Separating Counting from Non-Counting in Fragments of Two-Variable First-Order Logic (Extended Abstract)

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Abstract

We consider the problem of deciding whether two disjoint classes of models defined in a fragment of first-order logic (FO) with counting can be separated in the same fragment but without counting. This problem turns out to be hard. We show that separation for the two-variable fragment FO^2 extended with counting quantifiers by means of plain FO^2 is undecidable, and the same is true of the pair $\mathcal{ALCOIQ}/\mathcal{ALCOI}$ of description logics. On the other hand, we establish 2ExpTime-completeness of the separation problem for the pairs $\mathcal{ALCQ}^u/\mathcal{ALC}^u$ and $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$.

Keywords

Separation, two-variable first-order logic, counting quantifiers, bisimulation.

1. Introduction

Our concern in this paper is the following separation problems for a pair of languages L and L^s:

L/L^s**-separation:** given two mutually exclusive L-formulas φ and ψ, decide whether there exists an L^s-formula χ—a *separator* for φ and ψ—such that $\varphi \models \chi$ and $\chi \models \neg\psi$;

Craig L/L^s**-separation:** decide whether the given L-formulas φ and ψ have an L^s-separator χ that only contains common non-logical symbols (predicates and functions) of φ and ψ.

To illustrate, φ could be an ontology $\mathcal{O}$ and ψ a concept C that is not satisfiable with respect to $\mathcal{O}$, both given in an expressive language L. Then a separator ontology $\mathcal{O}'$ in a weaker, easier to comprehend language L^s potentially explains unsatisfiability as it inherits that $\mathcal{O} \models \mathcal{O}'$ and C is not satisfiable under $\mathcal{O}'$. Also, in the context of concept learning, φ and ψ could represent positive and negative examples for a target concept C. Then any separator in an appropriately chosen language L^s could represent the concept one aims to learn [1].

Separation generalises definability (aka membership), which asks whether a given L-formula (say, a datalog query) is equivalent to some L^s-formula (say, a first-order query), and is regarded as one of the main approaches to understanding the expressive power of L relative to L^s. For instance, studying separability of regular languages by smaller language classes (e.g., a star-free language) has brought major insights into the respective formal languages, with some fundamental open problems in the area cast as separation questions [2].

Craig L/L^s-separation generalises classical Craig interpolation in L [3] because a Craig L/L-separator for φ and ψ is a Craig interpolant for $\varphi \to \neg\psi$ in L.

Our aim in this paper is to investigate the decidability and complexity of the separation problem for certain fragments L of C^2—that is, the two-variable first-order logic FO^2 extended with counting quantifiers—and the same fragments L^s but without counting.

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Example 1. Consider the following C^2-formulas:

$$\varphi(x) = \exists^{=1} y\, R(x,y), \qquad \psi(x) = \exists^{=1} y\, \big(R(x,y) \wedge A(y)\big) \wedge \exists^{=1} y\, \big(R(x,y) \wedge \neg A(y)\big).$$

Then $\varphi \models \neg\psi$ and the FO^2-formula $\chi(x) = \forall y\, \big(R(x,y) \to A(y)\big) \vee \forall y\, \big(R(x,y) \to \neg A(y)\big)$ is a *separator* for $\varphi(x)$ and $\psi(x)$. For $\psi'(x) = \exists^{=2} y\, R(x,y)$, we also have $\varphi \models \neg\psi'$, but $\varphi(x)$ and $\psi'(x)$ are not separable in FO^2. On the other hand, there is no Craig FO^2-separator for $\varphi(x)$ and $\psi(x)$ as it would have to be defined using R only, and so separate $\varphi(x)$ and $\psi'(x)$ as well.

2. Logics

The logics we consider here can all be regarded as fragments of first-order logic, FO, and are defined as follows. Let σ be a *signature* containing unary and binary relation symbols and possibly constants. Fix a set *var* comprising two individual variables. Then

$\mathsf{FO}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}(\sigma)$, is defined as the set of formulas that are built from atoms $A(x)$, $R(x,y)$, and $x = y$ with unary $A \in \sigma$, binary $R \in \sigma$, and $x, y \in$ *var*, using the Boolean connectives $\wedge$ and $\neg$ and quantifier $\exists x$ with $x \in$ *var* (other Booleans and $\forall x$ are regarded as standard abbreviations);

$\mathsf{C}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}^2(\sigma)$ *with counting*, extends $\mathsf{FO}^2(\sigma)$ with the counting quantifiers $\exists^{<k} x$, for $k \in \mathbb{N}$ and $x \in$ *var* (other counting quantifiers $\exists^{=k} x$, $\exists^{\leq k} x$, $\exists^{\geq k} x$ can be introduced as abbreviations).

In this paper, we are only interested in formulas $\varphi(x)$ with one free variable $x \in$ *var*. The *signature* of φ is the set $sig(\varphi)$ of relation and constant symbols occurring in φ.

$\mathsf{FO}(\sigma)$ and its fragments are interpreted in σ*-structures* $\mathfrak{A} = (\mathrm{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R\in\sigma}, (c^{\mathfrak{A}})_{c\in\sigma})$ with a domain $\mathrm{dom}(\mathfrak{A}) \neq \emptyset$, relations $R^{\mathfrak{A}}$ on $\mathrm{dom}(\mathfrak{A})$ of the same arity as $R \in \sigma$, and elements $c^{\mathfrak{A}} \in \mathrm{dom}(\mathfrak{A})$. A *pointed structure* is a pair $\mathfrak{A}, a$ with $a \in \mathrm{dom}(\mathfrak{A})$.

We also consider a few fragments of C^2 that correspond to some standard description logics (DLs). In the context of DLs, unary relation symbols are called *concept names*, binary ones *role names*, and constants *individual names* [4]. A *role* is a role name r or its *inverse* r^-. The *universal role* is denoted by u. A *nominal* takes the form $\{c\}$ with an individual name c.

An $\mathcal{ALCOIQ}^u(\sigma)$*-concept* is defined by the grammar

$$C \ ::= \ \top \ \mid \ A \ \mid \ \{c\} \ \mid \ \neg C \ \mid \ C \sqcap C' \ \mid \ \geq k\; r.C \ \mid \ \exists u.C,$$

where $A \in \sigma$ is a concept, $c \in \sigma$ an individual, r a role name in σ or its inverse, and $k > 0$. We consider several fragments of $\mathcal{ALCOIQ}^u$. The weakest, $\mathcal{ALC}$, is obtained by dropping the universal role (indicated by omitting $\cdot^u$ from the name), inverse roles (indicated by omitting $\mathcal{I}$), nominals (indicated by omitting $\mathcal{O}$), and only admitting qualified number restrictions of the form $\exists r.C = (\geq 1\; r.C)$ (indicated by dropping $\mathcal{Q}$). The languages between $\mathcal{ALC}$ and $\mathcal{ALCOIQ}^u$ are now defined in the obvious way.

The semantics of DLs can be defined via the *standard translation* $\cdot^\sharp$ into C^2 with constants. For any $\mathcal{ALCOIQ}^u$-concept C, we denote by $C^\sharp_x$ the C^2-formula with constants and free variable $x \in$ *var* defined inductively by taking

$$\top^\sharp_x = (x = x), \quad A^\sharp_x = A(x), \quad \{c\}^\sharp_x = (x = c), \quad (\neg C)^\sharp_x = \neg C^\sharp_x, \quad (C \sqcap D)^\sharp_x = C^\sharp_x \wedge D^\sharp_x,$$
$$(\exists u.C)^\sharp_x = \top^\sharp_x \wedge \exists \bar{x}\, C^\sharp_{\bar{x}}, \quad (\geq k\; r.C)^\sharp_x = \exists^{\geq k} \bar{x}\, \big(r(x,\bar{x}) \wedge C^\sharp_{\bar{x}}\big),$$

where $\bar{x} = y$, $\bar{y} = x$ and $\{x, y\} =$ *var*.

The complexities of the satisfiability problems for the logics in question are as follows [4, 5]:

- FO^2, C^2, and $\mathcal{ALCOIQ}^u$ are all NExpTime-complete;
- $\mathcal{ALC}^u$, $\mathcal{ALCQ}^u$, $\mathcal{ALCIQ}^u$, and $\mathcal{ALCOI}^u$ are all ExpTime-complete.

3. Deciding Separation

Our main results are summarised in the next theorem:

Theorem 1. *The separation and Craig separation problems are*

- *undecidable for the pairs* $\mathsf{C}^2/\mathsf{FO}^2$ *and* $\mathcal{ALCOIQ}/\mathcal{ALCOI}$*,*
- 2ExpTime*-complete for the pairs* $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$ *and* $\mathcal{ALCQ}^u/\mathcal{ALC}^u$*.*

The proof of Theorem 1 is based on the following straightforward model-theoretic characterisation of separation in terms of bisimulations; see [6, 7, 8] and references therein:

Lemma 2. *Let* $\varphi(x)$ *and* $\psi(x)$ *be any* $\mathsf{C}^2(\sigma)$*-formulas,* $\varrho \subseteq \sigma$*, and let* L^s *be any of the languages introduced in Section 2. Then the following conditions are equivalent*:

- $\varphi(x)$ *and* $\psi(x)$ *do not have an* $L^s(\varrho)$*-separator*;
- *there are pointed* σ*-structures* $\mathfrak{A}, a$ *and* $\mathfrak{B}, b$ *such that*

$$\mathfrak{A} \models \varphi(a), \qquad \mathfrak{B} \models \psi(b), \qquad \mathfrak{A}, a \sim_{L^s(\varrho)} \mathfrak{B}, b.$$

For Craig separation, we additionally require that $\varrho \subseteq sig(\varphi) \cap sig(\psi)$*.*

Here, $\mathfrak{A}, a \sim_{L^s(\varrho)} \mathfrak{B}, b$ means that there is an $L^s(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(a, b) \in \boldsymbol{\beta}$, which implies that $\mathfrak{A} \models \phi(a)$ iff $\mathfrak{B} \models \phi(b)$, for all $L^s(\varrho)$-formulas $\phi(x)$ [6, 9, 5]. The proof of the characterisation in Lemma 2 is standard and similar to the characterisations of Craig interpolant nonexistence in [7, 8]. The undecidability proofs are by reduction of the halting problem for 2-register machines where the numbers in the registers are represented by the number of $L^s(\varrho)$-bisimilar nodes. The decidability proofs are based on novel adaptations of the mosaic technique for constructing $L^s(\varrho)$-bisimilar models [7, 8]. Preliminary detailed proofs are available at https://www.csc.liv.ac.uk/~frank/publ/publ.html.

4. Related Work

With the exception of separating modal μ-calculus formulas by plain modal formulas [10], separability has so far been mainly investigated for formal languages [11, 12, 13]. In contrast, definability has been investigated for many logics. For example, the problem of deciding whether a TBox given in a DL L can be equivalently expressed in another DL L' is considered in [14], the problem of deciding whether a guarded fragment (GF) formula or guarded negation fragment (GNF) formula is equivalent to an existential (or positive existential) GF formula or, respectively, GNF formula is considered in [15, 16], and there are many results on deciding whether fixpoints can be dropped from a second-order extension of a fragment of FO. Interestingly, the complexity of separation and definability of modal μ-calculus formulas by plain modal formulas are both ExpTime-complete [17, 10]. Variants of definability explored in description logic are approximation [18] and conservative rewritability [19].

Craig separators are a generalisation of Craig interpolants where $L^s = L$. If the logic L has the Craig interpolation property (CIP), then the Craig separator existence problem for φ and ψ reduces to checking whether $\varphi \models \psi$ and is thus not harder than entailment. Only recently the problem of deciding the existence of Craig interpolants has been considered for logics without the CIP [20, 21]. In fact, the bisimulation-based method employed here makes heavy use of techniques introduced for checking Craig interpolant existence [7, 8].

5. Discussion

We have started investigating the separation problem for fragments of FO with counting by formulas in the same fragments but without counting. Many problems remain to be addressed; we mention a few of them below:

1. Our decidability proofs are non-constructive, and it would be of interest to develop algorithms that construct separators whenever they exist and determine bounds on their size.

2. With the exception of $\mathcal{ALCQ}$, the logics with counting we considered do not have the finite model property. It would be of interest to investigate whether our results also hold on finite structures. In that case, the bisimulation criterion does not hold as formulated (because its proof uses compactness) and one has to employ a different criterion that holds on finite structures (say, bounded bisimulations).

3. Our logics have the universal role. We conjecture that without the universal role $\mathcal{ALCQ}/\mathcal{ALC}$- and $\mathcal{ALCIQ}/\mathcal{ALCI}$-separation becomes coNExpTime-complete.

4. Is definability less complex than separation for the pairs of languages considered here. For example, is $\mathsf{C}^2/\mathsf{FO}^2$-definability decidable?

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