On the of Limits of Decision: the Adjacent Fragment of First-Order Logic (Extended Abstract)

Bartosz Bednarczyk[1,2], Daumantas Kojelis[3] and Ian Pratt-Hartmann[3,4]

[1]Computational Logic Group, Technische Universität Dresden, Germany
[2]Institute of Computer Science, University of Wrocław, Poland
[3]Department of Computer Science, University of Manchester, UK
[4]Institute of Computer Science, University of Opole, Poland

Abstract

We define the *adjacent fragment* $\mathcal{AF}$ of first-order logic, obtained by restricting the sequences of variables occurring as arguments in atomic formulas. The adjacent fragment generalizes (after a routine renaming) two-variable logic as well as the fluted fragment. We show that the adjacent fragment has the finite model property, and that its satisfiability problem is no harder than for the fluted fragment (and hence is Tower-complete). We further show that any relaxation of the adjacency condition on the allowed order of variables in argument sequences yields a logic whose satisfiability and finite satisfiability problems are undecidable. Finally, we study the effect of the adjacency requirement on the well-known guarded fragment ($\mathcal{GF}$) of first-order logic. We show that the satisfiability problem for the guarded adjacent fragment ($\mathcal{GA}$) remains 2ExpTime-hard, thus strengthening the known lower bound for $\mathcal{GF}$.

Keywords

decidability, satisfiability, variable-ordered logics, complexity

The quest to find fragments of first-order logic for which satisfiability is algorithmically decidable has been a central undertaking of mathematical logic since the appearance of Hilbert and Ackermann's *Grundzüge der theoretischen Logik* [1, 2] almost a century ago. The great majority of such fragments so far discovered, however, belong to just three families: (i) quantifier prefix fragments [3], where we are restricted to prenex formulas with a specified quantifier sequence; (ii) two-variable logics [4], where the only logical variables occurring as arguments of predicates are x_1 and x_2; and (iii) guarded logics [5], where quantifiers are relativized by atomic formulas featuring all the free variables in their scope.

There is, however, a fourth family of decidable logics, originating in the work of W.V.O. Quine [6], and based on restricting the allowed sequences of variables occurring as arguments in atomic formulas. This family of logics, which includes the *fluted fragment*, the *ordered fragment* and the *forward fragment*, has languished in relative obscurity. In our paper, we investigate the potential for obtaining decidable fragments in this way, identifying a new fragment, which we call the *adjacent fragment*. This fragment not only includes the fluted, ordered and forward fragments, but also subsumes, in a sense we make precise, the two-variable fragment. We show that the satisfiability problem for the adjacent fragment is decidable, and determine bounds on its complexity.

To explain how restrictions on argument orderings work, we consider presentations of first-order logic without equality over purely relational signatures, employing individual variables from the alphabet $\{x_1, x_2, x_3, \ldots\}$. Any atomic formula in this logic has the form $p(\bar{x})$, where p is a predicate of arity $m \geq 0$ and $\bar{x}$ is a word over the alphabet of variables of length m. Call a first-order formula φ *index-normal* if, for any quantified sub-formula $Qx_k\,\psi$ of φ, ψ is a Boolean combination of formulas that are either atomic with free variables among $x_1, \ldots, x_k$, or have as their major connective a quantifier

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bartosz.bednarczyk@cs.uni.wroc.pl (B. Bednarczyk); daumantas.kojelis@manchester.ac.uk (D. Kojelis); ian.pratt@manchester.ac.uk (I. Pratt-Hartmann)
https://bartoszjanbednarczyk.github.io/ (B. Bednarczyk); https://daumantaskojelis.github.io/ (D. Kojelis); https://www.cs.man.ac.uk/~ipratt/ (I. Pratt-Hartmann)
0000-0002-8267-7554 (B. Bednarczyk); 0000-0002-1632-9498 (D. Kojelis); 0000-0003-0062-043X (I. Pratt-Hartmann)

binding x_{k+1}. By re-indexing variables, any first-order formula can easily be written as a logically equivalent index-normal formula. In the *fluted fragment*, denoted $\mathcal{FL}$, as defined by W. Purdy [7], we confine attention to index-normal formulas, but additionally insist that any atom occurring in a context in which x_k is quantified have the form $p(x_{k-m+1}\cdots x_k)$, i.e. $p(\bar{x})$ with $\bar{x}$ a *suffix* of $x_1\cdots x_k$. In the *ordered fragment*, due to A. Herzig [8], by contrast, we insist that $\bar{x}$ be a *prefix* of $x_1\cdots x_k$. In the *forward fragment* [9], we insist only that $\bar{x}$ be an *infix* of $x_1\cdots x_k$.

All these logics have the finite model property, and hence are decidable for satisfiability. Denoting by $\mathcal{FL}^k$ the sub-fragment of $\mathcal{FL}$ involving at most k variables (free or bound), the satisfiability problem for $\mathcal{FL}^k$ is known to be in $(k-2)$-NExpTime for all $k\geq 3$, and $\lfloor k/2\rfloor$-NExpTime-hard for all $k\geq 2$ [10]. Thus, satisfiability for the whole fluted fragment is Tower-complete, in the system of trans-elementary complexity classes due to [11]. By contrast, the satisfiability problem for ordered fragment is known to be PSpace-complete [8, 12]. On the other hand, the apparent liberalization afforded by the forward fragment yields no difference in expressive power [13].

Say that a word $\bar{x}$ over the alphabet $\{x_1,\ldots,x_k\}$ ($k\geq 0$) is *adjacent* if the indices of neighbouring letters differ by at most 1. For example, $x_3x_2x_1x_2x_2x_2x_3x_4x_3$ is adjacent, but $x_1x_3x_2$ is not. The *adjacent fragment*, denoted $\mathcal{AF}$, is analogous to the fluted, ordered and forward fragments, but we allow any atom $p(\bar{x})$ to occur in a context where x_k is available for quantification as long as $\bar{x}$ is an adjacent word over $\{x_1,\ldots,x_k\}$. As a simple example, the formula

$$\forall x_1\forall x_2\forall x_3\exists x_4\forall x_5\ \big(p(x_1x_2x_3x_2x_3x_4x_5)\rightarrow p(x_1x_2x_3x_4x_3x_4x_5)\big)$$

is a validity of $\mathcal{AF}$, as can be seen by assigning x_4 the same value as x_2. Evidently, $\mathcal{AF}$ includes the fluted, ordered and forward fragments; the inclusion is strict, since the formulas

$$\forall x_1\, r(x_1x_1),$$

$$\forall x_1x_2\big(r(x_1x_2)\rightarrow r(x_2x_1)\big),$$

stating that r is reflexive and symmetric, respectively, are in $\mathcal{AF}$. It is worth noting that the formula expressing transitivity; i.e.

$$\forall x_1x_2x_3\Big(\big(r(x_1x_2)\wedge r(x_2x_3)\big)\rightarrow r(x_1x_3)\Big),$$

is not in $\mathcal{AF}$ as the variable x_2 is skipped in the atom $r(x_1x_3)$.

To further aid intuition, we provide the following (possible) translations of english sentences into $\mathcal{AF}$. “Every student taking a programming course also takes some maths course” can be written as:

$$\forall x_1x_2\Big(\mathrm{Prog}(x_1)\wedge\mathrm{Stud}(x_2)\wedge\mathrm{Takes}(x_2,x_1)\rightarrow\exists x_3\big(\mathrm{Math}(x_3)\wedge\mathrm{Takes}(x_2,x_3)\big)\Big)$$

“Every languages student either recommends Norwegian to their peers or is recommended Norwegian by someone” may be written as:

$$\forall x_1x_2\Big(\mathrm{Stud}(x_1)\wedge\mathrm{Nor}(x_2)\rightarrow\big(\forall x_3\ \mathrm{Rec}(x_1,x_2,x_3)\big)\vee\big(\exists x_3\ \mathrm{Rec}(x_3,x_2,x_1)\big)\Big).$$

In the sequel we will define the fragment formally.

Let m and k be non-negative integers. For any integers i and j, we write $[i,j]$ to denote the set of integers h such that $i\leq h\leq j$. A function $f\colon[1,m]\rightarrow[1,k]$ is *adjacent* if $|f(i+1)-f(i)|\leq 1$ for all i ($1\leq i<m$). We write $\mathbf{A}_k^m$ to denote the set of adjacent functions $f\colon[1,m]\rightarrow[1,k]$. Since $[1,0]=\emptyset$, we have $\mathbf{A}_k^0=\{\emptyset\}$, and $\mathbf{A}_0^m=\emptyset$ if $m>0$. Let A be a non-empty set. A word $\bar{a}$ over the alphabet A is simply a tuple of elements from A. Accordingly, A^k denotes the set of words over A having length exactly k, and A^* is the set of all finite words over A. Any function $f\colon[1,m]\rightarrow[1,k]$ (adjacent or not) induces a natural map from A^k to A^m defined by $\bar{a}^f=a_{f(1)}\cdots a_{f(m)}$, where $\bar{a}=a_1\cdots a_k$. If $f\in\mathbf{A}_k^m$ (i.e. if f is adjacent), we may think of $\bar{a}^f$ as the result of a ‘walk’ on the tuple $\bar{a}$, starting at the element $a_{f(1)}$, and moving left, right, or remaining stationary according to the sequence of values

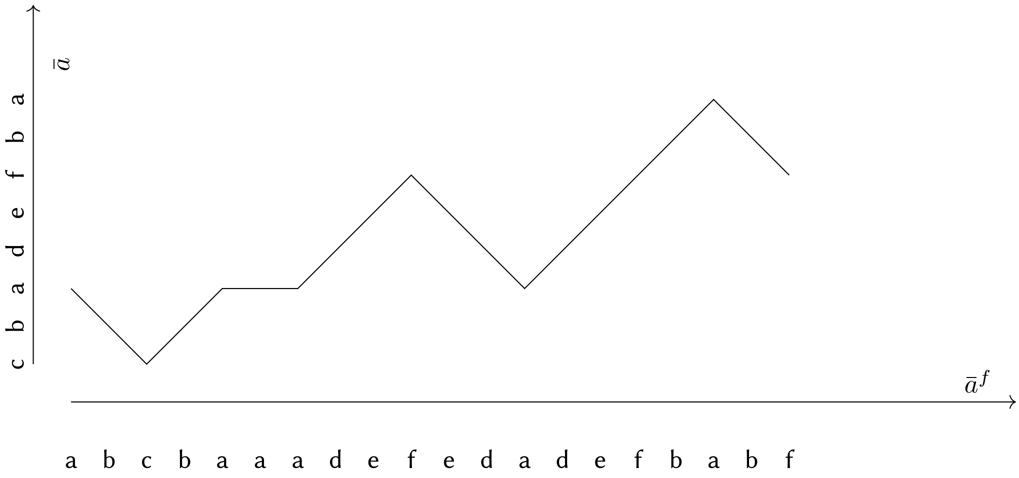

Figure 1: Generation of abcbaaadefedadefbabf from cbadefba.

$f(i+1)-f(i)$ $(1 \leq i < m)$. We may picture a walk as a piecewise linear function, with the generat*ed* word superimposed on the abscissa and the generat*ing* word on the ordinate, c.f. Figure 1.

For any $k \geq 0$, denote by $\mathbf{x}_k$ the fixed word $x_1 \cdots x_k$ (if $k = 0$, this is the empty word). A k-*atom* is an expression $p(\mathbf{x}_k^f)$, where p is a predicate of some arity $m \geq 0$, and $f\colon [1, m] \to [1, k]$. Thus, in a k-atom, each argument is a variable chosen from $\mathbf{x}_k$. If f is adjacent, we speak of an *adjacent* k-*atom*. Thus, in an adjacent k-atom, the indices of neighbouring arguments differ by at most one. When $k \leq 2$, the adjacency requirement is vacuous, and in this case we prefer to speak simply of k-*atoms*. Proposition letters (predicates of arity $m = 0$) count as (adjacent) k-atoms for all $k \geq 0$, taking f to be the empty function. When $k = 0$, we perforce have $m = 0$, since otherwise, there are no functions from $[1, m]$ to $[1, k]$; thus the 0-atoms are precisely the proposition letters.

We define the sets of first-order formulas $\mathcal{AF}^{[k]}$ by simultaneous structural induction:

1. every adjacent k-atom is in $\mathcal{AF}^{[k]}$;
2. $\mathcal{AF}^{[k]}$ is closed under Boolean combinations;
3. if φ is in $\mathcal{AF}^{[k+1]}$, $\exists x_{k+1}\, \varphi$ and $\forall x_{k+1}\, \varphi$ are in $\mathcal{AF}^{[k]}$.

Formally, we call $\mathcal{AF} = \bigcup_{k \geq 0} \mathcal{AF}^{[k]}$ the *adjacent fragment*. Note that formulas of $\mathcal{AF}$ contain no individual constants, function symbols or equality.

As every word over $\{x_1, x_2\}$ is adjacent, we may transform any formula of the two-variable fragment without equality, FO^2, in polynomial time, to a logically equivalent formula of $\mathcal{AF}$. The converse is true over signatures with predicates of arity at most two. Since the system of basic multimodal propositional logic is, under the standard translation to first-order logic, included within FO^2, this logic is similarly subsumed by $\mathcal{AF}$, as indeed is its notational variant, the description logic $\mathcal{ALC}$ (see, e.g. [14]).

We show that the satisfiability problem for the restriction of the adjacent fragment to formulas involving at most k variables (free or bound) is in $(k-2)$-NExpTime for all $k \geq 3$—and hence no more difficult than the k-variable fluted fragment, which it properly contains. The critical step in our analysis is [15, Theorem 3.1]–a theorem on the combinatorics of strings, which may be of independent interest. We also consider minimal relaxations of adjacency involving the fragment with just three variables, and show that, in all cases of interest, the satisfiability and finite satisfiability problems for the resulting logics are undecidable. Thus, adjacency is as far as we can go in seeking decidable fragments based on straightforward argument ordering restrictions of the type envisaged by Quine.

The adjacent fragment is incomparable in expressive power to the guarded fragment. Moreover, the satisfiability problem for the union of $\mathcal{GF}$ and $\mathcal{AF}$ is undecidable, as one can use adjacent formulas to introduce any k-ary universal relations, which makes $\mathcal{GF}$ as expressive as first-order logic. Therefore, we study the effect of the adjacency restriction on $\mathcal{GF}$. We investigate the complexity of satisfiability for the *guarded adjacent fragment* $\mathcal{GA}$, showing that the problem is 2ExpTime-complete, thus sharpening the existing 2ExpTime-hardness proof for $\mathcal{GF}$ [16].

Denoting $\mathcal{AF}^{\ell}$ for the ℓ-variable adjacent fragment we establish the following results using a variable reduction technique similar to that as for the fluted fragment.

Theorem 1. *The (finite) satisfiability problem for $\mathcal{AF}^{\ell}$ is in $(\ell - 2) -$ NExpTime and $\lfloor \ell/2 \rfloor -$ NExpTime-hard.*

This allows us to conclude the following about the whole fragment.

Theorem 2. *The (finite) satisfiability problem for $\mathcal{AF}$ is Tower-complete.*

We also sharpen existing lower bounds for the (finite) satisfiability of the guarded fragment by encoding an alternating turing machine running in exponential space in an adjacent way thus establishing the following.

Theorem 3. *The (finite) satisfiability problem for $\mathcal{GA}$ is 2ExpTime-complete.*

The full paper is published in the proceedings of the 50th International Colloquium on Automata, Languages, and Programming (ICALP 2023) [15]. The accompanying technical report with detailed proofs is available on arxiv [17].

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