On two-variable first-order logic with a partial order

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Abstract

The main motivation for this work is the open question of decidability of the satisfiability problem for the two-variable fragment of first-order logic, $\mathcal{FO}^2$, with one transitive relation. The problem can be reduced to the corresponding problem when the transitive relation is required to be a *partial* order. It is known that its finite satisfiability problem is decidable but the decidability of the general satisfiability problem has been resolved only for restricted variants. More precisely, the problem is decidable for the fragment with *transitive witnesses* in which existential quantifiers are required to be guarded by transitive atoms, a property in line with the standard translation of modal logic into first-order logic. We study the 'complementary' fragment with *free witnesses* where formulas, when written in negation normal form, contain existential quantifiers applied only to conjunctions of the form $x \sim y \wedge \psi$, where $x \sim y$ means that x and y are not comparable by the order.

We show that $\mathcal{FO}^2$ with a partial order and free witnesses is not locally finite. On the positive side, we show that the logic enjoys the finite antichain property that we believe is a crucial step towards showing decidability of its satisfiability problem. We also identify minimal syntactic restrictions needed to retain the finite model property.

Keywords

two-variable first-order logic, satisfiability problem, decidability, finite model property, transitivity, partial order, finite antichains, locally finite order

1. Introduction

Two-variable first-order logic, $\mathcal{FO}^2$, is the restriction of classical first-order logic, $\mathcal{FO}$, over relational signatures to formulae with at most two distinct variables x and y. The logic enjoys the finite model property [1], and its satisfiability (hence also finite satisfiability) problem is NExpTime-complete [2].

It is easy to show that $\mathcal{FO}^2$ is not able to express transitivity of a binary relation or related properties, such as that of being an order or an equivalence. In fact, the same limitation applies to many other decidable fragments of first-order logic, including the guarded fragment, $\mathcal{GF}$, and the fluted fragment, $\mathcal{FL}$. Hence, following the approach employed in modal logic, one program of research is to study properties of these logics over restricted classes of structures where some distinguished binary predicates are interpreted as transitive relations, orders or equivalences. Equivalently, one considers signatures containing *undistinguished* predicates and *distinguished* predicates having a predefined interpretation as a transitive relation, order, etc. In this setting, however, it is easy to write infinity axioms. For instance, the $\mathcal{FO}^2$-formula

$$\varphi_0 = \forall x \neg Txx \wedge \forall x \exists y Txy, \quad (1)$$

saying that T is serial and irreflexive, with T being a distinguished *transitive* predicate, is satisfiable but has only infinite models.

In this scenario decidability of the (finite) satisfiability problem usually depends on the number of distinguished relations, unless additional syntactic restrictions apply (cf. [3] for an overview and [4] for a comprehensive study). In particular, it has been established that three distinguished predicates interpreted as either transitive relations, equivalence relations or linear orders suffice to obtain undecidability of both the satisfiability and the finite satisfiability problems for each of the logics $\mathcal{GF}^2$, $\mathcal{FO}^2$ and $\mathcal{FL}^2$. These results can be strengthened: in case of $\mathcal{GF}^2$ and $\mathcal{FO}^2$ undecidability follows already

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over structures with one transitive and one equivalence relation [5], and in case of $\mathcal{FL}^2$ it suffices that one of the three transitive relation is an equality [6]. These undecidability results come mainly from the possible interactions between the transitive relations. When such interactions are restricted, decidability is retained even in the presence of arbitrarily many transitive relations, as in the guarded fragment with *transitive guards*, where transitive atoms are only allowed in guard positions [7].

On the positive side, $\mathcal{FO}^2$ with one equivalence relation has the finite model property, and its satisfiability (= finite satisfiability) problem is NExpTime-complete [8]; $\mathcal{FO}^2$ with two equivalence relations lacks the finite model property, but its satisfiability and finite satisfiability problems are both 2-NExpTime-complete [9].

When the distinguished predicates are required to be interpreted as linear orders, the picture is not fully transparent, as not all known complexity bounds are tight. Specifically, the satisfiability and finite satisfiability problems for $\mathcal{FO}^2$ together with one linear order are both NEXPTIME-complete [10]; the finite satisfiability problem for $\mathcal{FO}^2$ together with two linear orders is in 2-NExpTime [11] (falling to ExpSpace when all undistinguished predicates are unary [12]). Decidability of the satisfiability problem for $\mathcal{FO}^2$ with two linear orders was shown recently using an automata based approach by Toruńczyk and Zeume [13]. More precisely, the paper shows decidability of the countable satisfiability problem for two-variable logic in the presence of a tree order, a linear order, and arbitrary atoms that are definable from the tree order using monadic second-order formulas.

Turning towards transitive relations the following is known. The satisfiability problem for $\mathcal{GF}^2$ with one transitive relation is 2-ExpTime-complete [14]. Both the satisfiability and the finite satisfiability problems for $\mathcal{FL}$—the *full* fluted fragment—with one transitive relation remain decidable in the presence of equality and arbitrarily many undistinguished relations [6].

The case that is not yet fully understood is $\mathcal{FO}^2$ with one transitive relation, $\mathcal{FO}^2$1T. It is known that its finite satisfiability problem is decidable in 3-NExpTime [15] but the decidability of the general satisfiability problem has been resolved only for restricted variants. As mentioned above, the problem is decidable for $\mathcal{GF}^2$ with one transitive relation. Moreover, decidability of the satisfiability problem follows also for $\mathcal{GF}^2$ with one partial order, as (ir)reflexivity and antisymmetry of a binary relation are two-variable guarded properties. Decidability of the satisfiability problem is also known for $\mathcal{FO}^2$ with one linear order or with one tree order but the case of *any* transitive relation seems to be more intricate.

In 2013 it was advertised that Sat($\mathcal{FO}^2$1T) is decidable [16] but later it was discovered [17] that the proof works only for the fragment with *transitive witnesses* in which existential quantifiers are required to be guarded by transitive atoms, a property in line with the standard translation of modal logic into first-order logic. We remark that the infinity axiom (1) belongs to this fragment, as the existential quantifier in the subformula $\forall x \exists y Txy$ is guarded by the atom Txy. In this fragment transitive guards might also be of the form Tyx, so the syntax allows one to describe *tense* frames defined in modal logic by adding to usual Kripke frames $\langle W, R\rangle$ the converse of R.

In this paper we study the fragment $\mathcal{FO}^2$1T with *free witnesses* in which formulas, when written in negation normal form, contain existential quantifiers applied only to conjunctions of the form $\neg Txy \wedge \neg Tyx \wedge \psi$, enforcing that x and y are *incomparable* by T. We note that this pattern is neither guarded nor fluted. The logic can be seen as complementary to $\mathcal{FO}^2$1T with *transitive witnesses*.

Models for $\mathcal{FO}^2$1T-formulas, taking into account the interpretation of the transitive relation, can be seen as partitioned into cliques. In [15] it was observed that this logic enjoys the *small clique property*: every satisfiable formula has a model in which the size of cliques is bounded. This property allows one to reduce the (finite) satisfiability problem for $\mathcal{FO}^2$1T to the (finite) satisfiability problem for $\mathcal{FO}^2$ with one partial order encoding cliques by single elements satisfying some new unary predicates and connection types between cliques by pairs of elements satisfying new binary predicates. Hence, in the remaining part of the paper we concentrate on $\mathcal{FO}^2$1PO, the two-variable fragment of first-order logic with one partial order, and its subfragments.

Regarding expressive power it turns out that $\mathcal{FO}^2$1PO is able not only to enforce infinite models but also infinite antichains (cf. examples in [15, 17]). When restricting attention to the fragment with free witnesses, the finite model property is also lost [17]. However, as we show, the subfragment enjoys the *finite antichain property*: every satisfiable formula has a model in which the size of every antichain is exponentially bounded with respect to the length of the formula. The proof shows actually a bit more: every satisfiable formula has a model with a universe forming a simple block structure; the universe can be partitioned into a bounded number of chains consisting of small blocks containing elements of the same one-type. Elements inside the blocks are incomparable, and the partial order is induced by the order on the blocks. We believe that the finite antichain property will be crucial to show decidability. Notably, known undecidability proofs for elementary modal logics and for fragments of two-variable first-order logic usually encode variants of the tiling problems that are based on two-dimensional grids having unbounded antichains.

We also illustrate the expressive power of $\mathcal{FO}^2$1PO with free witnesses, showing that in this logic one can write formulas enforcing models that are not locally finite. (A structure is *locally finite* if for any two elements a, b of its domain, the interval $I(a, b)$ consisting of elements *between* a and b is finite.) We provide a formula using only unary undistinguished predicates, each of whose model embeds a copy of $\mathbb{Z}^k$ (k depends on the size of the signature). Such structures, for any $k>1$, have infinitely many infinite intervals. Moreover, we identify minimal syntactic restrictions on the universal part of formulas in the logic that allow us to rescue the finite model property.

The paper is structured as follows. Section 2 contains the necessary preliminaries. The proof that the logic $\mathcal{FO}^2$1PO with free witnesses is not locally finite is given in Section 3. The finite antichain property of the logic is established in Section 4 and the restrictions needed to retain the finite model property in Section 5. We conclude with some directions for future research.

Related work. Partially related is the research on logics for data words and data trees that can be seen as logics over signatures consisting of unary predicates corresponding to *data values* and (at least) two distinguished binary predicates: an equivalence relation used to compare data values and the linear order for words or the tree order relation(s) for trees, see e.g. [18, 19]. However, in these papers the structures (words or trees) are finite and, hence, the results concern only the finite satisfiability problem.

More closely related seems to be the research on automata on various classes of infinite partial orders, in particular, on *pomsets* (i.e. labelled partially ordered sets) investigated in the area of modelling concurrent systems. Decidability of the satisfiability problem for logics over these classes is often established by showing decidability of non-emptiness of the corresponding automata. This research contains the study of series-parallel pomsets, also called N-free pomsets, originated by [20, 21] for finite graphs, generalized to width-bounded infinite graphs in [22], scattered and countable posets in [23, 24], and to *synchronized* series-parallel graphs in [25]. Positive results are obtained usually in the presence of additional assumptions on the structures such as bounded-width, scattered, well-ordered, finite antichains. Since the structures enforced by $\mathcal{FO}^2$1PO-sentences are not necessarily N-free, the branching automata introduced for such structures cannot be directly applied to recognize models of the logics we are concerned with in this article.

Somewhat related is also research on certain modal logics. For instance, Humberstone [26] and Goranko [27] study the bimodal logic of *inaccessible worlds* determined by complementary frames of the form $\langle W, R, W^2{-}R\rangle$. In the logic the standard modal operator [ ] is used for worlds accessible by the relation R, and a second modal operator $[i]$ is used for inaccessible worlds: $[i]p$ holds at a world x, iff p is true in all worlds which are *not* accessible from x via R. This condition is not expressible in guarded logic but it is expressible in $\mathcal{FO}^2$ and in $\mathcal{FL}$. Decidability of both the global and the local satisfiability problem for the bimodal logic of inaccessible worlds over transitive frames can be inferred from the above mentioned decidability of $\mathcal{FL}$ with one transitive relation [6].

2. Preliminaries

We employ standard terminology and notation from model theory. Structures are denoted by (possibly decorated) fraktur letters $\mathfrak{A}$, $\mathfrak{B}$, and their domains by the corresponding Roman letters. Where a structure is clear from context, we frequently equivocate between predicates and their realizations, thus writing, for example, R in place of the technically correct $R^{\mathfrak{A}}$. If $\mathfrak{A}$ is a structure over a relational signature and $B \subseteq A$, then $\mathfrak{A}{\restriction}B$ denotes the (induced) substructure of $\mathfrak{A}$ with the universe B.

2.1. Poset terminology

Let $(X, <)$ be a poset, where $<$ denotes a *strict* partial order, i.e. a binary relation that is irreflexive and transitive. Elements $x, y \in X$ are said to be *comparable* if either $x < y$ or $y < x$. Elements $x, y \in X$ are said to be *incomparable*, denoted $x \sim y$, if they are not comparable and distinct. An element $x \in X$ is *maximal*, if there is no element $y \in X$ such that $x < y$.

Let $x \in X$ and $Y \subseteq X$. We write $x < Y$ ($Y < x$) if for every $y \in Y$ we have $x < y$ ($y < x$). For subsets $Y, Y' \subseteq X$ we write $Y < Y'$ if $y < y'$ holds for every $y \in Y$ and $y' \in Y'$. We also write $Y \sim Y'$ iff $Y \not< Y'$ and $Y' \not< Y$ hold.

A set $Y \subseteq X$ is an *antichain*, if all elements of Y are mutually incomparable; Y is a *chain*, if the partial order restricted to Y is total. An element $x \in X$ is an *upper bound* of Y if for every $y \in Y$, $y = x \vee y < x$. For any $x, y \in X$ we define $I(x, y) = \{z \in X | x < z < y\}$ as the *interval* of (x, y). A poset $(X, <)$ has the *finite antichain property* if every antichain in X is finite; it is *locally finite*, if for every $a, b \in X$ the interval $I(a, b)$ is finite.

We apply the above terminology to structures interpreting a signature σ in which one distinguished relation is a partial order $<$ in a natural way. For instance, let $\mathfrak{A}$ be a σ-structure and $B, C \subseteq A$. We write $B <^{\mathfrak{A}} C$, if for every $b \in B$, $c \in C$, $\mathfrak{A} \models b < c$. We say B is a *chain* in $\mathfrak{A}$, if for every $a, b \in B$, $\mathfrak{A} \models a = b \vee a < b \vee b < a$. We say C is an *antichain* in $\mathfrak{A}$, if for every $a, b \in C$, $\mathfrak{A} \models a = b \vee a \sim b$.

We say that a logic $\mathcal{L}$ interpreting a partial order has the *finite antichain property* if every satisfiable formula of $\mathcal{L}$ has a model that contains no infinite antichain. We say that $\mathcal{L}$ is *locally finite*, if every satisfiable formula of $\mathcal{L}$ has a model that contains no infinite intervals. Obviously, when $\mathcal{L}$ enjoys the finite model property then it is also locally finite and enjoys the finite antichain property.

2.2. Logics

We denote by $\mathcal{FO}^2$ the two-variable fragment of first-order logic (with equality) over relational signatures. As predicates having arity other than 1 or 2 add no effective expressive power in the context of $\mathcal{FO}^2$ and individual constants add no effective expressive power given the presence of the equality predicate, we shall take all signatures to consist only of unary and binary predicates.

By $\mathcal{FO}^2$1T, we understand the set of $\mathcal{FO}^2$-formulas over any signature $\sigma = \sigma_0 \cup \{T\}$, where T is a distinguished binary predicate letter. The semantics for $\mathcal{FO}^2$1T is as for $\mathcal{FO}^2$, subject to the restriction that T is always interpreted as a *transitive* relation. When the distinguished predicate T is additionally required to be irreflexive and antisymmetric (i.e. a *strict partial order*), we denote the corresponding set of $\mathcal{FO}^2$-formulas by $\mathcal{FO}^2$1PO. Note that the properties of a binary relation to be irreflexive and antisymmetric are two-variable formulas, hence $\mathcal{FO}^2$1PO is in fact a fragment of $\mathcal{FO}^2$1T. Finally, we define $\mathcal{FO}^2_u$1PO to be the subset of $\mathcal{FO}^2$1PO in which no binary predicates other than = and T appear. When working with $\mathcal{FO}^2$1PO and its fragments we often replace the predicate letter T by the more intuitive symbol $<$, written in infix notation.

Crucial for this paper are two restrictions of $\mathcal{FO}^2$1T depending on how the existential quantifiers are used; no restrictions are imposed on using universal quantifiers. The fragment *with transitive witnesses*, $\mathcal{FO}^2$1T$_{tw}$, consists of the formulas of $\mathcal{FO}^2$1T where, when written in negation normal form, existential quantifiers are 'guarded' by transitive atoms, i.e. they are applied to formulas with two free variables only of the form $\xi(x, y) \wedge \psi$, where $\xi(x, y)$ is one of the conjunctions: $Txy \wedge Tyx$, $Txy \wedge \neg Tyx$, or $Tyx \wedge \neg Txy$, and $\psi \in \mathcal{FO}^2$1T$_{tw}$. Similarly, the fragment *with free witnesses*, $\mathcal{FO}^2$1T$_{fw}$, consists of these formulas where, when written in negation normal form, existential quantifiers are applied to formulas with two free variables only of the form $\neg Txy \wedge \neg Tyx \wedge \psi$ with $\psi \in \mathcal{FO}^2$1T$_{fw}$. In case of $\mathcal{FO}^2$1PO the corresponding fragments are denoted by $\mathcal{FO}^2$1PO$_{tw}$ and $\mathcal{FO}^2$1PO$_{fw}$.

As an example, consider the formula $\forall x \exists y\, Txy$. Strictly speaking, the existential quantifier does not comply to any of the patterns mentioned above. However, using standard first-order tautologies, the subformula $\exists y\, Txy$ can be replaced by a disjunction of two formulas in $\mathcal{FO}^2$1T$_{tw}$. Hence, the infinity axiom (1) can be written in $\mathcal{FO}^2$1T$_{tw}$ but cannot be written in $\mathcal{FO}^2$1T$_{fw}$.

2.3. Types and normal forms

In previous work on $\mathcal{FO}^2$, several variants of the standard Scott normal form have been introduced that allow one to reduce the satisfiability problem to the satisfiability problem for formulas in the normal form. They are usually defined as a conjunction of sentences in prenex normal form with quantifier prefixes $\forall\forall$ and $\forall\exists$. Before we recall the ones useful for this paper we introduce some more notions.

Let σ be any relational signature. A *σ-literal* is a formula of the form $P\bar{x}$ or $\neg P\bar{x}$, where $\bar{x}$ is an n-tuple of variables and P is a predicate of arity n from σ. An (atomic) *1-type* is a maximal consistent set of σ-literals containing only one variable x and an (atomic) *2-type* is a maximal consistent set of σ-literals containing two variables x and y and featuring the formula $x \neq y$. We say that for a structure $\mathfrak{A}$ some element $a \in A$ *realizes a 1-type* α when α is the unique 1-type such that $\mathfrak{A} \models \alpha[a]$; we denote this 1-type by $\text{tp}^{\mathfrak{A}}[a]$. Similarly, we say that two distinct elements $a, b \in A$ *realize a 2-type* β when β is the unique 2-type such that $\mathfrak{A} \models \beta[a, b]$; we denote this 2-type by $\text{tp}^{\mathfrak{A}}[a, b]$. In addition, let $\boldsymbol{\alpha}$ be the set of all possible 1-types in σ and let $\boldsymbol{\beta}$ be the set of all possible 2-types in σ. Observe that both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are exponentially bounded in $|\sigma|$. For a given σ-structure $\mathfrak{A}$, let $\boldsymbol{\alpha}^{\mathfrak{A}}$ be the set of all 1-types realized in $\mathfrak{A}$ and let $\boldsymbol{\beta}^{\mathfrak{A}}$ be the set of all 2-types realized in $\mathfrak{A}$. Moreover, for each 1-type α realized in $\mathfrak{A}$, define A^{α} as the set of all elements in A which realize α.

Below we recall the normal forms from [15] tailored especially for the fragments we exploit in this paper. We employ the abbreviations:

$$T_{\equiv}(x,y) := Txy \wedge Tyx \wedge x \neq y, \qquad T_{\sim}(x,y) := \neg Txy \wedge \neg Tyx \wedge x \neq y,$$
$$T_{<}(x,y) := Txy \wedge \neg Tyx \wedge x \neq y, \qquad T_{>}(x,y) := \neg Txy \wedge Tyx \wedge x \neq y.$$

Definition 1. A formula φ of $\mathcal{FO}^2$1T (or of $\mathcal{FO}^2$1PO) is said to be in *transitive normal form* if it conforms to the pattern

$$\forall x \forall y\, \psi_0 \wedge \bigwedge_{i=1}^{m} \bigwedge_{d \in \{\sim,<,>,\equiv\}} \forall x\, (P_{i,d}x \rightarrow \exists y\, (T_d(x,y) \wedge \psi_{i,d}(x,y))), \tag{2}$$

where ψ_0 is quantifier-free, the $P_{i,d}$ are unary predicates and the $\psi_{i,d}(x,y)$ are quantifier- and equality-free formulas not featuring either of the atoms Txy or Tyx (they may contain the atoms Txx or Tyy).

Without loss of generality we assume that an $\mathcal{FO}^2$1PO-formula in transitive normal form features only $\forall\exists$-conjuncts with $T_d \in \{\sim, <, >\}$. Indeed, when T is a strict partial order, a formula $\forall x \exists y\, (P(x) \rightarrow T_{\equiv}(x,y) \wedge \psi(x,y))$ is only a complicated way of writing the logical constant `false`.

Additionally, normal form formulas of $\mathcal{FO}^2$1PO$_{tw}$ feature no conjuncts with $T_{\sim}$, and normal form formulas of $\mathcal{FO}^2$1PO$_{fw}$ feature only $\forall\exists$-conjuncts with $T_{\sim}$. So, denoting the transitive relation T by the symbol $<$ and employing the abbreviation

$$x \sim y := \neg(x < y) \wedge \neg(y < x) \wedge x \neq y,$$

any $\mathcal{FO}^2$1PO$_{fw}$-formula φ in transitive normal form (2) conforms to the pattern

$$\forall x \forall y\, \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x\, (P_i x \rightarrow \exists y\, (x \sim y \wedge \psi_i(x,y))), \tag{3}$$

where ψ_0 is quantifier-free, the P_is are unary predicates and the $\psi_i(x,y)$ are quantifier- and equality-free formulas not featuring $<$.

Lemma 1 ([15], Lemma 5.2)**.** *Let φ be an $\mathcal{FO}^2$1T-formula. There exists an $\mathcal{FO}^2$1T-formula φ^* in transitive normal form such that: (i) $\models \varphi^* \rightarrow \varphi$; (ii) every model of φ can be expanded to a model of φ^*; and (iii) the length of φ^* is bounded polynomially with respect to the length of φ.*

In case of $\mathcal{FO}^2_u$1PO one more normal form has been introduced in [15] to simplify reasoning concerning decidability of the finite satisfiability problem of the logic. We reuse the form with a slight modification so that it works for all domains not only finite ones.

Definition 2. A formula φ of $\mathcal{FO}^2_u$1PO is said to be in *basic normal form* if it is a conjunction of *basic formulas* of the form

$$\forall x(\alpha(x) \rightarrow \forall y(\alpha(y) \rightarrow x = y)) \tag{B1}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\alpha(y) \wedge x \neq y \rightarrow x \sim y)) \tag{B2a}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\beta(y) \rightarrow x \sim y)) \tag{B2b}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\beta(y) \rightarrow x < y)) \tag{B3}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\beta(y) \rightarrow x < y \vee x \sim y)) \tag{B4}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\alpha(y) \wedge x \neq y \rightarrow (x < y \vee y < x))) \tag{B5a}$$
$$\forall x(\alpha(x) \rightarrow \forall y(\beta(y) \wedge x \neq y \rightarrow (x < y \vee y < x))) \tag{B5b}$$
$$\forall x(\alpha(x) \rightarrow \exists y(x < y \wedge \mu(y))) \tag{B6}$$
$$\forall x(\alpha(x) \rightarrow \exists y(y < x \wedge \mu(y))) \tag{B7}$$
$$\forall x(\alpha(x) \rightarrow \exists y(x \sim y \wedge \mu(y))) \tag{B8}$$
$$\forall x.\mu(x) \tag{B9}$$
$$\exists x.\mu(x) \tag{B10}$$

where α, β are distinct 1-types and μ is a quantifier-free formula not featuring $=, <, \sim$.

Our modification concerns conjuncts (B6) and (B7) that in [15] had the forms $\forall x(\alpha(x) \rightarrow \exists y(\mu(y) \wedge \neg\alpha(y) \wedge x < y))$ and, respectively, $\forall x(\alpha(x) \rightarrow \exists y(\mu(y) \wedge \neg\alpha(y) \wedge y < x))$. These forms could be obtained by considering extremal elements satisfying the 1-type α that in infinite structures might not exist. We also omit the conjuncts (B1b): $\forall x(\alpha(x) \rightarrow \forall y(\beta(y) \rightarrow x = y))$ that for distinct 1-types α and β are equivalent to the logical constant `false` and can be expressed anyway.

The following Lemma can be proved exactly as Lemma 3.1 from [15].

Lemma 2. *Let φ be an $\mathcal{FO}^2_u$1PO-formula. There exists an $\mathcal{FO}^2_u$1PO-formula φ^* in basic normal form such that: (i) $\models \varphi^* \rightarrow \varphi$; (ii) every model of φ can be expanded to a model of φ^*, and (iii) the length of φ^* is bounded polynomially in the length of φ.*

We remark that the transformations in the proofs of Lemmas 1 and 2 do not influence the cardinalities of antichains or intervals in models.

3. Enforcing Infinite Intervals

In this section we show that $\mathcal{FO}^2_u1\mathrm{PO}_{fw}$ is not locally finite. The formula Φ we write builds on the infinity axiom presented in [17, Section 5]. It will contain several conjuncts of the form (B5b) with various α and β, so we introduce the abbreviation:

$$\alpha \bowtie \beta := \forall x(\alpha(x) \rightarrow \forall y(\beta(y) \rightarrow (x < y) \vee (y < x)));$$

we also say in such case that the 1-types α and β are *entangled*.

Let $I = \{0, 1, 2, 3, 4\}$ and $\sigma_0 = \{A_i | i \in I\} \cup \{B_i | i \in I\}$. Let Φ_0 be the formula saying that the unary predicates of σ_0 are mutually disjoint and exhaustive, with the exception of A_0 and B_0 that are equivalent (so, one is invited to think that A_0 and B_0 can be identified)

$$\forall x(A_0 x \leftrightarrow B_0 x) \wedge \forall x \bigvee_{C \in \sigma_0} Cx \quad \wedge \bigwedge_{C, C' \in \sigma_0 : C \neq C', \{C, C'\} \neq \{A_0, B_0\}} (Cx \rightarrow \neg C' x). \tag{4}$$

Let $\Phi_{spiral(A)}$ contain for every $i \in I$ the following conjuncts

$$A_i \bowtie A_{i+2}, \tag{5}$$

$$\forall x(A_i x \rightarrow \exists y(A_{i+1} y \wedge x \sim y)), \tag{6}$$

$$\forall x(A_i x \rightarrow \exists y(A_{i-1} y \wedge x \sim y)), \tag{7}$$

with the arithmetical operations in indices understood modulo 5.

Assume $\mathfrak{D} \models \Phi_0 \wedge \Phi_{spiral(A)}$. Without loss of generality we may assume that there is $a_0 \in D$ satisfying A_0. By (6) there is an incomparable element $a_1 \in D$ satisfying A_1. By (7), there is an element $a_{-1} \in D$ incomparable with a_0 satisfying A_4. By (5), a_1 and a_{-1} are comparable. Assume $\mathfrak{D} \models a_{-1} < a_1$. Now, the conjuncts (6) and (7) enforce existence of new elements a_2 incomparable with a_1 satisfying A_2 and a_{-2} incomparable with a_{-1} and satisfying A_3. Since, by (5), A_2 is entangled with A_0 and A_4, so we get $a_0 < a_2$ and $a_{-1} < a_2$. Here, we have no choice for the direction, as otherwise, by transitivity, a_1 and a_2 were comparable. Similarly, again by (5), A_3 is entangled with A_0 and A_1, we get $a_{-2} < a_0$ and $a_{-2} < a_1$. Hence, by transitivity, $a_{-2} < a_2$. Now, by (6), there must be a new element $a_3 \in D$ incomparable with a_2 satisfying A_3. Simple induction shows that there is an infinite sequence of distinct elements $a_k \in D$ ($k \in \mathbb{Z}$) such that, for every $k \in \mathbb{Z}$, a_k satisfies $A_{k (\text{mod } 5)}$, $a_k \sim a_{k+1}$ and, for every $k < l+1$, $a_k < a_l$. Let us call this sequence an A-spiral of $\mathfrak{D}$. Note that the argument repeats when $a_{-1} > a_1$, resulting in a spiral going in the opposite direction (for every $k < l+1$, $a_l < a_k$). Hence, the formula $\Phi_0 \wedge \Phi_{spiral(A)}$ is an axiom of infinity.

Now, let $\Phi_{spiral(B)}$ be a copy of the formula $\Phi_{spiral(A)}$ obtained by replacing all occurences of the A_is by corresponding B_is. Suppose $\mathfrak{D} \models \Phi_0 \wedge \Phi_{spiral(A)} \wedge \Phi_{spiral(B)}$. Starting with an element $a_0 (= b_0) \in D$ satisfying both A_0 and B_0, and repeating the above argument for the B-spiral, we see that $\mathfrak{D}$ contains an A-spiral $\{a_k\}_{k \in \mathbb{Z}}$ and a B-spiral $\{b_k\}_{k \in \mathbb{Z}}$ crossing at $a_0 = b_0$, each having a particular direction.

To enforce infinite intervals we add additional entanglements. Let Φ_{cross} be the formula containing for every $i \in I$ the entanglement:

$$A_2 \bowtie B_i. \tag{8}$$

This is consistent with the identification $A_0 = B_0$, since we already have $A_0 \bowtie A_2$ in (5).

Finally, let $\Phi = \Phi_0 \wedge \Phi_{spiral(A)} \wedge \Phi_{spiral(B)} \wedge \Phi_{cross}$ and let $\mathfrak{D} \models \Phi$. Suppose the direction of the A-spiral and of the B-spiral agrees with the natural order on the indices of its elements, i.e. $a_0 < a_2$ and $b_0 < b_2$. Simple reasoning with transitivity shows that all elements of the B-spiral lie below a_2. Note that a_5 (satisfying $A_0 = B_0$) generates a further B-spiral, all of whose elements are greater than a_2 and less than a_7. And so the process repeats. Note in this regard that, the elements b_{5k} ($k \in \mathbb{Z}$) which satisfy both A_0 and B_0 and, by (6), require a free witness satisfying A_1 may use a_1 as that free witness. Similarly for the higher B-spirals.

Hence, $\mathfrak{D}$ contains infinitely many mutually disjoint B-spirals sandwiched between a_{5k-3} and a_{5k+2}, i.e. the intervals $I(a_{5k-3}, a_{5k+2}) = \{d \in D | a_{5k-3} < d < a_{5k+2}\}$ are mutually disjoint and each of them contains a whole infinite B-spiral. The argument obviously repeats when the direction of the A-spiral is opposite. Fig. 1 depicts a possible model $\mathfrak{D}$ of Φ. Moreover, one could employ additional unary predicates to enforce a different C-spiral involving a_0 but 'going' in the third dimension, inducing infinitely many A-spirals, of which each induces infinitely many B-spirals, as described above. The process can be continued for any finite dimension. Hence, we have the following observation.

Theorem 3. *Let $k > 0$. There is a satisfiable formula $\varphi \in \mathcal{FO}^2_u 1\mathrm{PO}_{fw}$ such that every model of φ embeds a copy of $\mathbb{Z}^k$ (lexicographically ordered).*

Note that the formulas enforcing infinity axioms and infinite intervals presented in this section use only universal conjuncts of the form (B5b) that define entanglements between distinct 1-types. As we show in Section 5, this is not a coincidence.

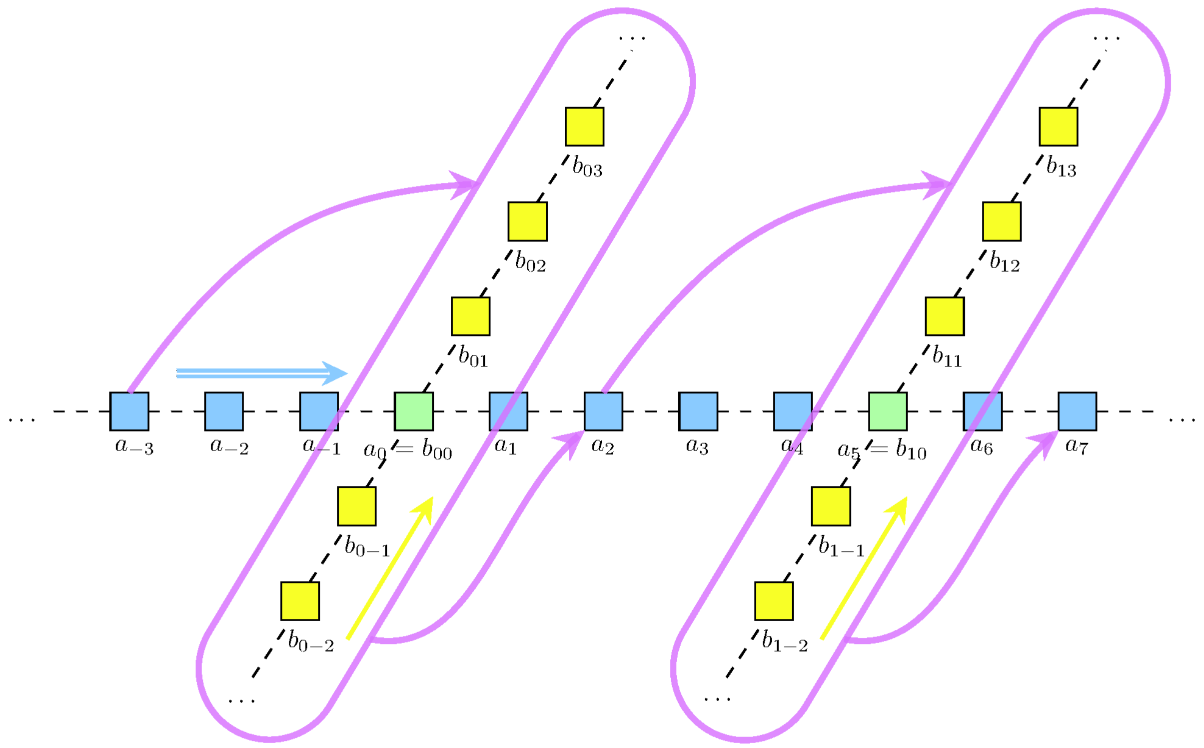

Figure 1: A structure $\mathfrak{D}$ which is a model of Φ inducing infinite intervals. The elements a_k with $k \in \mathbb{Z}$ form an A-spiral and the elements b_{kl} with $k, l \in \mathbb{Z}$ form B-spirals. Each element a_k realizes the 1-type $A_{k \text{ mod } 5}$ and each element b_{kl} realizes the 1-type $B_{l \text{ mod } 5}$. The straight thick blue arrow indicates the direction of the A-spiral; in particular, $a_0 < a_2$, $a_0 < a_3$ and $a_1 < a_3$. The straight thin yellow arrows indicate the directions of the B-spirals. Additionally, the curved arrows indicate that all elements of a given B-spiral crossing the A-spiral at the element a_{5j} with $j \in \mathbb{Z}$ are greater than a_{5j-3} and less than a_{5j+2} w.r.t. the partial order $<^{\mathfrak{D}}$.

4. Finite antichain property

It is not difficult to show that $\mathcal{FO}^2$1PO can enforce models with infinite antichains (cf. Section 5.1 in [17]). In this section we show that $\mathcal{FO}^2\text{1PO}_{fw}$ enjoys the finite antichain property. The proof of the finite antichain property for $\mathcal{FO}^2\text{1PO}_{fw}$ comprises two steps. We first show the property for the logic $\mathcal{FO}^2_u\text{1PO}_{fw}$ where signatures contain no binary predicates other than $<$ or $=$, and then generalize the proof to $\mathcal{FO}^2\text{1PO}_{fw}$.

We recall Zorn's lemma for posets that will be used in the ensuing argument.

Proposition 4 (Zorn). *Let $(X, <)$ be a partially ordered set. If every chain in X has an upper bound in X then, X has a maximal element.*

We also need some more notions. Let $\sigma = \sigma_0 \cup \{<\}$.

Definition 3. Let $\mathfrak{A}$ be a σ-structure. Define a *factorization of* $\mathfrak{A}$ as a set $\mathbb{P}$ of disjoint non-empty subsets of A such that:

(i) $\bigcup_{P \in \mathbb{P}} P = A$;
(ii) for every $P \in \mathbb{P}$, there exists a 1-type $\alpha \in \boldsymbol{\alpha}$, denoted tp$(P)$, such that for every $a \in P$ we have tp$^{\mathfrak{A}}[a] = \alpha$;
(iii) for every distinct $P, Q \in \mathbb{P}$ such that tp$(P) =$ tp(Q), we have $P <^{\mathfrak{A}} Q$ or $Q <^{\mathfrak{A}} P$.

We refer to the elements of $\mathbb{P}$ as *blocks*. A block of cardinality 1 is called a *unit*. A unit block of type α, where α is realized only once in $\mathfrak{A}$, is called a *king*. Moreover, if we assume that the size of the blocks in $\mathbb{P}$ other than kings is a constant M, we say that $\mathbb{P}$ is an *M-balanced factorization of* $\mathfrak{A}$.

Note that every partial order $\mathfrak{A}$ has the trivial factorization $\mathbb{P} = \{A^\alpha | \ \alpha \text{ is realized in } \mathfrak{A}\}$ in which elements realizing the same 1-type form a single block. If $\mathbb{P}$ and $\mathbb{Q}$ are two factorizations of $\mathfrak{A}$, then denote by $\mathbb{P} \sqsubseteq \mathbb{Q}$ the fact that for every block $Q \in \mathbb{Q}$, there exists a block $P \in \mathbb{P}$ such that $Q \subseteq P$. It is easily verifiable that $\sqsubseteq$ is a partial order on the set of all factorizations of $\mathfrak{A}$. We say that a factorization $\mathbb{P}$ is *maximal* if for every factorization $\mathbb{Q}$ of $\mathfrak{A}$, $\mathbb{P} \sqsubseteq \mathbb{Q}$ implies $\mathbb{P} = \mathbb{Q}$. The following lemma ensures us that maximal factorization exists.

Lemma 5. *Every σ-structure $\mathfrak{A}$ has got a maximal factorization.*

Proof. Recall that $\sqsubseteq$ is the partial order on the set of factorizations of $\mathfrak{A}$. We obtain the conclusion by an application of Zorn's lemma. Let $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$ be a chain w.r.t. $\sqsubseteq$ and define $\mathbb{P}^*$ as the sum of the intersections of the elements of the Cartesian product $\prod \mathbb{P}_i$, that is $\mathbb{P}^* = \bigcup_{\overline{P}\in\prod\mathbb{P}_i}\{\bigcap_{P\in\overline{P}} P\} \setminus \{\emptyset\}$. $\mathbb{P}^*$ is an upper bound of the chain because for each $i \in \mathbb{N}$ we have $\mathbb{P}_i \sqsubseteq \mathbb{P}^*$, which follows from the features of the intersection of sets. In addition, $\mathbb{P}^*$ is a factorization of $\mathfrak{A}$:

(i) We have $\bigcup_{P^*\in\mathbb{P}^*} P^* = A$ since every element $a \in A$ belongs to exactly one subset for each of the factorizations in the chain $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$.

(ii) Each block $P^* \in \mathbb{P}^*$ has only elements which share the same 1-type since each of the factorizations $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$ has only subsets in which elements share the same 1-type.

(iii) For each distinct blocks $P^*, Q^* \in \mathbb{P}^*$ such that $tp(P) = tp(Q)$, we have either $P^* <^{\mathfrak{A}} Q^*$ or $Q^* <^{\mathfrak{A}} P^*$. This is because for each $P^* \in \mathbb{P}^*$ there exists exactly one $P \in \mathbb{P}_i$ for each $i \in \mathbb{N}$ such that $P^* \subseteq P$ and similarly, for each $Q^* \in \mathbb{P}^*$ there exists exactly one $Q \in \mathbb{P}_i$ for each $i \in \mathbb{N}$ such that $Q^* \subseteq Q$. Hence, $P^* <^{\mathfrak{A}} Q^*$ or $Q^* <^{\mathfrak{A}} P^*$ follows directly from $P <^{\mathfrak{A}} Q$ or $Q <^{\mathfrak{A}} P$ that holds for $\mathbb{P}_i$.

□

The following lemma is crucial.

Lemma 6. *Let $\mathbb{P}$ be a maximal factorization of $\mathfrak{A}$. Then:*
(i) for every $P \in \mathbb{P}$, if P is not a unit then there exists $a, b \in P$ such that $a \sim^{\mathfrak{A}} b$;
(ii) for every distinct $P, Q \in \mathbb{P}$ such that $P \sim^{\mathfrak{A}} Q$ there exists $a \in P$ and $b \in Q$ such that $a \sim^{\mathfrak{A}} b$.

Proof. Let $\mathbb{P}$ be a maximal factorization of $\mathfrak{A}$.

(i) Let $P \in \mathbb{P}$ be such that for all distinct $a, b \in P$ we have either $a <^{\mathfrak{A}} b$ or $b <^{\mathfrak{A}} a$. Note that the elements in P are totally ordered w.r.t. $<^{\mathfrak{A}}$ since they are comparable. Let c be an element in P and define the subsets $P_< = \{d \in P | d <^{\mathfrak{A}} c\}$, $P_c = \{c\}$ and $P_> = \{d \in P | d >^{\mathfrak{A}} c\}$. Note that $P_< \cup P_> \neq \emptyset$ due to the fact that P is not a unit and hence $|P| \geq 2$. Obviously, $P = P_< \cup P_c \cup P_>$. Replacing in $\mathbb{P}$ the block P by three blocks $P_<$, P_c and $P_>$ we get $\mathbb{Q} = (\mathbb{P} \setminus \{P\}) \cup \{P_<, P_c, P_>\}$ which is a new factorization of $\mathfrak{A}$ such that $\mathbb{P} \sqsubseteq \mathbb{Q}$ and $\mathbb{P} \neq \mathbb{Q}$ which contradicts maximality of $\mathbb{P}$.

(ii) Let $P, Q \in \mathbb{P}$ be such that $P \neq Q$ and for all $a \in P$ and $b \in Q$ we have either $a <^{\mathfrak{A}} b$ or $b <^{\mathfrak{A}} a$. For every $a \in P$, define the subsets $Q_{<a} = \{b \in Q | \ b <^{\mathfrak{A}} a\}$ and $Q_{>a} = \{b \in Q | \ a <^{\mathfrak{A}} b\}$. Obviously, $Q_{<a} \cup Q_{>a} = Q$. Note that we have either $Q_{<a} = \emptyset$ or $Q_{>a} = \emptyset$. Otherwise, replacing in $\mathbb{P}$ the block Q by two non-empty blocks $Q_{<a}$ and $Q_{>a}$ we get $\mathbb{Q} = (\mathbb{P} \setminus \{Q\}) \cup \{Q_{>a}, Q_{<a}\}$—a new factorization of $\mathfrak{A}$ such that $\mathbb{P} \sqsubseteq \mathbb{Q}$ and $\mathbb{P} \neq \mathbb{Q}$ which contradicts maximality of $\mathbb{P}$.

Now, split P into the subsets $P_{<Q} = \{a \in P \mid Q_{<a} = Q\}$ and $P_{>Q} = \{a \in P \mid Q_{>a} = Q\}$. Again, we have either $P_{<Q} = \emptyset$ or $P_{>Q} = \emptyset$. Otherwise, replacing in $\mathbb{P}$ the block P by two blocks $P_{<Q}$ and $P_{>Q}$ we get a factorization $\mathbb{Q}' = (\mathbb{P} \setminus \{P\}) \cup \{P_{>Q}, P_{<Q}\}$ such that $\mathbb{P} \sqsubseteq \mathbb{Q}'$ and $\mathbb{P} \neq \mathbb{Q}'$ which again contradicts maximality of $\mathbb{P}$. Hence, it follows that for every $P, Q \in \mathbb{P}$ we have either $P <^{\mathfrak{A}} Q$ or $Q <^{\mathfrak{A}} P$ or there exist elements $a \in P$ and $b \in Q$ such that $a \sim^{\mathfrak{A}} b$, which terminates the proof. □

Now, we prove as follows.

Theorem 7. *Let φ be a satisfiable $\mathcal{FO}^2_u 1\mathrm{PO}_{fw}$-formula in basic normal form and let $M \geq 2$. Then, φ has a model with an M-balanced factorization.*

Proof. Let $\mathfrak{A} \models \varphi$ and $\mathbb{P}$ be a maximal factorization of $\mathfrak{A}$ that exists by Lemma 5.

We first mark at most M elements in every block of $\mathbb{P}$. Namely, if $|P| < M$, we mark all elements of P, otherwise we mark M distinct elements in P. We denote the set of marked elements in block P by A^P. Now, we use the marked elements to build a new structure $\mathfrak{B}$ with a partial order $<^{\mathfrak{B}}$ as follows:

(i) $B = \bigcup_{P \in \mathbb{P}} A^P$,
(ii) for each $P \in \mathbb{P}$, for each $a \in A^P$ set $\text{tp}^{\mathfrak{B}}[a] = tp(P)$,
(iii) for each $P \in \mathbb{P}$, for each $a, a' \in A^P$, if $a \neq a'$, set $a \sim^{\mathfrak{B}} a'$,
(iv) for each $P, Q \in \mathbb{P}$ such that $P <^{\mathfrak{A}} Q$ set $A^P <^{\mathfrak{B}} A^Q$,
(v) for each $P, Q \in \mathbb{P}$ such that $P \sim^{\mathfrak{A}} Q$, for each $a \in A^P$, $b \in A^Q$ set $a \sim^{\mathfrak{B}} b$.

In other words, (i) says that the domain of $\mathfrak{B}$ consists of all previously marked elements of $\mathfrak{A}$ and (ii) ensures that the 1-types are preserved. In particular,

$$\boldsymbol{\alpha}^{\mathfrak{B}} = \boldsymbol{\alpha}^{\mathfrak{A}} \text{ and } B^{\alpha} \subseteq A^{\alpha} \text{ for every 1-type } \alpha. \qquad (*)$$

Condition (iii) says that distinct elements within a block of $\mathbb{P}$ are not comparable in $\mathfrak{B}$. Condition (iv) says that the block order from $\mathbb{P}$ is preserved in $\mathfrak{B}$; and condition (v) means that if two blocks $P, Q \in \mathbb{P}$ are not comparable in $\mathfrak{A}$, then $A^P \times A^Q$ contains no pair of comparable elements in $\mathfrak{B}$. In particular, for every $a, b \in B$ we have: if $\mathfrak{B} \models a < b$ then $\mathfrak{A} \models a < b$. Note that Lemma 6 guarantees that for two blocks $P, Q \in \mathbb{P}$, $P \sim^{\mathfrak{A}} Q$ implies that there exist two elements $a \in P$ and $b \in Q$ such that $a \sim^{\mathfrak{A}} b$. Hence, $P' \sim^{\mathfrak{B}} Q'$ implies that the 2-type realized by $a' \in P'$ and $b' \in Q'$ is also realized in $\mathfrak{A}$. Hence, conditions (iii), (iv) and (v) ensure that all 2-types realized in $\mathfrak{B}$ are also realized in $\mathfrak{A}$:

$$\boldsymbol{\beta}^{\mathfrak{B}} \subseteq \boldsymbol{\beta}^{\mathfrak{A}}. \qquad (**)$$

To show that $<^{\mathfrak{B}}$ is antisymmetric, suppose there exist $a, b \in B$ such that $a <^{\mathfrak{B}} b$ and $b <^{\mathfrak{B}} a$. By (iii), $a \in A^P$, $b \in A^Q$ for some $P, Q \in \mathbb{P}$ such that $P \neq Q$. By construction, we have $P <^{\mathfrak{A}} Q$ and $Q <^{\mathfrak{A}} P$, which violates antisymmetry of $<^{\mathfrak{A}}$.

To show that $<^{\mathfrak{B}}$ is transitive, suppose there exist $a \in A^P, b \in A^Q, c \in A^R$ such that $\mathfrak{B} \models a < b \wedge b < c$. By construction, $P <^{\mathfrak{A}} Q$ and $Q <^{\mathfrak{A}} R$. Since $<^{\mathfrak{A}}$ is transitive, we have $P <^{\mathfrak{A}} R$. Hence, by (iv), $\mathfrak{B} \models a < c$.

Now, let $\mathbb{P}' = \bigcup_{P \in \mathbb{P}} \{A_P\}$. The above construction ensures that $\mathbb{P}'$ is a factorization of $\mathfrak{B}$. Moreover, it is a maximal factorization. Let $D \subseteq B$ be an antichain in $\mathfrak{B}$. Observe that if two elements of D realize the same 1-type in $\mathfrak{B}$, they belong to one block of $\mathbb{P}'$. Hence, since the size of every block in $\mathbb{P}'$ is bounded by M, the size of a maximal antichain in $\mathfrak{B}$ is bounded by $M \cdot |\boldsymbol{\alpha}|$.

Now, we show that $\mathfrak{B}$ satisfies φ.

All conjuncts of the form (B1), (B8), (B9) and (B10) are obviously true due to $(*)$.

All conjuncts of the form (B2a), (B2b), (B3), (B4), (B5a) and (B5b) are obviously true due to $(**)$.

We show that all conjuncts of the form (B8) are true in $\mathfrak{B}$. Let $\xi = \forall x(\alpha \rightarrow \exists y(\mu(y) \wedge x \sim y))$ be such a conjunct. Let $a \in B$ be an element such that $\text{tp}^{\mathfrak{B}}[a] = \alpha$. Since $\text{tp}^{\mathfrak{B}}[a] = \text{tp}^{\mathfrak{A}}[a]$ and $\mathfrak{A} \models \xi$, there is $b \in A$ such that $\mathfrak{A} \models \mu[b] \wedge a \sim^{\mathfrak{A}} b$. Let $P, Q \in \mathbb{P}'$ be the blocks containing, respectively, a and b. Note that by construction, $\mathbb{P}'$ is a maximal factorization. If $P = Q$, then P contains two elements and by part (i) of Lemma 6, there is $b' \in A^P$ such that $\mathfrak{B} \models a \sim b'$. In case, $P \neq Q$, by part (ii) of Lemma 6, there is $b' \in A^Q$ such that $\mathfrak{B} \models a \sim b'$. Hence, in any case there is an element $b' \in B$ such that $\mathfrak{B} \models \mu[b'] \wedge a \sim b'$.

Now, we show that the structure $\mathfrak{B}$ can be extended to $\mathfrak{B}''$ with an M-balanced factorization so that $\mathfrak{B}''$ satisfies φ.

Let $\mathbb{P}'$ be a maximal factorization of $\mathfrak{B}$. Note that each element $a \in B$ which belongs to a block $P \in \mathbb{P}'$ other than a king can be *copied*, that is we can construct a new structure $\mathfrak{B}'$ as follows: $B' = B \cup \{a'\}$, where a' is called a *copy* of a, $\mathbb{P}'' = (\mathbb{P}' \setminus \{P\}) \cup \{P' \cup \{a'\}\}$ for each $b, c \in B$ we set $\text{tp}^{\mathfrak{B}'}[b, c] = \text{tp}^{\mathfrak{B}}[b, c]$ and for each $b \in B$ we set $\text{tp}^{\mathfrak{B}'}[a', b] = \text{tp}^{\mathfrak{B}}[a, b]$. In the case if for all pairs of distinct elements $a, b \in A$ realizing the same 1-type α, we have $a <^{\mathfrak{B}} b$ or $b <^{\mathfrak{B}} a$, we set $a <^{\mathfrak{B}'} a'$. Such structure $\mathfrak{B}'$ is obviously

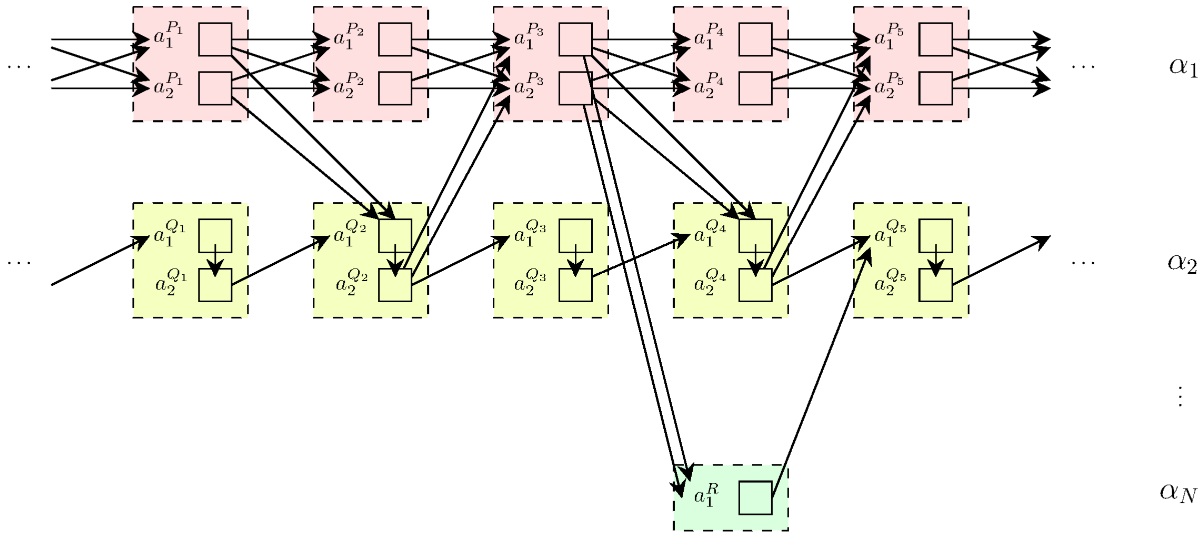

Figure 2: A σ-structure $\mathfrak{B}$ with its 2-balanced factorization. The blocks of $\mathbb{P}'$ are indicated with dashed rectangles, the relation $<$ is the transitive closure of the above directed graph and $\alpha_1, \alpha_2, ..., \alpha_N$ are 1-types. The elements which realize the 1-type α_2 are formed from blocks being units and the element which realizes the 1-type α_N is formed from a block being a king.

a model of φ. Note that $\mathbb{P}''$ might not be a maximal factorization. Repeating the operation of copying elements an appropriate number of times, we obtain a structure $\mathfrak{B}''$ with an M-balanced factorization which is a model of φ. $\square$

Observe that a model of the formula φ obtained according to Theorem 7 for $M = 2$ has antichains bounded in $2 \cdot |\boldsymbol{\alpha}|$. Thus, we obtain the following finite antichain property.

Corollary 8. *Let φ be a $\mathcal{FO}^2 1\mathrm{PO}_{fw}$-formula in basic normal form. If φ is satisfiable, then it has a model with bounded antichains.*

A possible σ-structure with its M-balanced factorization is shown in Fig. 2. Such structures play a major role in the proof of the finite antichain property for the whole logic $\mathcal{FO}^2 1\mathrm{PO}_{fw}$ that we provide now.

Theorem 9. *Let φ be a satisfiable $\mathcal{FO}^2 1\mathrm{PO}_{fw}$-sentence. Then it has a model with bounded antichains.*

Proof. Let Φ be a $\mathcal{FO}^2 1\mathrm{PO}_{fw}$-sentence in transitive normal form (3), where binary predicates other than $<$ and $=$ are allowed. For convenience, we recall the form below

$$\forall x \forall y\, \psi_0 \wedge \quad \bigwedge_{i=1}^{m} \forall x\, (P_i(x) \rightarrow \exists y\, (x \sim y \wedge \psi_i(x, y))). \tag{9}$$

We proceed similarly to the proof of Theorem 7, applying the following changes. First, we require that the constant M in the proof equals $3m$. Secondly, we modify $\mathfrak{B}''$ with its factorization $\mathbb{P}''$ as follows. Let $P, Q \in \mathbb{P}''$ be (not necessarily distinct) blocks. We put binary relations other than $<$, so that each element $a \in P \cup Q$ has all required witnesses for the $\forall\exists$-conjuncts in $P \cup Q$. Finally, we put binary relations other than $<$ so that only 2-types from $\mathfrak{A}$ appear in $P \cup Q$. It guarantees that we construct a structure $\mathfrak{B}''$ with its M-balanced factorization such that it does not realize a 2-type which is not realized in $\mathfrak{A}$ (universal constraints are still satisfied) and we can find for each element a of $\mathfrak{B}''$ all (at most m) appropriate witnesses b for the $\forall\exists$-conjuncts, no matter whether $a, b \in B''$ should share the same 1-type or not. In the former case, a witness b can be found inside the block P of a. Namely, the constant $M = 3m$ guarantees that we are able to define 2-types in $\mathfrak{B}''$ so that for each element $a' \in P$ there exist all witnesses (at most m) in the same block P. In the latter case, the fact that $P \sim^{\mathfrak{A}} Q$ implies $P \sim^{\mathfrak{B}''} Q$, guarantees that we are able to find all (at most m) witnesses b in $\mathfrak{B}''$. So, $M \geq 2m$

allows us to define 2-types in $\mathfrak{B}''$ so that for each element $a' \in P$ there exist all witnesses (at most m) in the block Q and for each element $b' \in Q$ there exist all witnesses (at most m) in the block P.

Hence, $\mathfrak{B}''$ is a model of Φ with antichains bounded in $3m \cdot |\boldsymbol{\alpha}|$. Since all the transformations required to obtain normal forms do not influence cardinalities of antichains, the above observation can be generalized to arbitrary $\mathcal{FO}^2 1\text{PO}_{fw}$-sentences. □

5. Retaining the finite model property

In this section we show that in the logic $\mathcal{FO}^2_u 1\text{PO}_{fw}$ the entanglements between distinct 1-types are crucial for losing the finite model property. Namely, we show the following theorem.

Theorem 10. *Let φ be an $\mathcal{FO}^2_u 1\text{PO}_{fw}$-formula in basic normal form not featuring conjuncts of the form* (B5b)*. If φ is satisfiable then it has a finite model.*

Proof. Let φ be a satisfiable $\mathcal{FO}^2_u 1\text{PO}_{fw}$-formula and let $\mathfrak{A}$ be a model of φ where $<^{\mathfrak{A}}$ is interpreted as the partial order.

For each pair of 1-types α and β, we say that α is *less than* β if φ contains a conjunct $\forall x(\alpha \rightarrow \forall y(\beta(y) \rightarrow x < y))$, being of the form (B3), and for each 1-type α, we say that α is *linearly ordered* if φ contains a conjunct $\forall x(\alpha \rightarrow \forall y(\alpha(y) \wedge x \neq y \rightarrow (x < y \vee y < x)))$, being of the form (B5a). Let $\prec^{\mathfrak{A}}$ be a binary relation on A such that for each pair of distinct elements $a, b \in A$ realizing the 1-types $\text{tp}^{\mathfrak{A}}[a] = \alpha$ and $\text{tp}^{\mathfrak{A}}[b] = \beta$, α is less than β. Let $\prec^+$ be the transitive closure of $\prec^{\mathfrak{A}}$. The relation $\prec^+$ is a partial order because it is transitive and $\prec^+$ is antisymmetric due to the fact that for each element $a, b \in A$, $a \prec^+ b$ implies $a <^{\mathfrak{A}} b$.

Now, for each $\alpha \in \boldsymbol{\alpha}^{\mathfrak{A}}$, define

$$g(\alpha) = \begin{cases} \{a_1^\alpha\} & \text{if } |A^\alpha| = 1 \text{ or } \alpha \text{ is linearly ordered,} \\ \{a_1^\alpha, a_2^\alpha\} & \text{otherwise,} \end{cases},$$

where a_1^α, a_2^α ($a_1^\alpha \neq a_2^\alpha$) are distinguished elements of A^α.

Define $B = \bigcup_{\alpha \in \boldsymbol{\alpha}^{\mathfrak{A}}} g(\alpha)$, $\mathfrak{B} = \mathfrak{A}\restriction B$ and $<^{\mathfrak{B}} = \prec^+ \cap B^2$. Of course, since $\prec^+$ is a partial order, $<^{\mathfrak{B}}$ is also a partial order.

It suffices to prove that $\mathfrak{B}$ satisfies φ.

All conjuncts of the form (B1), (B9), (B10) are true because $\boldsymbol{\alpha}^{\mathfrak{B}} = \boldsymbol{\alpha}^{\mathfrak{A}}$ and $B^\alpha \subseteq A^\alpha$ for every 1-type α.

All conjuncts of the form (B5a) are true since by the definition of $\mathfrak{B}$, for all 1-types α which are linearly ordered, $|B^\alpha| = 1$. Otherwise, if $|A^\alpha| > 1$, then $|B^\alpha| = 2$ and for each distinct $a, b \in B^\alpha$ we have $a \sim^{\mathfrak{B}} b$. Hence, all conjuncts of the form (B2a) are also true in $\mathfrak{B}$.

Let us prove (B2b), (B3), (B4). By the definition of $\mathfrak{B}$, the relation $\prec^+$ cannot imply the situation that for any distinct 1-types $\alpha, \beta \in \boldsymbol{\alpha}^{\mathfrak{B}}$, there exist $a' \in g(\alpha)$ and $b' \in g(\alpha)$ such that the 2-type $tp^{\mathfrak{B}}[a', b']$ contradicts any of the conjuncts of the form (B2b), (B3), (B4). In the case of (B2b) and (B4), the definition of $\mathfrak{B}$ implies that $a' \sim^{\mathfrak{B}} b'$ and in the case of (B3), the definition of $\mathfrak{B}$ implies that $a' <^{\mathfrak{B}} b'$.

We show that all conjuncts of the form (B8) are true in $\mathfrak{B}$. Let $\xi = \forall x(\alpha \rightarrow \exists y(\mu(y) \wedge x \sim y))$ be such a conjunct. For each $a \in A$ with its 1-type $\text{tp}^{\mathfrak{A}}[a] = \alpha$, there exists a distinct element $b \in A$ with its (not necessarily distinct) 1-type $\text{tp}^{\mathfrak{A}}[b] = \beta$ such that $\mathfrak{A} \models \mu[b] \wedge a \sim^{\mathfrak{A}} b$. Consider two cases.

1. $\alpha \neq \beta$. We have $a \sim^{\mathfrak{A}} b$ and a, b are not connected by the relation $\prec^+$. Then, by the definition of $\mathfrak{B}$, for all $a' \in g(\alpha)$ with their 1-type α and for all $b' \in g(\beta)$ with their 1-type β (where $a' \neq b'$), we have $a' \sim^{\mathfrak{B}} b'$. This implies that for each $a' \in g(\alpha)$ there exists a witness $b'' \in g(\beta)$ such that $\mathfrak{B} \models \mu[w] \wedge a' \sim b''$.
2. $\alpha = \beta$. By the definition of $\mathfrak{B}$, we know that $|g(\alpha)| = 2$ since α cannot be linearly ordered and $|A^\alpha| \geq 2$. So, for each $a' \in g(\alpha)$ there exists a witness $b' \in g(\alpha)$ such that $a' \neq b'$ and $\mathfrak{B} \models \mu[b'] \wedge a' \sim b'$.

Thus, we conclude that $\mathfrak{B}$ is a finite model of φ which has the size at most $2 \cdot |\boldsymbol{\alpha}|$. □

Discussion

One might argue that the restricted fragment identified in Theorem 10 that enjoys the finite model property is very limited. However, it allows one to express the mutual exclusion property of events in concurrent systems: for two events x and y the formula $\neg\exists x\exists y(Cx \wedge Cy \wedge x \sim y)$ says that x and y cannot access the same critical section at the same time. This formula, when written in negation normal form, constitutes an acceptable conjunct of the form (B5a). In this fragment one can also express cross product of two classes of elements corresponding to natural statements such as "elephants are bigger than mice" that are not guarded (cf. e.g. [28] for more motivation and results about description logics with such statements).

In view of the finite model property established in Theorem 10 it is worth pointing out that the problematic universal formulas are the entanglements of the form $\alpha \bowtie \beta$, with $\alpha \neq \beta$, that require to order every pair of elements from two distinct classes but, in contrast to the situation between elephants and mice, the order is not predefined. Formulas of this form define models that are not N-free. For instance, the spirals defined in Section 3 contain elements that are exactly in the forbidden configuration, e.g. in the structure $\mathfrak{D}$ depicted in Fig. 1 we have $a_2 > a_0 < a_3 > a_1$. Hence, branching automata mentioned in the Introduction, developed to deal with N-free posets, are not directly applicable to recognize models of such formulas.

So, in this article, we have also identified a minimal fragment of the two-variable logic with one partial order that is critical for answering the question on decidability of Sat$(\mathcal{FO}^2 1\mathrm{PO})$. Namely, it suffices to concentrate on: (i) signatures without equality comprising arbitrarily many unary predicates and one binary predicate required to be interpreted as a partial order; (ii) sentences in basic normal form as given in Definition 2 where only entanglements $\alpha \bowtie \beta$ of the form (B5b) and conjuncts of the form (B8) appear. We believe that the fragment is decidable and we plan to study its satisfiability problem on scattered structures. The finite antichain property established in this paper suggests that some automata techniques might be generalized to handle such structures.

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