Adomian Decomposition Method in The Theory of Nonlinear Periodic Boundary Value Problems with Delay

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Abstract

Among numerous studies of functional-differential equations, research on periodic boundary value problems for differential equations with concentrated delay holds a special place. This is primarily due to the wide application of periodic boundary value problems for differential equations with concentrated delay in physics, economics [3], biology [4], and mechanics [5]. By applying the Adomian decomposition method, we have derived the necessary and sufficient conditions for the existence of solutions to the weakly nonlinear periodic boundary value problem for a system of differential equations with concentrated delay in the critical case.

Keywords [1]

Functional-differential equations, differential equations with concentrated delay, periodic boundary value problems, weakly nonlinear boundary value problems, Adomian decomposition method.

1. Introduction

We studied the problem of constructing approximations to the T-periodic solution [1, 2]

$$z(t,\varepsilon):\ z(\cdot,\varepsilon)\ \in C^1[0,T], z(t,\cdot)\ \in\ C[0,\varepsilon_0]$$

of a system of differential equations with concentrated delay

$$dz(t,\varepsilon)/dt = A(t)z(t,\varepsilon) + B(t)z(t-\Delta,\varepsilon) + f(t) + \varepsilon\, Z(z(t,\varepsilon), z(t\ -\Delta,\varepsilon), t, \varepsilon). \qquad (1)$$

The solution of the periodic problem for equation (1) is sought in a small neighborhood of the T-periodic solution

$$z_0(\mathrm{t}) \in \mathbb{C}^1[0,T]$$

of the generating system

$$dz_0/dt\ =\ A(t)z_0(t)\ +\ B(t)z_0(t\ -\ \Delta)\ +\ f(t),\ \Delta \in \mathbb{R}^1. \quad (2)$$

Where $A(t)$, $B(t)$ are continuous T-periodic $(n \times n)$-matrices, $f(t)$ is continuous T-periodic vector-function, $Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon)$ is nonlinear vector function, which is analytic in a small neighborhood of the generating problem (2), continuous and T-periodic with the respect to the variable t, and also analytic with respect to the small parameter ε on the interval $[0,\varepsilon_0]$. As is known, in the critical case [2], specifically, in the presence of T-periodic solutions

Dynamical System Modeling and Stability Investigation (DSMSI-2023), December 19-21, 2023, Kyiv, Ukraine
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$$z_0(t, c_r) = X_r(t)c_r, \ \ c_r \in \mathbb{R}^r$$

of the homogeneous part

$$dz_0/dt = A(t)z_0(t) + B(t)z_0(t - \Delta) \qquad (3)$$

of system (2), and in the case of constant matrices $A(t) \equiv A$ and $B(t) \equiv B$, with the presence of purely imaginary roots

$$\lambda_j = \pm ik_j T, \quad i = \sqrt{-1}, \ \ j \in \mathbb{N}$$

of the characteristic equation

$$\det\left[A + Be^{-\lambda\Delta} - \lambda I_n\right] = 0,$$

the generating periodic problem for equation (2) is solvable not for all vector functions $f(t)$. In the critical case, the adjoint system

$$dy(t)/dt = -A^*(t)y(t) - B^*(t)y(t + \Delta)$$

has a family of T-periodic solutions of the form

$$y(t, c_r) = H_r(t)c_r, \ \ c_r \in \mathbb{R}^r.$$

Periodic problem for the equation (2) is solvable iff

$$\int_0^T H_r^*(s)f(s)ds = 0. \qquad (4)$$

Here $H_r(t)$ is (n × r)- matrix formed by r linearly independent T-periodic solutions of the adjoint system. Let us assume condition (4) is satisfied; in this case, the general solution of the generating T-periodic problem for equation (2) has the form

$$z_0(t, c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r,$$

where $G[f(s)](t)$ is a particular solution of the generating T-periodic problem for equation (2), $X_r(t)$ is (n × r)- matrix formed by r linearly independent T-periodic solutions of the system (2). To construct a particular solution $G[f(s)](t)$ of the generating T-periodic problem for equation (2), provided its solvability, the method of least squares [6] is applicable.

2. The necessary and sufficient conditions for solvability

Similarly to [2], we obtain the necessary condition for the solvability of the T-periodic problem for equation (2).

Lemma. *Let us assume that for the generating periodic problem for equation (2), a critical case occurs, and the solvability condition (4) is satisfied. In this case, the periodic problem for equation (2) has a family of T-periodic solutions in the form*

$$z_0(t, c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r.$$

Let us also assume that the T-periodic problem for equation (1) has a T-periodic solution

$$z(t, \varepsilon): \ z(\cdot, \varepsilon) \in \mathbb{C}^1[0, T], \ \ z(t, \cdot) \in \mathbb{C}[0, \varepsilon_0],$$

which, at $\varepsilon = 0$, transforms unto the generating solution

$$z(t, 0) = z_0(t, c_r^*).$$

Under these conditions, the vector $c_r^* \in \mathbb{R}^r$ satisfies the equation for the generating amplitudes

$$F(c_r^*) := \int_0^T H_r^*(s)Z(z_0(s, c_r^*), z_0(s - \Delta, c_r^*), s, \varepsilon)ds = 0. \qquad (5)$$

We will refer to equation (5) as the equation for the generating amplitudes of the nonlinear periodic boundary value problem for equation (1). The roots $c_r^* \in \mathbb{R}^r$ of equation (5) determine the generating solutions $z_0(t, c_r^*)$, in a neighborhood of which the sought solutions to the original nonlinear T-periodic boundary value problem for equation (1) may exist. However, if equation (5) has no real solutions for $c_r^* \in \mathbb{R}^r$, then the original nonlinear T-periodic boundary value problem for equation (1) has no sought-after solutions.

Let us denote an (r × r)- matrix

$$B_0 := \int_0^T H_r^*(s)\big[A_{1,0}(s)X_r(s) + A_{0,1}(s)X_r(s-\Delta)\big]ds;$$

here

$$A_{1,0}(t) = \left.\frac{\partial Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon)}{\partial z(t,\varepsilon)}\right|_{\substack{z(t,\varepsilon) = z_0(t, c_r^*) \\ z(t-\Delta,\varepsilon) = z_0(t-\Delta, c_r^*)}}$$

and

$$A_{0,1}(t) = \left.\frac{\partial Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon)}{\partial z(t-\Delta,\varepsilon)}\right|_{\substack{z(t,\varepsilon) = z_0(t, c_r^*) \\ z(t-\Delta,\varepsilon) = z_0(t-\Delta, c_r^*)}}$$

Are (n × n)- matrices. The traditional solvability condition for the nonlinear periodic boundary value problem for equation (1) in a small neighborhood of the generating solution $z_0(t, c_r^*)$ is the requirement for the simplicity of the roots [2, 6]

$$\det B_0 \neq 0$$

of equation (5) for the generating amplitudes. We will demonstrate that the requirement for the simplicity of the roots of equation (5) for the generating amplitudes is a sufficient condition for the solvability of the nonlinear periodic boundary value problem for equation (1) in a small neighborhood of the generating solution

$$z_0(t, c_r^*) = X_r c_r^* + G[f(s)](t), \quad c_r^* \in \mathbb{R}^r.$$

In the article [6], we found constructive necessary and sufficient conditions for solvability, along with a scheme for constructing solutions of the nonlinear T-periodic boundary value problem for equation (1). Based on the method of simple iterations, we developed a convergent iterative scheme to find approximations to the solutions of this problem. However, in the process of constructing solutions to the nonlinear T-periodic boundary value problem for equation (1) using the least squares method, the issue of impossibility of finding solutions in terms of elementary functions arises, which, in turn, leads to significant errors in solving nonlinear boundary value problems.

Furthermore, the construction of solutions for nonlinear boundary value problems using the method of simple iterations [2] and the least squares method is significantly complicated by the computation of derivatives of nonlinearities. Given this, simplifying the computation of nonlinear derivatives and the potential to find solutions for nonlinear boundary value problems, including periodic boundary value problems, in elementary functions can be achieved using the Adomian decomposition method [7, 8]. Additionally, the use of the Adomian decomposition method significantly simplifies the proof of convergence of iterative schemes for constructing solutions to nonlinear boundary value problems. An example of such simplification will be provided below. Thus, the purpose of this article is to find constructive solvability conditions and a scheme for constructing solutions to the nonlinear T-periodic boundary value problem for equation (1) using the Adomian decomposition method.

Fixing one of the solutions of equation (5), we approach the problem of finding analytical solutions for the nonlinear T-periodic boundary value problem for equation (1) in a small neighborhood of the generating solution $z_0(t, c_r^*)$. We seek the solution of the nonlinear T-periodic boundary value problem for equation (1) in the critical case in the form

$$z(t,\varepsilon) := z_0(t, c_r) + u_1(t,\varepsilon) + \ \dots + u_k(t,\varepsilon) + \ \dots .$$

The nonlinear vector function $Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon)$ is analytic in a small neighborhood of the generating solution of problem (2); therefore, in the specified neighborhood, occurs an expansion in the form [7, p. 502]

$$\begin{aligned} Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon) &\\ &= Z(z_0(t,c_r^*), z_0(t-\Delta,c_r^*), t, 0) \\ &+ Z_1(z_0(t,c_r^*), u_1(t,\varepsilon), z_0(t-\Delta,c_r^*), u_1(t-\Delta,\varepsilon), t, \varepsilon) + \ \dots + \end{aligned}$$
$$+ Z_k(z_0(t,c_r^*), u_1(t,\varepsilon), \dots, u_k(t,\varepsilon), z_0(t-\Delta,c_r^*), u_1(t-\Delta,\varepsilon), \dots, u_k(t-\Delta,\varepsilon), t, \varepsilon) + \ \dots .$$

First approximation to solution of nonlinear T- periodic boundary value problem for equation (1) in critical case

$$z_1(t,\varepsilon) \coloneqq z_0(t,c_r^*) + u_1(t,\varepsilon), \ \ u_1(t,\varepsilon) = X_r(t)c_1(\varepsilon) + (t,\varepsilon), \ \ c_1(\varepsilon) \in \mathbb{R}^r$$

determines the solution of the T- periodic boundary value problem for equation

$$\frac{du_1(t,\varepsilon)}{dt} = A(t)u_1(t,\varepsilon) + B(t)u_1(t-\Delta,\varepsilon) + \varepsilon Z(z_0(t,c_r^*), z_0(t-\Delta,c_r^*), t, 0);$$

Here

$$u_1^{(1)}(t,\varepsilon) = \varepsilon G[Z(z_0(s,c_r^*), z_0(s-\Delta,c_r^*), s, 0)](t)$$

is a particular solution of this problem. The solvability of the T-periodic boundary value problem in the first approximation is guaranteed by choosing the root c_r^* of equation (5) for the generating amplitudes of the nonlinear periodic boundary value problem for equation (1). The second approximation to the solution of the nonlinear T-periodic boundary value problem for equation (1) in the critical case

$$z_2(t,\varepsilon) \coloneqq z_0(t,c_r^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \ \ u_2(t,\varepsilon) = X_r(t)c_2(\varepsilon) + u_2^{(1)}(t,\varepsilon), \ \ c_2(\varepsilon) \in \mathbb{R}^r$$

determines the solution of the T-periodic boundary value problem for the equation

$$\frac{du_2(t,\varepsilon)}{dt} = A(t)u_2(t,\varepsilon) + B(t)u_2(t-\Delta,\varepsilon) +$$
$$+\varepsilon Z_1(z_0(t,c_r^*), u_1(t,\varepsilon), z_0(t-\Delta,c_r^*), u_1(t-\Delta,\varepsilon), t, \varepsilon);$$

Here

$$u_2^{(1)}(t,\varepsilon) = \varepsilon G[Z_1(z_0(s,c_r^*), u_1(s,\varepsilon), z_0(s-\Delta,c_r^*), u_1(s-\Delta,\varepsilon), s, \varepsilon)](t)$$

is a particular solution of this problem. The solvability of the T-periodic boundary value problem in the second approximation guarantees the solvability of the equation

$$F_1\big(c_1(\varepsilon)\big) \coloneqq \int_0^T H_r^*(s) Z_1(z_0(s,c_r^*), u_1(s,\varepsilon), z_0(s-\Delta,c_r^*), u_1(s-\Delta,\varepsilon), s, \varepsilon)ds = 0.$$

Unlike the equation for the generating amplitudes, the last equation is linear:

$$F_1\big(c_1(\varepsilon)\big) = B_0 c_1(\varepsilon) + \delta_1(c_r^*, \varepsilon) = 0,$$

and also solvable, provided the roots of the equation for the generating amplitudes are simple; here

$$B_0 = F_1'\big(c_1(\varepsilon)\big) \in \mathbb{R}^{r\times r}, \ \ \delta_1(c_r^*) \coloneqq F_1\big(c_1(\varepsilon)\big) - B_0 c_1(\varepsilon).$$

In order to prove this, let us denote the vector function [8, 9]

$$v(t,\mu) \coloneqq z_0(t,c_r^*) + \mu u_1(t,\varepsilon) + \ \dots + \mu^k u_k(t,\varepsilon) + \ \dots ;$$

in this case

$$F_1\big(c_1(\varepsilon)\big) \coloneqq \int_0^T H_r^*(s) Z_1(z_0(s,c_r^*), u_1(s,\varepsilon), z_0(s-\Delta,c_r^*), s, 0, u_1(s-\Delta,\varepsilon), \varepsilon)ds \coloneqq$$
$$= \int_0^T H_r^*(s) Z_\mu'\,(v(s,\mu), v(s-\Delta,\mu), s, \varepsilon)ds\Big|_{\mu=0} =$$
$$= \int_0^T H_r^*(s)[A_{1,0}(s)u_1(s,\varepsilon) + A_{0,1}(s)u_1(s-\Delta,\varepsilon)]\,ds,$$

and thus

$$B_0 = F_1'\big(c_1(\varepsilon)\big), \ \ \delta_1(c_r^*,\varepsilon) = \int_0^T H_r^*(s)\left[A_{1,0}(s)u_1^{(1)}(s,\varepsilon) + A_{0,1}(s)u_1^{(1)}(s-\Delta,\varepsilon)\right]ds.$$

Therefore, assuming the simplicity of the roots of the equation for the generating amplitudes (5), we obtain a unique solution to the boundary value problem in the first approximation

$$u_1(t,\varepsilon) = X_r(t)c_1(\varepsilon) + u_1^{(1)}(t,\varepsilon),$$

$$c_1(\varepsilon) = -B_0^{-1}\int_0^T H_r^*(s)\left[A_{1,0}(s)u_1^{(1)}(s,\varepsilon) + A_{0,1}(s)u_1^{(1)}(s-\Delta,\varepsilon)\right]ds.$$

Third approximation to solution of nonlinear T- periodic boundary value problem for equation (1) in critical case

$$z_3(t,\varepsilon) := z_0(t,c_r^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon) + u_3(t,\varepsilon),$$

$$u_3(t,\varepsilon) = X_r(t)c_3(\varepsilon) + u_3^{(1)}(t,\varepsilon), \ \ c_3(\varepsilon) \in \mathbb{R}^r$$

determines the solution of the T-periodic boundary value problem for the equation

$$\frac{du_3(t,\varepsilon)}{dt} = A(t)u_3(t,\varepsilon) + B(t)u_3(t-\Delta,\varepsilon) +$$

$$+\varepsilon Z_2(z_0(t,c_r^*),u_1(t,\varepsilon),u_2(t,\varepsilon),z_0(t-\Delta,c_r^*),u_1(t-\Delta,\varepsilon),u_2(t-\Delta,\varepsilon),t,\varepsilon);$$

Here

$$u_3^{(1)}(t,\varepsilon) =$$

$$= \varepsilon G[Z_2(z_0(s,c_r^*),u_1(s,\varepsilon),u_2(s,\varepsilon),z_0(s-\Delta,c_r^*),u_1(s-\Delta,\varepsilon),u_2(s-\Delta,\varepsilon),s,\varepsilon)](t)$$

is a particular solution of this problem. The solvability of the T-periodic boundary value problem in the second approximation guarantees the solvability of the equation

$$F_2(c_2(\varepsilon),\varepsilon) := \int_0^T H_r^*(s)\,\times$$

$$\times\, Z_2(z_0(s,c_r^*),u_1(s,\varepsilon),u_2(s,\varepsilon),z_0(s-\Delta,c_r^*),u_1(s-\Delta,\varepsilon),u_2(s-\Delta,\varepsilon),s,\varepsilon)ds = 0.$$

Unlike the equation for the generating amplitudes, the last equation is linear:

$$F_2\big(c_2(\varepsilon)\big) = B_0c_2(\varepsilon) + \delta_2(c_r^*,c_1(\varepsilon),\varepsilon) = 0,$$

and also solvable, provided that the roots of the equation for the generating amplitudes are simple. Here

$$B_0 = F_2'\big(c_2(\varepsilon)\big) \in \mathbb{R}^{r\times r}, \ \ \delta_2(c_r^*,c_1(\varepsilon),\varepsilon) := F_2\big(c_2(\varepsilon)\big) - B_0c_2(\varepsilon).$$

Denote $(n \times n)$ – matrices

$$\mathrm{A}_{2,0}(\mathrm{t},\mathrm{u}_1(t,\varepsilon)) :=$$

$$= \frac{\partial}{\partial z(t,\varepsilon)}\left[\frac{\partial \mathrm{Z}(\mathrm{z}(\mathrm{t},\varepsilon),\mathrm{z}(\mathrm{t}-\Delta,\varepsilon),\mathrm{t},\varepsilon)}{\partial \mathrm{z}(\mathrm{t},\varepsilon)}u_1(t,\varepsilon)\right]\Bigg|_{\substack{\mathrm{z}(\mathrm{t},\varepsilon) = \mathrm{z}_0(\mathrm{t},\mathrm{c}_\mathrm{r}^*)\\ \mathrm{z}(\mathrm{t}-\Delta,\varepsilon) = \mathrm{z}_0(\mathrm{t}-\Delta,\mathrm{c}_\mathrm{r}^*)}},$$

$$\mathrm{A}_{1,1}(\mathrm{t},\mathrm{u}_1(t-\Delta,\varepsilon)) :=$$

$$= \frac{\partial}{\partial z(t,\varepsilon)}\left[\frac{\partial \mathrm{Z}(\mathrm{z}(\mathrm{t},\varepsilon),\mathrm{z}(\mathrm{t}-\Delta,\varepsilon),\mathrm{t},\varepsilon)}{\partial \mathrm{z}(\mathrm{t}-\Delta,\varepsilon)}u_1(t-\Delta,\varepsilon)\right]\Bigg|_{\substack{\mathrm{z}(\mathrm{t},\varepsilon) = \mathrm{z}_0(\mathrm{t},\mathrm{c}_\mathrm{r}^*)\\ \mathrm{z}(\mathrm{t}-\Delta,\varepsilon) = \mathrm{z}_0(\mathrm{t}-\Delta,\mathrm{c}_\mathrm{r}^*)}},$$

and

$$\mathrm{A}_{0,2}(\mathrm{t},\mathrm{u}_1(t-\Delta,\varepsilon)) :=$$

$$= \frac{\partial}{\partial z(t-\Delta,\varepsilon)}\left[\frac{\partial \mathrm{Z}(\mathrm{z}(\mathrm{t},\varepsilon),\mathrm{z}(\mathrm{t}-\Delta,\varepsilon),\mathrm{t},\varepsilon)}{\partial \mathrm{z}(\mathrm{t}-\Delta,\varepsilon)}u_1(t-\Delta,\varepsilon)\right]\Bigg|_{\substack{\mathrm{z}(\mathrm{t},\varepsilon) = \mathrm{z}_0(\mathrm{t},\mathrm{c}_\mathrm{r}^*)\\ \mathrm{z}(\mathrm{t}-\Delta,\varepsilon) = \mathrm{z}_0(\mathrm{t}-\Delta,\mathrm{c}_\mathrm{r}^*)}},$$

Indeed,

$$F_2(c_2(\varepsilon),\varepsilon) = \frac{1}{2!}\int_0^T H_r^*(s)Z''_{\mu^2}(v(s,\mu),v(s-\Delta,\mu),s,\varepsilon)ds\Bigg|_{\mu = 0} =$$

$$= \int_0^T H_r^*(s)\big[A_{1,0}(s)u_2(s,\varepsilon) + A_{0,1}(s)u_2(s-\Delta,\varepsilon)\big]ds +$$

$$+\frac{1}{2!}\int_0^T H_r^*(s)[A_{2,0}\big(s,u_1(s,\varepsilon)\big)+2A_{1,1}\big(s,u_1(s,\varepsilon)\big)u_1(s-\Delta,\varepsilon)+$$
$$+A_{0,2}\big(s,u_1(s-\Delta,\varepsilon)\big)u_1(s-\Delta,\varepsilon)]ds,$$

thus

$$B_0=F_2'(c_2(\varepsilon),\varepsilon),$$

furthermore

$$\delta_2(c_r^*,c_1(\varepsilon),\varepsilon)=\int_0^T H_r^*(s)\left[A_{1,0}(s)u_2^{(1)}(s,\varepsilon)+A_{0,1}(s)u_2^{(1)}(s-\Delta,\varepsilon)\right]ds+$$
$$+\frac{1}{2!}\int_0^T H_r^*(s)[A_{2,0}\big(s,u_1(s,\varepsilon)\big)u_1(s,\varepsilon)+2A_{1,1}\big(s,u_1(s,\varepsilon)\big)u_1(s-\Delta,\varepsilon)+$$
$$A_{0,2}\big(s,u_1(s-\Delta,\varepsilon)\big)u_1(s-\Delta,\varepsilon)]ds.$$

Thus, provided that the roots of the equation for the generating amplitudes (5) are simple, we obtain a unique solution to the boundary value problem in the second approximation

$$u_2(t,\varepsilon)=X_r(t)c_2(\varepsilon)+u_2^{(1)}(t,\varepsilon),$$
$$c_2(\varepsilon)=-B_0^{-1}\int_0^T H_r^*(s)\left[A_{1,0}(s)u_2^{(1)}(s,\varepsilon)+A_{0,1}(s)u_2^{(1)}(s-\Delta,\varepsilon)\right]ds-$$
$$-\frac{1}{2!}B_0^{-1}\int_0^T H_r^*(s)[A_{2,0}\big(s,u_1(s,\varepsilon)\big)u_1(s,\varepsilon)+2A_{1,1}\big(s,u_1(s,\varepsilon)\big)u_1(s-\Delta,\varepsilon)+$$
$$+A_{0,2}\big(s,u_1(s-\Delta,\varepsilon)\big)u_1(s-\Delta,\varepsilon)]ds.$$

Solvability of the T-periodic boundary value problem in $k+1$ approximation guarantees the solvability of the equation

$$F_{k+1}\big(c_k(\varepsilon)\big):=\int_0^T H_r^*(s)Z_k(z_0(s,c_r^*),u_1(s,\varepsilon),\ldots,u_{k+1}(s,\varepsilon),$$
$$z_0(s-\Delta,c_r^*),u_1(s-\Delta,\varepsilon),\ldots,u_{k+1}(s-\Delta,\varepsilon),s,\varepsilon)\,ds=0.$$

The sequence of approximations to the solution of the nonlinear T-periodic boundary value problem for equation (1) in the critical case is determined by the iterative scheme

$$z_1(t,\varepsilon):=z_0(t,c_r^*)+u_1(t,\varepsilon),\;\; u_1(t)=X_r(t)c_1(\varepsilon)+u_1^{(1)}(t,\varepsilon),$$

$$\mathrm{u}_1^{(1)}(t,\varepsilon)=\varepsilon G[Z(z_0(s,c_r^*),z_0(s-\Delta,c_r^*),s,0)](t),$$

$$c_1(\varepsilon)=-B_0^{-1}\int_0^T H_r^*(s)\left[A_{1,0}(s)u_1^{(1)}(s,\varepsilon)+A_{0,1}(s)u_1^{(1)}(s-\Delta,\varepsilon)\right]ds,$$
$$z_2(t,\varepsilon):=z_0(t,c_r^*)+u_1(t,\varepsilon)+u_2(t,\varepsilon),\;\; u_2(t,\varepsilon)=X_r(t)c_2(\varepsilon)+u_2^{(1)}(t,\varepsilon),$$

$$u_2^{(1)}(t,\varepsilon)=\varepsilon G[Z_1(z_0(s,c_r^*),u_1(s,\varepsilon),z_0(s-\Delta,c_r^*),u_1(s-\Delta,\varepsilon),s,\varepsilon))](t),$$

$$c_2(\varepsilon)=-B_0^{-1}\int_0^T H_r^*(s)\left[A_{1,0}(s)u_2^{(1)}(s,\varepsilon)+A_{0,1}(s)u_2^{(1)}(s-\Delta,\varepsilon)\right]ds-$$
$$-\frac{1}{2!}B_0^{-1}\int_0^T H_r^*(s)[A_{2,0}\big(s,u_1(s,\varepsilon)\big)u_1(s,\varepsilon)+2A_{1,1}\big(s,u_1(s,\varepsilon)\big)u_1(s-\Delta,\varepsilon)+$$
$$+A_{0,2}\big(s,u_1(s-\Delta,\varepsilon)\big)u_1(s-\Delta,\varepsilon)]\,ds,\ldots,$$

$$z_{k+1}(t,\varepsilon):=z_0(t,c_r^*)+u_1(t,\varepsilon)+\;\ldots\;+u_{k+1}(t,\varepsilon),\qquad (6)$$

$$u_{k+1}(t,\varepsilon) = X_r(t)c_{k+1}(\varepsilon) + u_{k+1}^{(1)}(t,\varepsilon),\ \delta_{k+1}(c_r^*, c_1(\varepsilon), \varepsilon) = F_{k+1}\big(c_2(\varepsilon)\big) - B_0 c_{k+1}(\varepsilon),$$

$$u_{k+1}^{(1)}(t,\varepsilon) = \varepsilon G[Z_k(z_0(s,c_r^*), u_1(s,\varepsilon), \dots, u_k(s,\varepsilon),$$

$$z_0(s-\Delta, c_r^*), s, \varepsilon, u_1(s-\Delta,\varepsilon), \dots, u_k(s-\Delta,\varepsilon))](t),$$

$$c_{k+1}(\varepsilon) = -B_0^{-1}\delta_{k+1}(c_r^*, c_{k+1}(\varepsilon), \varepsilon),\ \ k = 1,2, \dots\ .$$

Theorem. *In the critical case, the periodic problem for equation (2) with concentrated delay, under condition (4), has an r-parametric family of solutions*

$$z_0(t,c_r^*) = X_r(t)c_r + G[f(s)](t),\ \ c_r \in \mathbb{R}^r.$$

Assuming $\det B_0 \neq 0$ *the simplicity of the roots of the equation (5) for the generating amplitudes in a small neighborhood of the generating solution* $z_0(t,c_r^*)$*, the nonlinear periodic boundary value problem for equation (1) has a unique solution*

$$z(t,\varepsilon):\ z(\cdot,\varepsilon) \in C^1[0,T], z(t,\cdot) \in C[0,\varepsilon_0].$$

The sequence of approximations to the solution of the nonlinear periodic boundary value problem for equation (1) is determined by the iterative scheme (6). If there exists a constant $0 < \gamma < 1$*, for which the inequalities*

$$\|u_1(t,\varepsilon)\| \le \gamma \|z_0(t,c_r^*)\|,\ \ \|u_{k+1}(t,\varepsilon)\| \le \gamma \|u_k(t,\varepsilon)\|,\ \ k = 1,2,\dots \qquad (7)$$

hold, then the iterative scheme (6) converges to the solution of the nonlinear periodic boundary value problem for equation (1) with concentrated delay.

3. Finding approximations to the periodic solution of the equation modeling a non-isothermal chemical reaction

Let us apply the iterative scheme (6) in order to find approximations to the periodic solution of the equation with concentrated delay, which models a non-isothermal chemical reaction [10, 11].

Example. The conditions of the proven theorem hold in the case of a 2π-periodic boundary value problem with concentrated delay

$$dz(t,\varepsilon)/dt = A(t)z(t,\varepsilon) + B(t)z(t-\Delta,\varepsilon) + f(t) + \varepsilon Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon); \quad (8)$$

here

$$A(t) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \ B(t) := 0,\ \ f(t) := \begin{pmatrix} \cos t \\ 0 \end{pmatrix},\ \ \Delta := \frac{\pi}{2},$$

and also

$$Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), \varepsilon) := \big(1 + x(t,\varepsilon)\big)e^{-\frac{\varepsilon}{1+y(t-\Delta,\varepsilon)}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \ z(t,\varepsilon) := \begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \end{pmatrix}.$$

For the generating periodic problem for equation (8), there is a critical case [2, 12], and condition (4) is satisfied, therefore, it is solvable:

$$z_0(t,c_r) = X_r(t)c_r + G[f(s)](t),\ \ c_r \in \mathbb{R}^1;$$

here

$$X_r(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \ G[f(s)](t) = \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

The equation for the generating amplitudes (5) in the case of a problem of finding a periodic solution for equation (8) has a simple root $c_r^* = 1$, which determines the generating solution

$$z_0(t, c_r^*) = -\begin{pmatrix} 1 + \sin t \\ 0 \end{pmatrix}.$$

The matrix $B_0 = 2\pi$ is non-singular, so according to the proven theorem, the 2π-periodic problem for equation (8) with concentrated delay is uniquely solvable. Thus, we obtain the first approximation

$$z_1(t, \varepsilon) := z_0(t, c_r^*) + u_1(t, \varepsilon), \;\; u_1(t, \varepsilon) = X_r(t)c_1(\varepsilon) + u_1^{(1)}(t, \varepsilon), \;\; c_1(\varepsilon) = -\varepsilon$$

to the solution of the periodic problem for equation (8); here

$$u_1^{(1)}(t, \varepsilon) = \varepsilon G[Z(z_0(s, c_r^*), z_0(s - \Delta, c_r^*), s, 0)](t) = \varepsilon \begin{pmatrix} 1 - \sin t - \cos t \\ -\cos t \end{pmatrix},$$

and also

$$Z(z_0(t, c_r^*), z_0(t - \Delta, c_r^*), t, 0) = (1 + c_r^* + \sin t\,) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, we obtain the second approximation to the solution of the nonlinear periodic boundary value problem for equation (8) in the critical case

$$z_2(t, \varepsilon) := z_0(t, c_r^*) + u_1(t, \varepsilon) + u_2(t, \varepsilon), \;\; u_2(t, \varepsilon) = X_r(t)c_2(\varepsilon) + u_2^{(1)}(t, \varepsilon);$$

here

$$u_2^{(1)}(t, \varepsilon) = \varepsilon G\big[Z_1\big(z_0(s, c_r^*), u_1(s, \varepsilon), z_0(s - \Delta, c_r^*)\big), u_1(s - \Delta, \varepsilon), s, \varepsilon)\big](t) =$$
$$= \varepsilon^2 \begin{pmatrix} \sin t + 3\cos t - 3 \\ 2\cos t - \sin t \end{pmatrix}, \;\; c_2(\varepsilon) = \frac{7\varepsilon^2}{2}.$$

and also

$$Z_1(z_0(t, c_r^*), u_1(t, \varepsilon), z_0(t - \Delta, c_r^*), u_1(t - \Delta, \varepsilon), t, \varepsilon) =$$
$$= (1 + c_1(\varepsilon) - 2\sin t - \cos t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In the same way, we obtain the third approximation to the solution of the nonlinear periodic boundary value problem for equation (8) in the critical case

$$z_3(t, \varepsilon) := z_0(t, c_r^*) + u_1(t, \varepsilon) + u_2(t, \varepsilon) + u_3(t, \varepsilon), \;\; c_3(\varepsilon) = -\frac{65\varepsilon^3}{8},$$

$$u_3(t, \varepsilon) = X_r(t)c_3(\varepsilon) + u_3^{(1)}(t, \varepsilon), \;\; u_3^{(1)}(t, \varepsilon) =$$
$$= \varepsilon G\big[Z_2\big(z_0(s, c_r^*), u_1(s, \varepsilon), u_2(s, \varepsilon), z_0(s - \Delta, c_r^*)\big), u_1(s - \Delta, \varepsilon), u_2(s - \Delta, \varepsilon), s, \varepsilon)\big](t) =$$
$$= \frac{\varepsilon^3}{8} \begin{pmatrix} 53 - 52\cos t - \cos 2t + 12\sin t + 2\sin 2t \\ 2(16\sin t + \sin 2t - 10\cos t) \end{pmatrix};$$

and also

$$Z_2(z_0(t, c_r^*), u_1(t, \varepsilon), u_2(t, \varepsilon), z_0(t - \Delta, c_r^*), u_1(t - \Delta, \varepsilon), u_2(t - \Delta, \varepsilon), t, \varepsilon) =$$
$$= \frac{\varepsilon^2}{2}(7 - 2c_2(\varepsilon) - 8\cos t - \cos 2t - 5\sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the obtained approximations to the periodic solution of equation (8), the inequalities

$$\|u_1(t, \varepsilon)\| \le \gamma \|z_0(t, c_r^*)\|, \;\; \|u_{k+1}(t, \varepsilon)\| \le \gamma \|u_k(t, \varepsilon)\|, \;\; \gamma \approx 0{,}131\,256, \;\; k = 0{,}1{,}2,$$

hold, indicating the practical convergence of the obtained approximations to the periodic solution of the equation (8) for

$$\varepsilon \in [0, \varepsilon_0], \;\; \varepsilon_0 \approx 0{,}25.$$

The accuracy of the obtained approximations to the periodic solution of equation (8) is determined by the residuals

$$\Delta_k(\varepsilon) := ||dz_k(t, \varepsilon)/dt - A(t)z_k(t, \varepsilon) - B(t)z_k(t - \Delta, \varepsilon) - f(t) -$$
$$-\varepsilon Z(z_k(t, \varepsilon), z_k(t - \Delta, \varepsilon), t, \varepsilon)||_{\,\mathbb{C}[0;\, 2\pi]}, \;\; k = 0, 1, 2, 3.$$

In particular,

$$\Delta_0(0,1) \approx 0,0904\,837,\ \ \Delta_1(0{,}1) \approx 0,0213\,474,$$
$$\Delta_2(0,1) \approx 0,00\,469\,105,\ \ \Delta_3(0{,}1) \approx 0,00\,112\,528;$$
$$\Delta_0(0,01) \approx 0,0099\,005,\ \ \Delta_1(0{,}01) \approx 0,000\,222\,616,$$
$$\Delta_2(0,01) \approx 4,97\,520\ \times 10^{-6},\ \ \Delta_3(0{,}01) \approx 1,21\,626\ \times 10^{-7}.$$

The research scheme proposed in the article for investigating solvability conditions and constructing approximations to the periodic solution of equation (1) can be transferred to matrix boundary value problems, including those with concentrated delay [13 - 16].

The authors of the article express their sincere gratitude to the Managing Director of the Max Planck Institute for Dynamics of Complex Technical Systems in Magdeburg, Professor Peter Benner, for his support and discussion of the obtained results.

4. References

[1] J. K. Hale, Theory of Functional Differential Equations, Springer – Verlag. York, Heidelberg, Berlin, 1975.

[2] A. A. Boichuk, A.M. Samoilenko, Generalized inverse operators and Fredholm boundary-value problems, Utrecht, Boston: VSP, 2004.

[3] R. Bellman, K. L .Cooke, Differential-Difference Equations. Mathematics in Science and Engineering, New York, Academic Press, 1963.

[4] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics. Mathematics and Its Applications, Dordrecht: Kluwer Academic Publishers, 1992.

[5] Th. Erneux, Applied Delay Differential Equations. Surveys and Tutorials in the Applied Mathematical Sciences, Vol. 3, Springer Science+Business Media, New York, 2009.

[6] S.M. Chuiko, A.S. Chuiko, On the approximate solution of periodic boundary value problems with delay by the least-squares method in the critical case, Nonlinear Oscillations (N.Y.) 14, 3 (2012), pp. 445-460.

[7] G. Adomian, A review of the decomposition method in applied mathematics, Journ. of Math. Math. Anal. and Appl., 135 (1988), pp. 501-544.

[8] G. Adomian, R. Rach, Nonlinear Stochastic Differential Delay Equations, Journal of Math. analysis and appl., 91 (1983), pp. 94-101.

[9] M. Mac, C. S. Leung, T. Harko, A brief introducion to the Adomian decomposition method, Romanian Astron. Journ., 1, (2019), pp. 1-41.

[10] P. Benner, A. Seidel-Morgenstern, A. Zuyev, Periodic switching strategies for an isoperimetric control problem with application to nonlinear chemical reactions, Applied Mathematical Modelling, 69 (2019), pp. 287-300.

[11] A. Zuyev, A. Seidel-Morgenstern and P. Benner, An isoperimetric optimal control problem for a non-isothermal chemical reactor with periodic inputs, Chemical Engineering Science 161 (2017), 206-214..

[12] O.A. Boichuk and A.M. Samoilenko, Generalized inverse operators and Fredholm boundary-value problems, Boston: De Gruyter, 2016.

[13] A.A. Boichuk, S.A. Krivosheya, Criterion of the solvability of matrix equations of the Lyapunov type, Ukrainian Mathematical Journal, 50, 8 (1998), № 8, 1162-1169.

[14] P. Lancaster, M. Tismenetsky, The Theory of Matrices, New York, Academic Press,1985.

[15] S.M. Chuiko, Nonlinear matrix differential-algebraic boundary value problem, Lobachevskii Journal of Mathematics, 38 (2018), 236–244.

[16] A.A. Boichuk, J. Diblik, D. Ya. Khusainov, M. Ruzickova, Fredholm's boundary-value problems for differential systems with a single delay, Nonlinear Anal., 72 (2010), pp. 2251-2258.