that achieves good experimental performances. Approximate strategyproof mechanisms (w.r.t. the utilitarian social welfare) for HGs and the GASP with bounded coalition size are also worth considering.

All in all, the study of strategyproof mechanisms is clearly of interest in all those scenarios where strategic agents have to be split by a central authority into coalitions. So far, very little is known in the case of HGs and the GASP, and many questions are open.

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**References**


