

Modal Logics of Partial Quasiary Predicates with Equality

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Abstract

The purpose of this paper is to study new classes of program-oriented logical formalisms of the modal type. We propose transitional modal logics (TML) of partial quasiary predicates without monotonicity condition and enriched with equality predicates. The work describes semantic models and languages of pure first-order TML with strong equality predicates, the properties of the equality predicates are given. A feature of these logics is the use of the extended renomination composition. The semantic basis of TML is transitional modal systems (TMS). Important varieties of such systems are specified – general TMS (GMS), temporal TMS (TmMS), multimodal TMS (MMS). The languages of GMS, TmMS and MMS are described, interpretation mappings of formulas on semantic models are defined. The properties of transitional modal systems are considered, the interaction of modal compositions with renominations and quantifiers is investigated. A number of logical consequence relations are determined on sets of formulas specified with states (irrefutability, truth, falsity and strong logical consequence relations). The properties of these relations, in particular, properties related to equality predicates, are described. On the basis of these properties, a construction of sequent type calculi is planned for the proposed logics.

Keywords ¹

Modal logic, transitional modal system, partial predicate, equality predicate, logical consequence relation

1. Introduction

The apparatus of modal logics is used with great success to describe and model various subject areas and aspects of human activity (see, e.g., [1–5]). Temporal logics are used for dynamic systems modeling, specification and verification of programs [1, 2, 6–8], a number of specification systems and languages have been developed on the basis of these logics. Epistemic logics are successfully used to describe artificial intelligence systems, information and expert systems, databases and knowledge bases with incomplete information. Traditional modal logics are based on classical logic of predicates. However, classical logic has a number of limitations, it does not adequately take into account the partiality, incompleteness, structuredness of information about the subject area. The fundamental limitations of classical logic of predicates bring to the fore the task of building new, program-oriented classes of logical formalisms of modal type. Such are composition nominative modal logics [9], built on the basis of a common for logic and programming composition nominative approach. Composition nominative modal logics (CNML) combine capabilities of composition nominative logics of partial quasiary predicates [10] and traditional modal logics. A very important class of CNML is transitional modal logics (TML), they reflect the aspect of change and evolution of subject areas. Traditional modal logics – alethic, temporal, epistemic, deontic, etc – can be naturally

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considered within TML. Multimodal and temporal composition nominative logics are subclasses of TML; within the multimodal TML, general TML and TML of epistemic type are further singled out. A number of TML classes have been investigated, in particular, in [9].

In this paper we study new classes of program-oriented logical formalisms of modal type – pure first-order transitional modal logics of partial non-monotonic quasiary predicates with equality. Such logics use the extended renomination composition and 0-ary compositions \equiv_{xy} (strong equality predicates). Transitional modal systems of these logics (TMS) and their important varieties – general (GMS), temporal (TmMS) and multimodal (MMS) TMS – are described. We specify languages of the introduced systems and present their properties. Special attention is paid to properties related to the strong equality predicates and interaction of quantifiers with modal compositions. For formalization of logical consequence in TML for sets of formulas specified with states, we define irrefutability, truth, falsity and strong logical consequence relations and provide their main properties, in particular, ones related to the equality predicates and elimination of modalities.

2. Quasiary predicates and their compositions

To facilitate reading, we will provide the definitions necessary for the further presentation. Concepts that are not defined here are interpreted in the sense of works [9, 10].

In this paper, we will follow the notation used in [9, 10]. In particular, modalities “necessarily” and “possibly” are denoted by the operators \Box (“box”) and \Diamond (“diamond”) respectively, their symbols in the language by \Box and \Diamond ; temporal operators are denoted by $\Box\uparrow$, $\Box\downarrow$, $\Diamond\uparrow$, $\Diamond\downarrow$, and their symbols in the language by $\Box\uparrow$, $\Box\downarrow$, $\Diamond\uparrow$, $\Diamond\downarrow$.

Quasiary predicates. Data in the logics of quasiary predicates have a structure of nominative sets. Functions and predicates defined on nominative sets are called quasiary.

A nominative set is a set of named values, i.e. set of pairs (name, value). Formally, a V -A-nominative set (V -A-NS) is defined as a single-valued function of the form $d : V \textcircled{R} A$. We interpret V and A as sets of subject names (variables) and subject values respectively. Next, V -A-NS is presented in the form $[v_1 \mapsto a_1, \dots, v_n \mapsto a_n, \dots]$; here $v_i \in V$, $a_i \in A$, $v_i \neq v_j$ when $i \neq j$. We denote the set of all V -A-nominative sets by ${}^V A$.

For single-valued functions, we will further write:

- $f(d)\downarrow$, if the value of $f(d)$ is defined,
- $f(d)\uparrow$, if the value of $f(d)$ is undefined.

For V -A-NS we introduce operations $\|Z$ and $\|_{-Z}$, where $Z \subseteq V$:

$$d \| Z = \{v \mapsto a \in d \mid v \in Z\}; \quad d \|_{-Z} = \{v \mapsto a \in d \mid v \notin Z\}.$$

Operation $r_{x_1, \dots, x_n, \perp, \dots, \perp}^{v_1, \dots, v_n, u_1, \dots, u_m} : {}^V A \textcircled{R} {}^V A$ of the extended renomination, where all $v_i, x_i, u_j \in V$, and the symbol $\perp \notin V$ means no value, is specified as follows:

$$r_{x_1, \dots, x_n, \perp, \dots, \perp}^{v_1, \dots, v_n, u_1, \dots, u_m}(d) = d \|_{-\{v_1, \dots, v_n, u_1, \dots, u_m\}} \cup [v_1 \mapsto d(x_1), \dots, v_n \mapsto d(x_n)].$$

In case $d(x_i)\uparrow$, the component with name v_i is absent.

We introduce an abbreviated notation \bar{y} for y_1, \dots, y_n : instead of $r_{x_1, \dots, x_n, \perp, \dots, \perp}^{v_1, \dots, v_n, u_1, \dots, u_m}$, we will write $r_{\bar{x}, \perp}^{\bar{v}, \bar{u}}$.

Thus, given $d(z)\uparrow$, we obtain $r_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(d) = r_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(d)$ and $r_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, z}(d) = r_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(d)$.

Sequential application of operations $r_{\bar{y}, \perp}^{\bar{v}, \bar{z}}$ and $r_{\bar{x}, \perp}^{\bar{u}, \bar{t}}$ can be interpreted as one renomination operation called their *convolution* and denoted by $r_{\bar{y}, \perp}^{\bar{v}, \bar{z}} \square_{\bar{x}, \perp}^{\bar{u}, \bar{t}}$. Hence $r_{\bar{x}, \perp}^{\bar{u}, \bar{t}}(r_{\bar{y}, \perp}^{\bar{v}, \bar{z}}(d)) = r_{\bar{y}, \perp}^{\bar{v}, \bar{z}} \square_{\bar{x}, \perp}^{\bar{u}, \bar{t}}(d)$.

A partial function of the form $Q : {}^V A \textcircled{R} \{T, F\}$ is called a V -A-*quasiary predicate*. Here $\{T, F\}$ is the set of truth values. In this paper, we will consider modal logics of *single-valued* quasiary predicates, or P -predicates.

We will denote by PrP^{V-A} the class of V -A-*quasiary predicates*.

Each P -predicate S is determined by two sets:

- the *truth* domain $T(S) = \{d \mid S(d) = T\}$;
- the *falsity* domain $F(S) = \{d \mid S(d) = F\}$.

The single-valuedness of the predicate S means that $T(S) \cap F(S) = \emptyset$.

The name $x \in V$ is *unessential* for the predicate S , if for all $d_1, d_2 \in {}^V A$ we have:

$$d_1 \parallel_{-x} = d_2 \parallel_{-x} \Rightarrow Q(d_1) = Q(d_2).$$

The unessentiality of x means that the predicate S is “insensitive” to the data component named x .
 V - A -quasiary P -predicate S is:

- irrefutable (partially true), if $F(S) = \emptyset$;
- constantly *true* (denoted T), if $F(S) = \emptyset$ and $T(S) = {}^V A$;
- constantly *false* (denoted F), if $T(S) = \emptyset$ and $F(S) = {}^V A$;
- totally undefined (denoted E), if $T(S) = F(S) = \emptyset$.

Therefore, we have three constant P -predicates: T, F, and E.

For single-valued predicates, the concept of monotonicity is specified as *equitonicity* – preservation of the already accepted value after data expansion.

P -predicate S is *equitone* if the conditions $S(d) \downarrow$ and $d \subseteq d'$ imply $S(d') \downarrow = S(d)$.

Variable assignment predicates Ez which indicate whether a component with a corresponding name $z \in V$ has a value in the input data are used for quantifier elimination in logics of non-monotonic predicates [10]. In this paper, instead of Ez , we will use a more descriptive notation $z \downarrow$. We define the predicates $z \downarrow$ as follows:

$$T(z \downarrow) = \{d \mid d(z) \downarrow\}; \quad F(z \downarrow) = \{d \mid d(z) \uparrow\}.$$

Predicates $z \downarrow$ are total, single-valued, non-monotonic.

Every $x \in V$ such that $x \neq z$, is unessential for $z \downarrow$.

Equality predicates. In logics of quasiary predicates, two types of equality predicates can be considered: weak (up to definability) equality and strong equality. Weak equality predicates are inherent for logics of monotonic (equitone) predicates, while strong equality predicates are inherent for logics of non-monotonic predicates.

In this work, we consider modal logics of P -predicates without monotonicity restrictions with strong equality predicates $\equiv_{\{x,y\}}$. We specify these predicates as follows (see [10]):

$$\begin{aligned} T(\equiv_{\{x,y\}}) &= \{d \mid d(x) \downarrow, d(y) \downarrow \text{ and } d(x) = d(y)\} \cup \{d \mid d(x) \uparrow \text{ and } d(y) \uparrow\}, \\ F(\equiv_{\{x,y\}}) &= \{d \mid d(x) \downarrow, d(y) \downarrow, d(x) \neq d(y)\} \cup \{d \mid d(x) \downarrow, d(y) \uparrow \text{ or } d(x) \uparrow, d(y) \downarrow\}. \end{aligned}$$

Further on the predicates $\equiv_{\{x,y\}}$ will be denoted as \equiv_{xy} . We will also use the traditional notation $x \equiv y$. Note that \equiv_{xy} and \equiv_{yx} denote the *same* predicate!

Predicates $\equiv_{\{x\}}$ – a special case of $\equiv_{\{x,y\}}$ when x and y are the same name – will be denoted by \equiv_{xx} (or $x \equiv x$ in the traditional form).

Every $z \in V \setminus \{x, y\}$ is unessential for \equiv_{xy} ; every $z \in V \setminus \{x\}$ is unessential for \equiv_{xx} .

Predicates \equiv_{xy} are total single-valued and non-monotonic.

For \equiv_{xx} we have $T(\equiv_{xx}) = {}^V A$ and $F(\equiv_{xx}) = \emptyset$, therefore \equiv_{xx} is a constant predicate T.

Basic logical compositions. Basic logical compositions of pure first-order modal logics of quasiary predicates are logical connectives \neg and \vee , renomination $R_{\bar{x}, \bar{u}}^{\bar{v}, \bar{u}}$, and existential quantification $\exists x$. Let us briefly recall their definitions (see [10]).

Renomination composition is specified using a corresponding operation:

$$R_{\bar{x}, \bar{u}}^{\bar{v}, \bar{u}}(Q)(d) = Q(r_{\bar{x}, \bar{u}}^{\bar{v}, \bar{u}}(d)) \text{ for all } d \in {}^V A.$$

We define compositions \neg , \vee , $\exists x$ through the truth and falsity domains of the corresponding predicates:

$$\begin{array}{ll}
T(\neg P) = F(P); & F(\neg P) = T(P); \\
T(P \vee Q) = T(P) \cup T(Q); & F(P \vee Q) = F(P) \cap F(Q); \\
T(\exists x P) = \bigcup_{a \in A} \{d \mid Q(d \parallel_{-x} \cup x \mapsto a) = T\}; & F(\exists x P) = \bigcap_{a \in A} \{d \mid Q(d \parallel_{-x} \cup x \mapsto a) = F\}.
\end{array}$$

Properties of propositional compositions and quantifiers not related to renominations are similar to the properties of classical logical connectives and quantifiers. Properties associated with renominations are specified as follows:

$$\begin{array}{l}
\text{R)} \mathbf{R}(Q) = Q; \\
\text{RI)} \mathbf{R}_{z, \bar{x}, \perp}^{z, \bar{v}, \bar{u}}(Q) = \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q) \text{ – convolution of an identical pair of names in renomination;} \\
\text{RU)} z \in V \text{ is unessential for the predicate } Q \Rightarrow \mathbf{R}_{y, \bar{x}, \perp}^{z, \bar{v}, \bar{u}}(Q) = \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q); \\
\text{R}\uparrow) \text{ given } d(z) \uparrow, \text{ then } \mathbf{R}_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(Q)(d) = \mathbf{R}_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(Q)(d) \text{ and } \mathbf{R}_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, z}(Q)(d) = \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q)(d); \\
\text{R}\neg) \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\neg Q) = \neg \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q); \\
\text{R}\vee) \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(P \vee Q) = \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(P) \vee \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q); \\
\text{RR)} \mathbf{R}_{\bar{y}, \perp}^{\bar{v}, \bar{z}}(\mathbf{R}_{\bar{x}, \perp}^{\bar{u}, \bar{t}}(Q)) = \mathbf{R}_{\bar{y}, \perp}^{\bar{v}, \bar{z}} \circ_{\bar{x}, \perp}^{\bar{u}, \bar{t}}(Q); \\
\text{Ren)} \text{ given } z \text{ is unessential for } Q \text{ then } \exists y Q = \exists z \mathbf{R}_z^y(Q); \\
\text{R}\exists) \text{ given } y \notin \{\bar{v}, \bar{x}, \bar{u}\} \text{ then } \exists y \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Q) = \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\exists y Q); \\
\text{R}\exists) \text{ given } z \text{ is unessential for } Q \text{ and } z \notin \{\bar{v}, \bar{x}, \bar{u}\} \text{ then } \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\exists y P) = \exists z \mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}} \circ_z^y(P).
\end{array}$$

The quantifier elimination is based on the following properties:

$$\begin{array}{l}
\text{T}\exists\vee) T(\mathbf{R}_{\bar{v}, \perp, y}^{\bar{u}, \bar{w}, x}(Q)) \cap T(y \downarrow) \subseteq T(\mathbf{R}_{\bar{v}, \perp}^{\bar{u}, \bar{w}}(\exists x Q)); \\
\text{F}\exists\vee) F(\mathbf{R}_{\bar{v}, \perp}^{\bar{u}, \bar{w}}(\exists x Q)) \cap T(y \downarrow) \subseteq F(\mathbf{R}_{\bar{v}, \perp, y}^{\bar{u}, \bar{w}, x}(Q)).
\end{array}$$

Let the predicate $\neg z \downarrow$ be denoted by $z \uparrow$. Then we have $z \downarrow \vee z \uparrow = T$.

Let us consider properties of equality predicates. Note that the symmetry of equality is virtually technical as \equiv_{xy} and \equiv_{yx} is the same predicate.

The reflexivity of equality predicates follows from the fact that $T(\equiv_{xx}) = \forall A$ and $F(\equiv_{xx}) = \emptyset$:

$$\text{Rf}\equiv) \equiv_{xx} = T, \text{ i.e. each } \equiv_{xx} \text{ is a constant predicate } T.$$

The transitivity of equality predicates:

$$\text{Tr}\equiv) \equiv_{xy} \& \equiv_{yz} \rightarrow \equiv_{xz} = T.$$

Renomination properties of equality predicates (properties of the $\mathbf{R}\equiv$ type):

$$\begin{array}{l}
\mathbf{R}\equiv_{xx}) \mathbf{R}_{\bar{w}, \perp, z}^{\bar{v}, \bar{u}, x}(\equiv_{xx}) = \equiv_{zz}; \\
\mathbf{R}\equiv_0) \text{ given } x, y \notin \{\bar{u}, \bar{v}\}, \text{ then } \mathbf{R}_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xy}) = \equiv_{xy}; \\
\text{in particular, given } x \notin \{\bar{u}, \bar{v}\}, \text{ then } \mathbf{R}_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xx}) = \equiv_{xx}; \\
\mathbf{R}\equiv_1) \text{ given } y \notin \{\bar{u}, \bar{v}\} \text{ and } x \neq y, \text{ then } \mathbf{R}_{\bar{w}, \perp, z}^{\bar{v}, \bar{u}, x}(\equiv_{xy}) = \equiv_{zy}; \\
\mathbf{R}\equiv_2) \mathbf{R}_{\bar{w}, \perp, z, s}^{\bar{v}, \bar{u}, x, y}(\equiv_{xy}) = \equiv_{zs}; \\
\mathbf{R}\equiv\uparrow_1) \text{ given } y \notin \{\bar{u}, \bar{v}\} \text{ and } x \neq y, \text{ then } \mathbf{R}_{\bar{w}, \perp, \perp}^{\bar{v}, \bar{u}, x}(\equiv_{xy}) = y \uparrow; \\
\mathbf{R}\equiv\uparrow_2) \mathbf{R}_{\bar{w}, \perp, z, \perp}^{\bar{v}, \bar{u}, x, y}(\equiv_{xy}) = z \uparrow;
\end{array}$$

$\mathbf{R} \equiv \mathbf{R}_{\bar{w}, \perp, \perp, \perp}^{\bar{v}, \bar{u}, x, y} (\equiv_{xy}) = \mathbf{T}$; in particular, $\mathbf{R}_{\bar{w}, \perp, \perp}^{\bar{v}, \bar{u}, x} (\equiv_{xx}) = \mathbf{T}$.

The substitution of equals in renomination: for all $Q \in PrP^{V-A}$ and $d \in {}^V A$ we have

$$\equiv_{rp} \equiv_{xy}(d) = T \Rightarrow \mathbf{R}_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(Q)(d) = \mathbf{R}_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, z}(Q)(d).$$

Derivative property of the elimination of a pair of equals: for all $Q \in PrP^{V-A}$ and $d \in {}^V A$ we have

$$\equiv_{eR} \equiv_{xy}(d) = T \Rightarrow \mathbf{R}_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, x}(Q)(d) = \mathbf{R}_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(Q)(d).$$

3. Transitional modal systems and their varieties

The concept of composition nominative modal system [9] is the central notion of CNML. Such systems describe possible worlds of modal logic and are models of these worlds.

Composition nominative modal system (CNMS) is an object of the form $\mathbf{M} = (Cms, Ds, Dns)$, where:

- Cms is a composition modal system (CMS); it defines semantic aspects of the world;
- Ds is a descriptive system; it defines a set of standard descriptions Fm – formulas of the CNML language;
- Dns is a denotation system; it determines values of standard descriptions (language formulas) on semantic models; an interpretation mapping Im of formulas on states of the world is usually used for this.

CMS are relational semantic models. They have the form $Cms = (St, \mathbf{R}, Pr, C)$, where:

- St is a set of states of the world;
- \mathbf{R} is a set of relations on St of the form $\mathbf{R} \subseteq St \times St^n$;
- Pr is a set of predicates on states of the world;
- C is a set of compositions on Pr .

Therefore, CNMS can be interpreted as objects $\mathbf{M} = ((St, \mathbf{R}, Pr, C), Fm, Im)$.

For the first-order CNMS, a set of states of the world St is specified as a set of algebraic structures $\alpha = (A_\alpha, Pr_\alpha)$, where A_α is a set of basic data of the state α , Pr_α is a set of quasiary predicates ${}^V A_\alpha \rightarrow \{T, F\}$, called *predicates of the state* α .

Let us call *global* the predicates of the form ${}^V A \rightarrow \{T, F\}$, where $A = \bigcup_{\alpha \in S} A_\alpha$.

A set of compositions in CNMS is determined by basic *logical* compositions of the corresponding level and basic *modal* compositions.

We consider logical connectives \neg and \vee , compositions of extended renomination $\mathbf{R}_{\bar{x}, \perp}^{\bar{v}, \bar{u}}$, quantification compositions $\exists x$ and special 0-ary compositions – strong equality predicates \equiv_{xy} – to be basic logical compositions of pure first-order KNMS with equality.

Transitional modal systems. An important class of CNML, transitional modal logics (TML) reflect the aspect of change and evolution of subject areas, describing transitions from one state of the world to another. At the heart of TML lies the concept of transitional modal system [9].

Transitional modal system (TMS) are CNMS in which the set \mathbf{R} consists of relations of the form $\mathbf{R} \subseteq St \times St$. We interpret them as state transition relations.

Special cases of TMS are general transitional, temporal and multimodal systems.

TMS in which \mathbf{R} consists of a single binary relation \triangleright and with a basic modal composition \square ("necessary") are called *general* (GMS).

TMS in which \mathbf{R} consists of a single binary relation \triangleright and with basic modal compositions $\square \uparrow$ ("it will always be the case") and $\square \downarrow$ ("it has always been the case") are called *temporal* (TmMS).

TMS with a set of relations $\mathbf{R} = \{\triangleright_i \mid i \in I\}$ and basic modal compositions M_i , $i \in I$, in which each $\triangleright_i \in \mathbf{R}$ is matched with a corresponding modal composition M_i , are called *multimodal* (MMS).

GMS is a special case of MMS, with \mathbf{R} consisting of a single relation \triangleright and a single basic modal composition M identical to \Box .

For GMS, the derivative composition \Diamond ("possible") is traditionally specified as:

$$\Diamond P \text{ means } \neg\Box\neg P.$$

For MMS, each M_i is identical to \Box with respect to its relation \triangleright_i , $i \in I$.

For TmMS, the derivative compositions $\Diamond\uparrow$ ("it will sometimes be the case") and $\Diamond\downarrow$ ("it was sometime the case") are defined as:

$$\Diamond\uparrow P \text{ means } \neg\Box\uparrow\neg P; \quad \Diamond\downarrow P \text{ means } \neg\Box\downarrow\neg P.$$

Further on TMS will be abbreviated denoted $\mathbf{M} = (St, \mathbf{R}, A, Im)$, where $A = \bigcup_{\alpha \in S} A_\alpha$.

GMS languages. Let us describe a language of a first-order GMS with equality. The alphabet:

- a set V of subject names (variables);
- a set Ps of predicate symbols (language signature);
- a set of basic logical compositions' symbols $\{\neg, \vee, R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}, \exists x, \equiv_{xy}\}$;
- a set of basic modal compositions' symbols (modal signature), which consists of one symbol: $M_S = \{\Box\}$.

The set Fm of language formulas is determined inductively:

FA) every $p \in Ps$ and every symbol \equiv_{xy} is a formula; such formulas are *atomic*;

FP) let Φ and Ψ be formulas; then $\neg\Phi$ and $\vee\Phi\Psi$ are formulas;

FR) let Φ be a formula; then $R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi)$ is a formula;

F \exists) let Φ be a formula; then $\exists x\Phi$ is a formula;

F \Box) let Φ be a formula, then $\Box\Phi$ is a formula.

We will use the derived modal composition \Diamond , and thus write $\Diamond\Phi$ instead of $\neg\Box\neg\Phi$.

For each $p \in Ps$, a set of its unessential names is specified using a total mapping $v : Ps \rightarrow 2^V$. Let us limit ourselves to the set $V_T \subseteq V$ of names unessential for all $p \in Ps$ – called a set of *totally unessential* names, therefore $v(p) \supseteq V_T$ for each $p \in Ps$. To define sets of *guaranteed to be unessential* names for formulas, we continue v to the mapping $v : Fm \rightarrow 2^V$ as follows:

$$- v(\equiv_{xy}) = V \setminus \{x, y\};$$

$$- v(\neg\Phi) = v(\Phi);$$

$$- v(\vee\Phi\Psi) = v(\Phi) \cap v(\Psi);$$

$$- v(R_{x_1, \dots, x_n, \perp, \dots, \perp}^{v_1, \dots, v_n, u_1, \dots, u_m} \Phi) = (v(\Phi) \cup \{v_1, \dots, v_n, u_1, \dots, u_m\}) \setminus \{x_i \mid v_i \notin v(\Phi), i \in \{1, \dots, n\}\};$$

$$- v(\exists x\Phi) = v(\Phi) \cup \{x\};$$

$$- v(\Box\Phi) = v(\Phi).$$

The pair $\sigma = (Ps, v)$ is an extended GMS signature.

A GMS type is determined by an extended signature and properties of the relation \triangleright .

Let us define an interpretation mapping of formulas on states of the world.

First, we specify an interpretation mapping of atomic formulas $Im : Ps \times St \rightarrow Pr$, where a condition $Im(p, \alpha) \in Pr_\alpha$ should hold (basic predicates are predicates of states). Compositions' symbols (in particular, \equiv_{xy} symbols) are interpreted as corresponding compositions (in particular, corresponding equality predicates \equiv_{xy}). The mapping Im is continued to an interpretation mapping of formulas on states $Im : Fm \times St \rightarrow Pr$ as follows:

$$IP) Im(\neg, \alpha) = \neg(Im(\Phi, \alpha)); \quad Im(\vee\Phi\Psi, \alpha) = \vee(Im(\Phi, \alpha), Im(\Psi, \alpha));$$

$$IR) Im(R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi), \alpha) = R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(Im(\Phi, \alpha));$$

$$I\exists) \text{Im}(\exists x\Phi, \alpha)(d) = \begin{cases} T, & \text{if } \text{Im}(\Phi, \alpha)(d \parallel_{-x} \cup x \mapsto a) = T \text{ for some } a \in A_\alpha, \\ F, & \text{if } \text{Im}(\Phi, \alpha)(d \parallel_{-x} \cup x \mapsto a) = F \text{ for any } a \in A_\alpha, \\ \text{else undefined.} \end{cases}$$

I□) for each $\alpha \in St$ we set

$$\text{Im}(\Box\Phi, \alpha)(d) = \begin{cases} T, & \text{if } \text{Im}(\Phi, \delta)(d) = T \text{ for any } \delta \in St : \alpha \triangleright \delta, \\ F, & \text{if exists } \delta \in St : \alpha \triangleright \delta \text{ and } \text{Im}(\Phi, \delta)(d) = F, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha \triangleright \beta$, then for each $d \in VA$ we define $\text{Im}(\Box\Phi, \alpha)(d) \uparrow$.

Note that for abbreviations of formulas of the form $\Diamond\Phi$, we define the mapping Im as follows:

I◇) for each $\alpha \in St$ we specify:

$$\text{Im}(\Diamond\Phi, \alpha)(d) = \begin{cases} T, & \text{if exists } \delta \in St : \alpha \triangleright \delta \text{ and } \text{Im}(\Phi, \delta)(d) = T, \\ F, & \text{if } \text{Im}(\Phi, \delta)(d) = F \text{ for any } \delta \in St : \alpha \triangleright \delta, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha \triangleright \beta$, then for each $d \in VA$ we define $\text{Im}(\Diamond\Phi, \alpha)(d) \uparrow$.

Predicates which are values of formulas that do not use modalities belong to predicates of states.

A predicate $\text{Im}(\Phi, \alpha)$, which is a value of the formula Φ in the state α , is denoted by Φ_α .

A formula Φ is *valid on state* α (denoted $\alpha \models \Phi$), if Φ_α is a valid (irrefutable) predicate.

A formula Φ is *valid in GMS* M (denoted $M \models \Phi$), if for each $\alpha \in St$ the predicate Φ_α is valid.

Let \mathcal{M} be a GMS class of a given type. The formula Φ is called *\mathcal{M} -valid* (denoted $\mathcal{M} \models \Phi$), if $M \models \Phi$ for all GMS $M \in \mathcal{M}$.

Depending on conditions imposed on the relation \triangleright , different classes of GMS can be specified. Traditionally, cases where \triangleright can be reflexive, symmetric or transitive are considered. In case \triangleright is reflexive/transitive/symmetric, then we will write the corresponding symbol $R/T/S$ in the GMS name. Thus we get the following GMS classes:

R -GMS, T -GMS, S -GMS, RT -GMS, RS -GMS, TS -GMS, RTS -GMS.

Note that R -GMS, RS -GMS, RT -GMS, RTS -GMS are similar to classical T -model, B -model, $S4$ -model and $S5$ -model structures.

TmMS languages. Let us describe a language of a first-order TmMS with equality. The alphabet is identical to the alphabet of a first-order GMS language with equality, with a set M_s specified as follows: $M_s = \{\Box \uparrow, \Box \downarrow\}$.

The set Fm of language formulas is determined according to the described above items FA, FP, FR, F◇, but instead of $F\Box$ we have:

$F\Box \uparrow \downarrow$) let Φ be a formula, then $\Box \uparrow \Phi$ and $\Box \downarrow \Phi$ are formulas.

We will use the derived modal compositions $\Diamond \uparrow \Phi$ and $\Diamond \downarrow \Phi$ to write formulas of the form $\neg \Box \uparrow \neg \Phi$ or $\neg \Box \downarrow \neg \Phi$.

When we define a mapping Im , instead of I□) for formulas $\Box \uparrow \Phi$ and $\Box \downarrow \Phi$ we have:

I□ $\uparrow \downarrow$) for each $\alpha \in St$ we specify:

$$\text{Im}(\Box \uparrow \Phi, \alpha)(d) = \begin{cases} T, & \text{if } \text{Im}(\Phi, \delta)(d) = T \text{ for any } \delta \in St : \alpha \triangleright \delta, \\ F, & \text{if exists } \delta \in St : \alpha \triangleright \delta \text{ and } \text{Im}(\Phi, \delta)(d) = F, \\ \text{else undefined.} \end{cases}$$

$$\text{Im}(\Box \downarrow \Phi, \alpha)(d) = \begin{cases} T, & \text{if } \text{Im}(\Phi, \delta)(d) = T \text{ for any } \delta \in St : \delta \triangleright \alpha, \\ F, & \text{if exists } \delta \in St : \delta \triangleright \alpha \text{ and } \text{Im}(\Phi, \delta)(d) = F, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha \triangleright \beta$, then for each $d \in VA$ we define $\text{Im}(\Box \uparrow \Phi, \alpha)(d) \uparrow$.

Given for $\alpha \in St$ there is no β such that $\beta \triangleright \alpha$, then for each $d \in VA$ we define $\text{Im}(\Box \downarrow \Phi, \alpha)(d) \uparrow$.

Depending on conditions imposed on \triangleright , different classes of TmMS can be defined. Considering the cases when \triangleright can be reflexive, symmetric or transitive, we obtain the following classes:

R -TmMS, T -TmMS, S -TmMS, RT -TmMS, RS -TmMS, TS -TmMS, RTS -TmMS.

MMS languages. Let us describe a language of a first-order MMS language with equality. The alphabet is identical to the alphabet of a first-order GMS language with equality, with a set M_s is specified as follows: $M_s = \{M_i \mid i \in I\}$.

The set Fm of language formulas is determined according to the described above items FA, FP, FR, F \exists , but instead of F \Box we have:

FM) let Φ be a formula, $M_i \in M_s$; then $M_i\Phi$ is a formula.

When we define a mapping Im , instead of I \Box for formulas of the form $M_i\Phi$ we have:

IM) for each $\alpha \in St$ we specify:

$$Im(M_i\Phi, \alpha)(d) = \begin{cases} T, & \text{if } Im(\Phi, \delta)(d) = T \text{ for any } \delta \in St: \alpha \triangleright_i \delta, \\ F, & \text{if exists } \delta \in St: \alpha \triangleright_i \delta \text{ and } Im(\Phi, \delta)(d) = F, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha \triangleright_i \beta$, then for each $d \in {}^V A$ we define $Im(M_i\Phi, \alpha)(d) \uparrow$.

Depending on properties of relations \triangleright_i , different classes of MMCs can be defined. We consider cases where \triangleright_i can be reflexive, symmetric or transitive, and all \triangleright_i are of the same type. Thus we obtain the following MMS pure types:

R -MMS, T -MMS, S -MMS, RT -MMS, RS -MMS, TS -MMS, RTS -MMS.

At the same time, much more complex, mixed types are possible here. For example, the relation \triangleright_1 is transitive and reflexive, \triangleright_2 is symmetric, \triangleright_3 is transitive, etc.

We will call MMS with finite sets of same-type relations \triangleright_i *epistemic MMS* (EpMS). Therefore, GMS can be interpreted as a special case of EpMS.

Traditional systems of epistemic logic of knowledge can be naturally considered within EpMS. In particular, R -EpMS, RT -EpMS, RTS -EpMS are generalizations of epistemic $T_{(n)}$, $S4_{(n)}$, $S5_{(n)}$ -systems respectively.

4. Properties of transitional modal systems

Let $d \notin {}^V A_\delta$. The question arises: how to set the value of $\Phi_\delta(d)$?

Let given $d \notin {}^V A_\delta$ we have $\Phi_\delta(d) \uparrow$, hence $\Phi_\delta(d) \downarrow \Rightarrow d \in {}^V A_\delta$. Such a property is called a *strong condition of undefinedness on states*. In this case, from $(\Box\Phi)_\alpha(d) = T$ follows that $d \in {}^V A_\delta$ for all δ such that $\alpha \triangleright \delta$. This means that basic objects (basic data) cannot disappear when transitioning to a successor state, which imposes strong restrictions on semantic models. In addition, equitonicity of predicates is violated under a modal composition. Therefore, the strong undefinedness on states condition is too restrictive.

The *general condition of undefinedness on states* seems more organic:

given $d \notin {}^V A_\delta$, we specify $\Phi_\delta(d) = \Phi_\delta(d_\delta)$.

Here d_δ denotes NS $[\nu \rightarrow a \in d \mid a \in A_\delta]$.

Informally, this means that a predicate on states δ "perceives" only such components $\nu \rightarrow a$ that $a \in A_\delta$. Hence for TMS with a general condition of undefinedness on states, we have $\Phi_\delta(d) = \Phi_\delta(d_\delta)$ for all $d \in {}^V A$.

It is not difficult to show that in GMS, TmMS, MMS with a general condition of undefinedness on states, basic modal compositions preserve the predicates' equitonicity.

Thus, further on we will consider TMS with a general condition of undefinedness on states.

Predicates that are values of non-modalized formulas, belong to predicates of state; they are defined as follows: for each $d \in {}^V A$ we have $\Phi_\delta(d) = \Phi_\delta(d_\delta)$.

Predicates which are values of modalized formulas, belong to global predicates.

Interaction of modal compositions with renominations and quantifiers. Symbols of TMS modal compositions can be carried through renominations:

Theorem 1. $R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\mathfrak{K}(\Phi))(d) = \mathfrak{K}(R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi))(d)$ for arbitrary $\mathfrak{K} \in M_s$, $\Phi \in Fm$, $d \in {}^V A$.

The statement of Theorem 1 is specified for the cases of GMS, TmMS, MMS.

Here we consider:

$M_S = \{\Box\}$ for the GMS case,

$M_S = \{\Box\uparrow, \Box\downarrow\}$ for the TmMS case,

$M_S = \{M_i \mid i \in I\}$ for the MMS case.

Consequence 1. Formulas $R_{x,\perp}^{\bar{v},\bar{u}}(\mathfrak{K}(\Phi)) \leftrightarrow \mathfrak{K}(R_{x,\perp}^{\bar{v},\bar{u}}(\Phi))$, where $\mathfrak{K} \in M_S$, are irrefutable.

The interaction of modal compositions and quantifiers in TMS is more interesting. We will consider the interaction of \Box and \Diamond with $\exists x$ and $\forall x$ in GMS, cases of TmMS and MMS are similar.

Theorem 2. Let M be a GMS of equitone predicates, then for each $\Phi \in Fm$ we have:

1) $M \models \exists x \Box \Phi \rightarrow \Box \exists x \Phi$;

2) $M \models \Box \forall x \Phi \rightarrow \forall x \Box \Phi$.

Consequence 2. Let M be a GMS of equitone predicates, then for each $\Phi \in Fm$ we have:

1) $M \models \Diamond \forall x \Phi \rightarrow \forall x \Diamond \Phi$;

2) $M \models \exists x \Diamond \Phi \rightarrow \Diamond \exists x \Phi$.

Thus, in a GMS of equitone predicates, formulas $\exists x \Box \Phi \rightarrow \Box \exists x \Phi$, $\Box \forall x \Phi \rightarrow \forall x \Box \Phi$, $\Diamond \forall x \Phi \rightarrow \forall x \Diamond \Phi$, $\exists x \Diamond \Phi \rightarrow \Diamond \exists x \Phi$ are irrefutable (valid).

In contrast to the statement of Theorem 2, in a GMS without monotonicity restrictions, formulas $\exists x \Box \Phi \rightarrow \Box \exists x \Phi$ та $\Box \forall x \Phi \rightarrow \forall x \Box \Phi$ are refutable (see also [9]).

Example 1. Formula $\exists x \Box \Phi \rightarrow \Box \exists x \Phi$ is refuted in a GMS without monotonicity restrictions.

Let $St = \{\alpha, \beta\}$, $R = \{\alpha \triangleright \beta\}$, $A_\alpha = \{a, b\}$, $A_\beta = \{b\}$. We have $[x \mapsto a, y \mapsto b]_\beta = [y \mapsto b]$.

Let us take $p \in Ps$ from $v(p) = V \setminus \{x\}$. Let us set $p_\beta([y \mapsto b]) = T$, $p_\beta([x \mapsto b, y \mapsto b]) = F$.

Given $A_\beta = \{b\}$ and $p_\beta([x \mapsto b, y \mapsto b]) = F$, we have $(\exists x p)_\beta([y \mapsto b]) = F \Rightarrow (\Box \exists x p)_\alpha([y \mapsto b]) = F$. We obtain $p_\beta([x \mapsto a, y \mapsto b]_\beta) = p_\beta([y \mapsto b]_\beta) = T \Rightarrow (\Box p)_\alpha([x \mapsto a, y \mapsto b]) = T \Rightarrow (\exists x \Box p)_\alpha([y \mapsto b]) = T$. So, $(\exists x \Box p \rightarrow \Box \exists x p)_\alpha([y \mapsto b]) = F$, therefore $\alpha \not\models \exists x \Box p \rightarrow \Box \exists x p$.

Example 2. Formula $\Box \forall x \Phi \rightarrow \forall x \Box \Phi$ is refuted in a GMS without monotonicity restrictions.

Let $St = \{\alpha, \beta\}$, $R = \{\alpha \triangleright \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b\}$. Let us take $p \in Ps$ from $v(p) = V \setminus \{x\}$.

Let us set $p_\alpha(\emptyset) = F$, $p_\beta(\emptyset) = F$, $p_\beta([x \mapsto b]) = T$.

We have $p_\beta([x \mapsto b]) = T \Rightarrow (\forall x p)_\beta(\emptyset) = T \Rightarrow (\Box \forall x p)_\alpha(\emptyset) = F$.

We obtain $[x \mapsto a]_\beta = \emptyset \Rightarrow p_\beta([x \mapsto a]_\beta) = p_\beta(\emptyset) = F \Rightarrow (\Box p)_\alpha([x \mapsto a]) = F \Rightarrow (\forall x \Box p)_\alpha(\emptyset) = F$.

Therefore, $(\Box \forall x p \rightarrow \forall x \Box p)_\alpha(\emptyset) = F \Rightarrow \alpha \not\models \Box \forall x p \rightarrow \forall x \Box p$.

Note that in the examples 1 and 2, the predicate p_β is non-equitone.

Consequence 3. Formulas $\exists x \Diamond \Phi \rightarrow \Diamond \exists x \Phi$ and $\Diamond \forall x \Phi \rightarrow \forall x \Diamond \Phi$ are refutable.

Example 3. Formulas $\Box \exists x \Phi \rightarrow \exists x \Box \Phi$ and $\forall x \Box \Phi \rightarrow \Box \forall x \Phi$ are refuted in a GMS of equitone predicates.

Let $St = \{\alpha, \beta\}$, $R = \{\alpha \triangleright \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{a, b\}$.

Let us take $p, q \in Ps$ from $v(p) = v(q) = V \setminus \{x\}$.

Let us set $p_\alpha([x \mapsto a]) = F$, $p_\beta([x \mapsto a]) = F$, $p_\beta([x \mapsto b]) = T$.

Given $A_\alpha = \{a\}$, we have $p_\beta([x \mapsto a]) = F \Rightarrow (\Box p)_\alpha([x \mapsto a]) = F \Rightarrow (\exists x \Box p)_\alpha([x \mapsto a]) = F$.

However, given $p_\beta([x \mapsto b]) = T$, we obtain $(\exists x p)_\beta([x \mapsto a]) = T \Rightarrow (\Box \exists x p)_\alpha([x \mapsto a]) = T$.

Therefore, $(\Box \exists x p \rightarrow \exists x \Box p)_\alpha([x \mapsto a]) = F \Rightarrow \alpha \not\models \Box \exists x p \rightarrow \exists x \Box p$.

Let us set $q_\alpha([x \mapsto a]) = T$, $q_\beta([x \mapsto a]) = T$, $q_\beta([x \mapsto b]) = F$.

We have $q_\beta([x \mapsto b]) = F \Rightarrow (\forall x q)_\beta([x \mapsto a]) = F \Rightarrow (\Box \forall x q)_\alpha([x \mapsto a]) = F$.

Given $q_\beta([x \mapsto a]) = T$, we have $(\Box q)_\beta([x \mapsto a]) = T \Rightarrow (\forall x \Box q)_\beta([x \mapsto a]) = T$.

Therefore, $(\forall x \Box q \rightarrow \Box \forall x q)_\alpha([x \mapsto a]) = F$, whence $\alpha \not\models \forall x \Box q \rightarrow \Box \forall x q$.

Consequence 4. Formulas $\forall x \Diamond \Phi \rightarrow \Diamond \forall x \Phi$ and $\Diamond \exists x \Phi \rightarrow \exists x \Diamond \Phi$ are refutable in a GMS of equitone

predicates.

Characteristic features of equality predicates in TML. In the classical logic of predicates, formulas of the form $x=y$ are refutable under ≥ 2 -element interpretations, but they are irrefutable under 1-element interpretations. However, this is not the case for formulas \equiv_{xy} of logic of quasiary predicates, they are also refutable under 1-element interpretations. Indeed, let us take an 1-element interpretation $A = (\{a\}, I)$ and $d = [x \mapsto a]$. Then $d(x) = a$ and $d(y) \uparrow$, therefore $\equiv_{xy}([x \mapsto a]) = F$.

For TML, this gives the following result.

Statement 1. Formulas \equiv_{xy} and $\Box \equiv_{xy}$ can be refuted on GMS with 1-element states.

At the same time, formulas \equiv_{xx} are always interpreted as a constant predicate T. Hence

Statement 2. $M \models \equiv_{xx}$ and $M \models \Box \equiv_{xx}$ for each GMS M .

Note that $\Box \equiv_{xx}$ is not always interpreted as a constant predicate T!

Example 4. Let us consider GMS with $St = \{\alpha, \beta\}$ and $R = \{\alpha \triangleright \beta\}$. Since there is no state η such that $\beta \triangleright \eta$, then $(\Box \equiv_{xx})_\beta(d) \uparrow$ for each d .

Example 5. Formulas $\equiv_{xy} \rightarrow \Box \equiv_{xy}$ and $\Box \equiv_{xy} \rightarrow \equiv_{xy}$ are refuted in the specified above GMS.

Let $St = \{\alpha, \beta\}$, $R = \{\alpha \triangleright \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b\}$.

Let us take $d = [x \mapsto b, z \mapsto a] \Rightarrow d_\alpha = [z \mapsto a]$, $d_\beta = [x \mapsto b]$. Hence $d_\alpha(x) \uparrow$, $d_\alpha(y) \uparrow$, $d_\beta(x) = b$, $d_\beta(y) \uparrow$.

We have $(\equiv_{xy})_\alpha(d) = (\equiv_{xy})_\alpha(d_\alpha) = T$.

At the same time, $(\equiv_{xy})_\beta(d) = (\equiv_{xy})_\beta(d_\beta) = F \Rightarrow (\Box \equiv_{xy})_\alpha(d) = (\Box \equiv_{xy})_\alpha(d_\alpha) = F$.

Therefore, $(\equiv_{xy} \rightarrow \Box \equiv_{xy})_\alpha(d) = F \Rightarrow \alpha \not\models \equiv_{xy} \rightarrow \Box \equiv_{xy}$.

Let us take $h = [x \mapsto a, z \mapsto b] \Rightarrow h_\alpha = [x \mapsto a]$, $h_\beta = [z \mapsto b]$. Hence $h_\alpha(x) = a$, $h_\alpha(y) \uparrow$, $h_\beta(x) \uparrow$, $h_\beta(y) \uparrow$.

We have $(\equiv_{xy})_\alpha(h) = (\equiv_{xy})_\alpha(h_\alpha) = F$.

At the same time, $(\equiv_{xy})_\beta(h) = (\equiv_{xy})_\beta(h_\beta) = T \Rightarrow (\Box \equiv_{xy})_\alpha(h) = (\Box \equiv_{xy})_\alpha(h_\alpha) = T$.

Therefore, $(\Box \equiv_{xy} \rightarrow \equiv_{xy})_\alpha(h) = F \Rightarrow \alpha \not\models \Box \equiv_{xy} \rightarrow \equiv_{xy}$.

5. Logical consequence relations in TML

We will define logical consequence relations in TML on a set of formulas specified with names of states (or simply, specified with states).

Let us denote a formula specified with a state by Φ^α , where Φ is a formula of the language, and $\alpha \in S$ is its specification (S is a set of names of states of the world). Hence, the specification indicates the state of the world on which the formula is considered.

To eliminate quantifiers in the logics of non-monotonic predicates, variable assignment predicates $z \downarrow$ will be used. Therefore, further on we will consider *extended* sets of formulas specified with states, in which special formulas of the form $z \downarrow^\alpha$ can appear: although the set of language formulas is expanded with predicate symbols $z \downarrow$, formulas $z \downarrow$ are not used for construction of more complex formulas. In fact it is possible to consider $z \downarrow$ as full-fledged atomic formulas, using them for construction of complex formulas, however this does not actually change semantic properties of the language.

Let Σ be a set of formulas specified with states with a specifications' set S . We will call a set Σ *consistent* with a TMS $M = (St, R, A, Im)$, provided that an injection of S into St is defined.

On sets of formulas specified with states, we will introduce the relations of irrefutability, truth, falsity and strong (denoted by IR , T , F and TF respectively) logical consequence. Such relations are specified similarly to the corresponding relations in the traditional logics of quasiary predicates (see [10]).

Let Γ and Δ be sets of formulas specified with states.

Further on, when writing $\Gamma \models_M \Delta$, we will assume by default that Γ and Δ are consistent with the TMS M .

Δ is an IR -consequence of Γ in a consistent with them TMS M (denoted $\Gamma \models_{M|IR} \Delta$), if for all $d \in {}^V A$:

$$\Phi_\alpha(d) = T \text{ for all } \Phi^\alpha \in \Gamma \Rightarrow \Psi_\beta(d) \neq F \text{ for some } \Psi^\beta \in \Delta.$$

Δ is a *logical IR-consequence* of Γ with respect to a TMS of a type \mathcal{M} if $\Gamma_M \models_{IR} \Delta$ for all TMS $M \in \mathcal{M}$. We will denote this by $\Gamma \overset{\mathcal{M}}{\models}_{IR} \Delta$, or $\Gamma \models_{IR} \Delta$, if \mathcal{M} is implied.

Therefore, $\Gamma \not\models_{IR} \Delta \Leftrightarrow$ a TMS M consistent with Γ and Δ and $d \in {}^V A$ exist such that:

$$\Phi_\alpha(d) = T \text{ for all } \Phi^\alpha \in \Gamma \text{ and } \Psi_\beta(d) = F \text{ for all } \Psi^\beta \in \Delta.$$

Δ is a *T-consequence* of Γ in a consistent with them TMS M (denoted $\Gamma_M \models_T \Delta$), if for all $d \in {}^V A$:

$$\Phi_\alpha(d) = T \text{ for all } \Phi^\alpha \in \Gamma \Rightarrow \Psi_\beta(d) = T \text{ for some } \Psi^\beta \in \Delta.$$

Δ is a *logical T-consequence* of Γ with respect to a TMS of a type \mathcal{M} if $\Gamma_M \models_T \Delta$ for all TMS $M \in \mathcal{M}$. $\Gamma \overset{\mathcal{M}}{\models}_T \Delta$, or $\Gamma \models_T \Delta$, if \mathcal{M} is implied.

Therefore, $\Gamma \not\models_T \Delta \Leftrightarrow$ a TMS M consistent with Γ and Δ and $d \in {}^V A$ exist such that:

$$\text{for all } \Phi^\alpha \in \Gamma \text{ we have } \Phi_\alpha(d) = T \text{ and for all } \Psi^\beta \in \Delta \text{ we have } \Psi_\beta(d) \neq T.$$

Δ is a *F-consequence* of Γ in a consistent with them TMS M (denoted $\Gamma_M \models_F \Delta$), if for all $d \in {}^V A$:

$$\Psi_\beta(d) = F \text{ for all } \Psi^\beta \in \Delta \Rightarrow \Phi_\alpha(d) = F \text{ for some } \Phi^\alpha \in \Gamma.$$

Δ is a *logical F-consequence* of Γ with respect to a TMS of a type \mathcal{M} if $\Gamma_M \models_F \Delta$ for all TMS $M \in \mathcal{M}$. We will denote this by $\Gamma \overset{\mathcal{M}}{\models}_F \Delta$, or $\Gamma \models_F \Delta$, if \mathcal{M} is implied.

Therefore, $\Gamma \not\models_F \Delta \Leftrightarrow$ a TMS M consistent with Γ and Δ and $d \in {}^V A$ exist such that:

$$\text{for all } \Psi^\beta \in \Delta \text{ we have } \Psi_\beta(d) = F \text{ and for all } \Phi^\alpha \in \Gamma \text{ we have } \Phi_\alpha(d) \neq F.$$

Δ is a *TF-consequence* of Γ in a consistent with them TMS M (denoted $\Gamma_M \models_{TF} \Delta$), if:

$$\Gamma_M \models_T \Delta \text{ and } \Gamma_M \models_F \Delta.$$

Therefore, $\Gamma_M \not\models_{TF} \Delta \Leftrightarrow \Gamma_M \not\models_T \Delta$ or $\Gamma_M \not\models_F \Delta$.

Δ is a *logical TF-consequence* of Γ with respect to a TMS of a type \mathcal{M} if $\Gamma_M \models_{TF} \Delta$ for all TMS $M \in \mathcal{M}$. This will be denoted by $\Gamma \overset{\mathcal{M}}{\models}_{TF} \Delta$, or $\Gamma \models_{TF} \Delta$, if \mathcal{M} is implied.

It is obvious that $\Gamma \models_{TF} \Delta \Leftrightarrow \Gamma \models_T \Delta$ and $\Gamma \models_F \Delta$.

Therefore, $\Gamma \not\models_{TF} \Delta \Leftrightarrow$ a TMS M consistent with Γ and Δ exists such that:

$$\Gamma_M \not\models_T \Delta \text{ or } \Gamma_M \not\models_F \Delta.$$

The non-modal properties of the specified relations repeat the corresponding properties of the same-named relations for sets of formulas of the traditional logic of quasiary predicates.

These properties are:

- decompositions of formulas $\neg\neg_L, \neg\neg_R, \vee_L, \vee_R, \neg\vee_L, \neg\vee_R$, and also \neg_L, \neg_R ;
- properties that guarantee the corresponding consequence relation;
- simplifications and equivalent transformations induced by properties of predicates R, RI, RU, R \neg , R \vee , RR, R \uparrow . Each such property R* (except R \uparrow) produces 4 corresponding properties R*_L, R*_R, \neg R*_L, \neg R*_R for the logical consequence relation when the selected formula or its negation is on the left or right side of this relation, and R \uparrow produces 8 properties;
- properties related to the quantifier elimination;
- properties related to equality predicates.

As an example, let us give the properties produced by R \uparrow :

$$R\uparrow_{1L}) R_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, \Gamma_M \models^* \Delta, z \downarrow^\alpha \Leftrightarrow R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi), \Gamma_M \models^* \Delta, z \downarrow^\alpha;$$

$$\neg R\uparrow_{1L}) \neg R_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, \Gamma_M \models^* \Delta, z \downarrow^\alpha \Leftrightarrow \neg R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi), \Gamma_M \models^* \Delta, z \downarrow^\alpha;$$

$$R\uparrow_{1R}) \Gamma_M \models^* \Delta, R_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, z \downarrow^\alpha \Leftrightarrow \Gamma_M \models^* \Delta, R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, z \downarrow^\alpha;$$

$$\neg R\uparrow_{1R}) \Gamma_M \models^* \Delta, \neg R_{\bar{x}, \perp, z}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, z \downarrow^\alpha \Leftrightarrow \Gamma_M \models^* \Delta, \neg R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, z \downarrow^\alpha;$$

$$R\uparrow_{2L}) R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, \Gamma_M \models^* \Delta, y \downarrow^\alpha \Leftrightarrow R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi)^\alpha, \Gamma_M \models^* \Delta, y \downarrow^\alpha;$$

$$\neg R\uparrow_{2L}) \neg R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, \Gamma_M \models^* \Delta, y \downarrow^\alpha \Leftrightarrow \neg R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi)^\alpha, \Gamma_M \models^* \Delta, y \downarrow^\alpha;$$

$$R\uparrow_{2R} \Gamma_M \models^* \Delta, R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, y \downarrow^\alpha \Leftrightarrow \Gamma_M \models^* \Delta, R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi)^\alpha, y \downarrow^\alpha;$$

$$\neg R\uparrow_{2R} \Gamma_M \models^* \Delta, \neg R_{\bar{x}, \perp, \perp}^{\bar{v}, \bar{u}, y}(\Phi)^\alpha, y \downarrow^\alpha \Leftrightarrow \Gamma_M \models^* \Delta, \neg R_{\bar{x}, \perp}^{\bar{v}, \bar{u}}(\Phi)^\alpha, y \downarrow^\alpha.$$

Equality properties. Let us consider properties related to equality predicates.

Predicates \equiv_{xy} are total single-valued, so \neg can be removed similarly to \neg_L and \neg_R :

$$\neg_{\equiv L} \neg \equiv_{xy}^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \equiv_{xy}^\alpha, \Delta;$$

$$\neg_{\equiv R} \Gamma_M \models^* \neg \equiv_{xy}^\alpha, \Delta \Leftrightarrow \equiv_{xy}^\alpha, \Gamma_M \models^* \Delta;$$

$$\neg_{R \equiv L} \neg R_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xy})^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* R_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xy})^\alpha, \Delta;$$

$$\neg_{R \equiv R} \Gamma_M \models^* \neg R_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xy})^\alpha, \Delta \Leftrightarrow R_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xy})^\alpha, \Gamma_M \models^* \Delta.$$

Transitivity of equality:

$$Tr \equiv \equiv_{xy}^\alpha, \equiv_{yz}^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \equiv_{xz}^\alpha, \Gamma_M \models^* \Delta.$$

Basing on $R \equiv_{xx}$, $R \equiv_0$, $R \equiv_1$, $R \equiv_2$, $R \equiv \uparrow_1$, $R \equiv \uparrow_2$, we obtain the corresponding simplification properties; let us give an example of the properties induced by $R \equiv \uparrow_1$ and $R \equiv \uparrow_2$:

$$R \equiv \uparrow_{1L} \text{ given } y \notin \{\bar{u}, \bar{v}\} \text{ and } x \neq y \text{ then } R_{\bar{w}, \perp, \perp}^{\bar{v}, \bar{u}, x}(\equiv_{xy})^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \Delta, y \downarrow^\alpha;$$

$$R \equiv \uparrow_{1R} \text{ given } y \notin \{\bar{u}, \bar{v}\}, x \neq y \text{ then } \Gamma_M \models^* \Delta, R_{\bar{w}, \perp, \perp}^{\bar{v}, \bar{u}, x}(\equiv_{xy})^\alpha \Leftrightarrow y \downarrow^\alpha, \Gamma_M \models^* \Delta;$$

$$R \equiv \uparrow_{2L} R_{\bar{w}, \perp, z, \perp}^{\bar{v}, \bar{u}, x, y}(\equiv_{xy})^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \Delta, z \downarrow^\alpha;$$

$$R \equiv \uparrow_{2R} \Gamma_M \models^* \Delta, R_{\bar{w}, \perp, z, \perp}^{\bar{v}, \bar{u}, x, y}(\equiv_{xy})^\alpha \Leftrightarrow z \downarrow^\alpha, \Gamma_M \models^* \Delta.$$

Basing on $R \equiv_{xx}$ and $R \equiv$, we obtain the elimination properties for the constant T-formula:

$$El \equiv_{xx} \equiv_{xx}^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \Delta;$$

$$ElR \equiv_{xx} R_{\bar{w}, \perp, \perp}^{\bar{v}, \bar{u}, x}(\equiv_{xx})^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \Delta;$$

$$ElR \equiv R_{\bar{w}, \perp, \perp, \perp}^{\bar{v}, \bar{u}, x, y}(\equiv_{xy})^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \Gamma_M \models^* \Delta.$$

Property \equiv_{rp} induces the following properties of substitution of equals:

$$\equiv_{rpL} \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, R_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Gamma_M \models^* \Delta;$$

$$\equiv_{rpR} \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \Gamma_M \models^* R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Delta \Leftrightarrow \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \Gamma_M \models^* R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, R_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Delta;$$

$$\equiv_{\neg rpL} \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \neg R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Gamma_M \models^* \Delta \Leftrightarrow \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \neg R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \neg R_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Gamma_M \models^* \Delta;$$

$$\equiv_{\neg rpR} \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \Gamma_M \models^* \neg R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Delta \Leftrightarrow \equiv_{xy}^\alpha, x \downarrow^\alpha, y \downarrow^\alpha, \Gamma_M \models^* \neg R_{\bar{w}, \perp, x}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \neg R_{\bar{w}, \perp, y}^{\bar{v}, \bar{u}, z}(\Phi)^\alpha, \Delta.$$

Basing on Rf , $R \equiv_{xx}$ and $R \equiv_\Gamma$, we obtain the properties which guarantee each of the consequence relations:

$$C_{Rf} \Gamma_M \models^* \Delta, \equiv_{xx}^\alpha;$$

$$C_{R \equiv_{xx}} \Gamma_M \models^* \Delta, R_{\bar{w}, \perp}^{\bar{v}, \bar{u}}(\equiv_{xx})^\alpha;$$

$$C_{R \equiv} \Gamma_M \models^* \Delta, R_{\bar{x}, \perp, \perp, \perp}^{\bar{v}, \bar{u}, x, z}(\equiv_{xz})^\alpha.$$

The property related to $z \downarrow$ and \equiv_{xy} which guarantees each of the consequence relations:

$$C \downarrow_{\equiv} \equiv_{xy}^\alpha, x \downarrow^\alpha, \Gamma_M \models^* y \downarrow^\alpha, \Delta.$$

Properties related to modal compositions. For TMS of non-monotonic predicates with equality, such properties are generally analogous to the corresponding properties of TMS without equality predicates (see, e.g., [9]). These are properties of carrying modalities over renominations and properties of modality elimination.

The properties of carrying modalities over renominations belong to the properties of equivalent transformations based on Theorem 1. In a general case, there are 4 such properties. We present these properties for the composition \square in GMS, for TmMS and MMS they are formulated similarly:

$$\begin{aligned} \mathbf{R}\square_{\text{L}}) \Gamma, R_{x,\perp}^{\bar{v},\bar{u}}(\square\Phi)^\alpha \vDash_M \Delta &\Leftrightarrow \Gamma, \square R_{x,\perp}^{\bar{v},\bar{u}}(\Phi)^\alpha \vDash_M \Delta; \\ \neg\mathbf{R}\square_{\text{L}}) \Gamma, \neg R_{x,\perp}^{\bar{v},\bar{u}}(\square\Phi)^\alpha \vDash_M \Delta &\Leftrightarrow \Gamma, \neg\square R_{x,\perp}^{\bar{v},\bar{u}}(\Phi)^\alpha \vDash_M \Delta; \\ \mathbf{R}\square_{\text{R}}) \Gamma \vDash_M \Delta, R_{x,\perp}^{\bar{v},\bar{u}}(\square\Phi)^\alpha &\Leftrightarrow \Gamma \vDash_M \Delta, \square R_{x,\perp}^{\bar{v},\bar{u}}(\Phi)^\alpha; \\ \neg\mathbf{R}\square_{\text{R}}) \Gamma \vDash_M \Delta, \neg R_{x,\perp}^{\bar{v},\bar{u}}(\square\Phi)^\alpha &\Leftrightarrow \Gamma \vDash_M \Delta, \neg\square R_{x,\perp}^{\bar{v},\bar{u}}(\Phi)^\alpha. \end{aligned}$$

The elimination properties for the composition \square in GMS have the following form:

$$\begin{aligned} \square_{\text{EL}_L}) \square\Phi^\alpha, \Gamma \vDash_M \Delta &\Leftrightarrow \{\Phi^\beta \mid \alpha \triangleright \beta\} \cup \Gamma \vDash_M \Delta; \\ \neg\square_{\text{EL}_R}) \Gamma \vDash_M \Delta, \neg\square\Phi^\alpha &\Leftrightarrow \Gamma \vDash_M \Delta \cup \{\neg\Phi^\beta \mid \alpha \triangleright \beta\}; \\ \square_{\text{EL}_R}) \Gamma \vDash_M \Delta, \square\Phi^\alpha &\Leftrightarrow \Gamma \vDash_M \Delta, \Phi^\beta \text{ for all states } \beta \in S \text{ such that } \alpha \triangleright \beta; \\ \neg\square_{\text{EL}_L}) \neg\square\Phi^\alpha, \Gamma \vDash_M \Delta &\Leftrightarrow \neg\Phi^\beta, \Gamma \vDash_M \Delta \text{ for all states } \beta \in S \text{ such that } \alpha \triangleright \beta. \end{aligned}$$

The properties of \square_{EL_L} and \square_{EL_R} are sufficient for the relation \vDash_{IR} .

The elimination properties for modalities can be similarly formulated for TmMS and MMS.

In the case of additional conditions imposed on the transition relation, the elimination properties for modalities will be modified accordingly.

For example, for the relation \vDash_{IR} in GMS, let us consider the cases when \triangleright can be transitive, reflexive or symmetric.

1. \triangleright reflexive. We have $\alpha \triangleright \alpha$, therefore, for \square_{EL_L} an additional condition $\Phi^\alpha \in \{\Phi^\beta \mid \alpha \triangleright \beta\}$.

2. \triangleright symmetric. We have $\alpha \triangleright \beta \Leftrightarrow \beta \triangleright \alpha$, therefore \square_{EL_L} and \square_{EL_R} are specified as follows:

$$s\square_{\text{EL}_L}) \square\Phi^\alpha, \Gamma \vDash_{IR} \Delta \Leftrightarrow \{\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\} \cup \Gamma \vDash_{IR} \Delta.$$

$$s\square_{\text{EL}_R}) \Gamma \vDash_{IR} \Delta, \square\Phi^\alpha \Leftrightarrow \Gamma \vDash_{IR} \Delta, \Phi^\beta \text{ for all states } \beta \in S \text{ such that } \alpha \triangleright \beta \text{ and } \beta \triangleright \alpha.$$

3. \triangleright reflexive and symmetric. We have $s\square_{\text{EL}_R}$ and since $\alpha \triangleright \alpha$ we obtain

$$rs\square_{\text{EL}_L}) \square\Phi^\alpha, \Gamma \vDash_{IR} \Delta \Leftrightarrow \{\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\} \cup \Gamma \vDash_{IR} \Delta \text{ and } \Phi^\alpha \in \{\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\}.$$

4. \triangleright transitive. We have \square_{EL_R} and

$$t\square_{\text{EL}_L}) \square\Phi^\alpha, \Gamma \vDash_{IR} \Delta \Leftrightarrow \{\Phi^\beta \mid \alpha \triangleright \beta\} \cup \{\square\Phi^\beta \mid \alpha \triangleright \beta\} \cup \Gamma \vDash_{IR} \Delta.$$

Here $\{\square\Phi^\beta \mid \alpha \triangleright \beta\} \cup \Gamma$ is necessary due to the transitivity of the relation \triangleright .

5. \triangleright transitive and reflexive. We have \square_{EL_R} and $t\square_{\text{EL}_L}$, and since $\alpha \triangleright \alpha$ for $t\square_{\text{EL}_L}$ we obtain the condition $\Phi^\alpha \in \{\Phi^\beta \mid \alpha \triangleright \beta\}$.

6. \triangleright transitive and symmetrical. We have $s\square_{\text{EL}_R}$ and

$$ts\square_{\text{EL}_L}) \square\Phi^\alpha, \Gamma \vDash_{IR} \Delta \Leftrightarrow \{\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\} \cup \{\square\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\} \cup \Gamma \vDash_{IR} \Delta.$$

7. \triangleright transitive, reflexive and symmetric. We have $s\square_{\text{EL}_R}$ and $ts\square_{\text{EL}_L}$, and since $\alpha \triangleright \alpha$ for $ts\square_{\text{EL}_L}$ we obtain the condition $\Phi^\alpha \in \{\Phi^\beta \mid \alpha \triangleright \beta \text{ or } \beta \triangleright \alpha\}$.

The above properties of relations \vDash_{IR} for sets of formulas in a *given* TMS \mathbf{M} induce analogous properties of the corresponding *logical* consequence relations \vDash^* .

Properties of logical consequence relations for sets of formulas specified with states are the semantic basis for constructing the corresponding sequent calculi.

6. Conclusion

In this paper we propose new classes of program-oriented logical formalisms – transitional modal logics of partial non-monotonic quasiary predicates with equality and extended renomination. We

describe the semantic models and languages of pure first-order TMLs with strong equality predicates, the properties of equality predicates are given. The following classes of transitional modal systems are specified: general TMS, temporal TMS, multimodal TMS. The languages of such systems are described, the semantic properties, in particular, properties related to equality predicates, are considered. The interaction of modal compositions with renominations and quantifiers is investigated. We define relations of irrefutability, truth, falsity and strong logical consequence on sets of formulas specified with states, and present their properties. These properties are the semantic basis for further constructing the sequent type calculi for the proposed logics.

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